General free particle quantum chains: para-fermionic Z(N) and fermionic XX quantum chains

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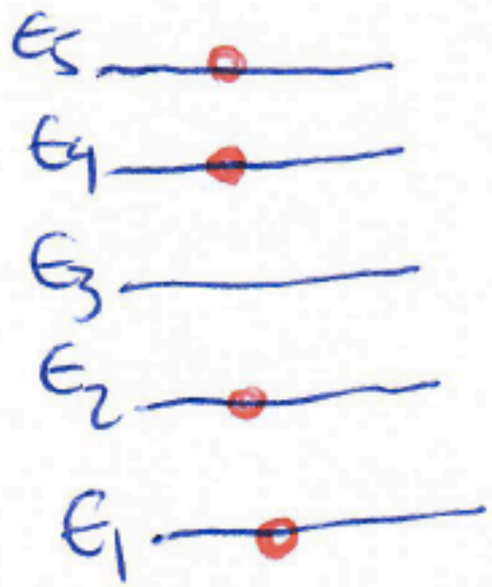
What is a free fermion and a free parafermion?

Remember: Stat Mech undergrad: "quantum gases"



$$E_{\{n_i\}} = n_1\epsilon_1 + n_2\epsilon_2 + \ldots + n_{\bar{M}}\epsilon_{\bar{M}} \,, \quad n_i = 0, 1$$

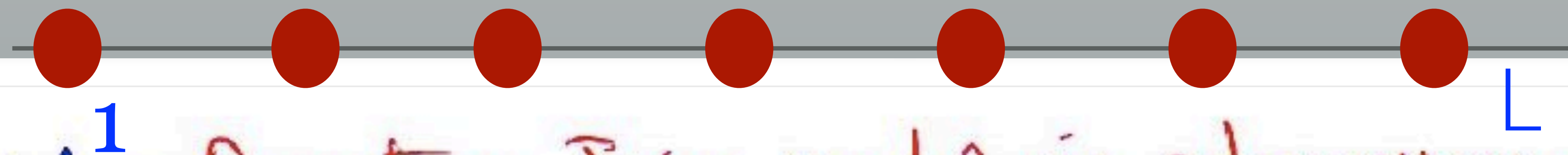
$$E_{\{n_i\}} \rightarrow 2E_{\{n_i\}} - (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{\bar{M}})$$

$$E_{\{n_i\}} = -(\omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 + \ldots + \omega^{s_{\bar{M}}}\epsilon_{\bar{M}}),$$

$$\omega = e^{i\frac{2\pi}{2}} = -1 \,, \quad s_i = 0, 1$$

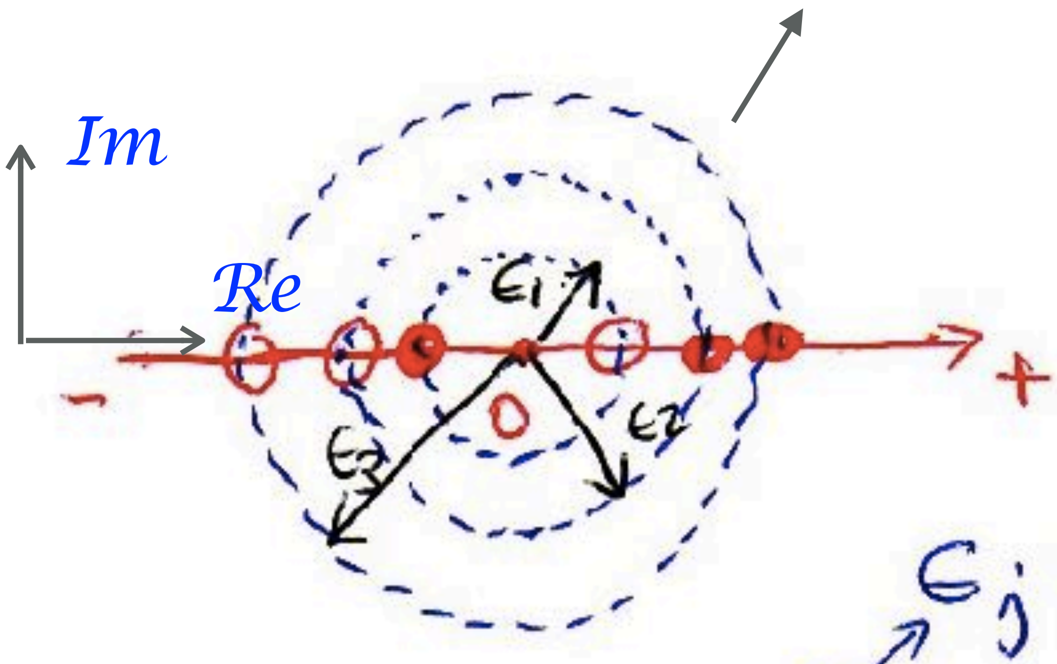
$$\boxed{\text{The fermions are free}}$$

$$Z(\beta) = (e^{-\beta\epsilon_1} + e^{\beta\epsilon_1}) \cdot (e^{-\beta\epsilon_2} + e^{\beta\epsilon_2}) \cdots (e^{-\beta\epsilon_{\bar{M}}} + e^{\beta\epsilon_{\bar{M}}})$$



Example: Quantum Ising model in a transverse field

"circle-repulsion"



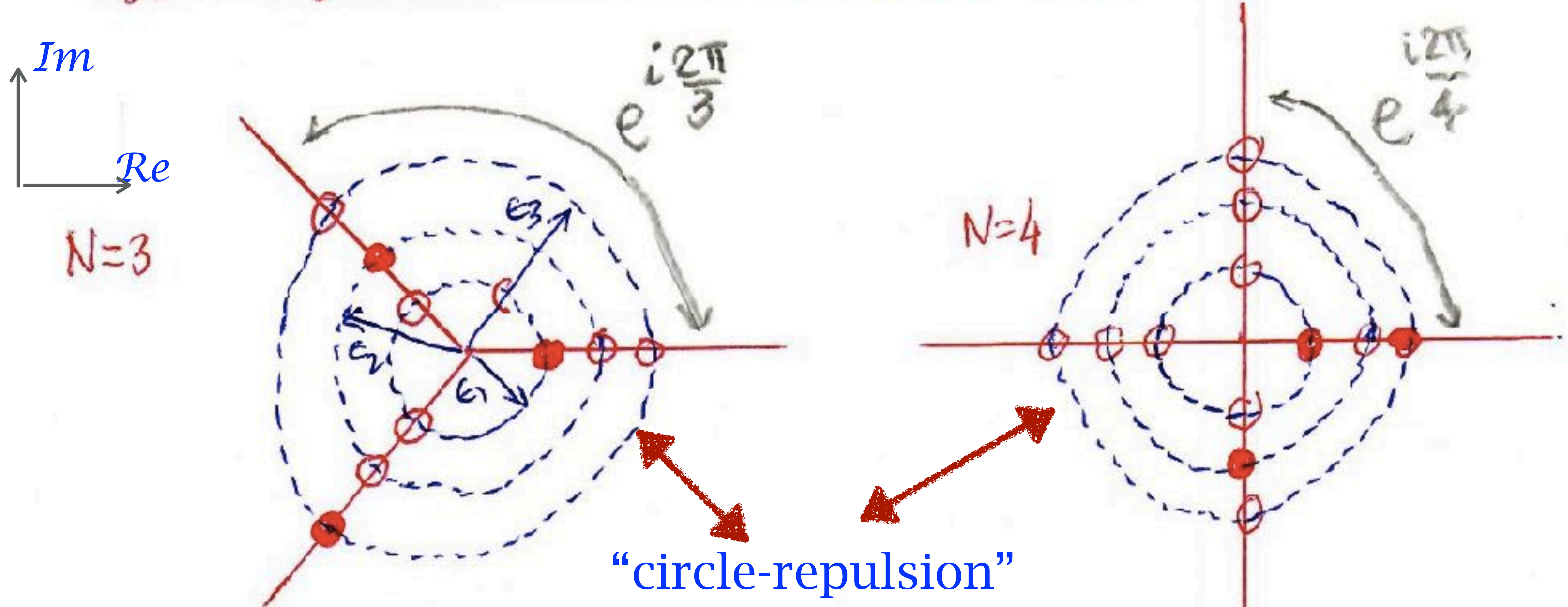
$$\mathcal{H} = -\lambda \sum_{i=1}^{L} \sigma_i^x - \sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z \qquad \text{(FBC)}$$

$$\epsilon_j = 2\cos\left(\frac{2\pi j}{2L+1}\right), \quad j = 1, 2, \ldots$$

Quasi-energies

$\lambda = 1$

How woud be free parafermions?



$$E_{\{s_i\}} = -(\omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 + \dots + \omega^{s_{\bar{M}}}\epsilon_{\bar{M}})$$

$$s_i = 0,1,\dots,N-1, \qquad \omega = e^{i\frac{2\pi}{N}}$$

$$\Xi_1(\beta) = \prod_{i=1}^{\bar{M}} \left(e^{\beta\epsilon_1} + e^{\beta\omega\epsilon_i} + e^{\beta\omega^2\epsilon_i} + \dots + e^{\beta\omega^{N-1}\epsilon_i} \right) \in \mathbb{R}$$

Example: Free-parafermionic $Z(N)$ Baxter Chain

$$\mathcal{H}_B = -\lambda \sum_{i=1}^{L} R_i - \sum_{i=1}^{L-1} S_i S_{i+1}^{+}$$

$$RS = \omega S R \quad , \quad S^N = R^N = 1 \quad , \quad S^{+} = S^{N-1} \leftarrow \text{N x N matrices}$$

$$\omega = e^{i2\pi/N}$$

$$\epsilon_i = \left(2\cos\left(\frac{\pi i}{2L+1}\right)\right)^{2/N} \quad (\text{for } \lambda = 1)$$

Ex $N=3$

$$\omega = e^{i2\pi/3}$$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

non-Hermitian (complex spectrum)

What is essential for fixing the eigenspectra?

Let us see in the Ising case

$$\mathcal{H}_{\mathrm{II}} = \lambda \sum_{i=1}^{L} \sigma_i^x - \sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z \qquad (C.P.C)$$

$$\mathcal{H}_{I} = - \sum_{j=1}^{M} h_j \qquad \text{recall: } \sigma^x \sigma^z = -\sigma^z \sigma^x$$

$$h_{2i-1} = \lambda \sigma_i^x \quad , \quad h_{2i} = \sigma_i^z \sigma_{i+1}^z$$

$$h_i h_{i+1} = \omega h_{i+1} h_i \quad , \quad [h_i, h_j] = 0 \quad (|i-j|>1),$$

$$\omega = e^{i\frac{2\pi}{2}} = -1$$

For the $\mathbb{Z}(N)$ – Baxter Chain

$$\mathcal{H} = -\sum_{i=1}^{M} h_i$$

$$h_1 = R_1 \,,\quad h_2 = \lambda S_1 S_2^{+} \,,\quad h_3 = R_2 \,, \ldots$$

$$\ldots\quad h_{M-1} = \lambda S_{M-1} S_M^{+} \,,\quad h_M = \lambda R_M$$

Algebra: $h_i h_{i+1} = \omega h_{i+1} h_i \,,\quad \omega = e^{i\frac{2\pi}{N}} \,,\quad h_i^N = \mathbb{C}$

$$[h_i, h_j] = 0 \text{ if } |j-i| > 1$$

Same kind of algebra

All those models are particular cases of the more general quantum chain - with parameter
$p = 1, 2, \ldots$

$$\mathcal{H}^{(p)} = - \sum_{i=1}^{M} h_i$$

$\{h_i\}$ $(i = 1, 2, \ldots, M)$, generators of the exchange algebra:

$$h_i h_j = \omega h_j h_i \text{ if } \alpha\, j - i \leq p$$

$$[h_i, h_j] = 0 \text{ if } |i - j| > p$$

$\omega = \rho e^{i\alpha}$ ← general complex c-number

(in the previous cases $h_i^N = \mathbb{C}$ number)

In the fermionic case (N=2) these models are particular cases of even more general free-fermion models in frustated graphs.

- Elman, Chapman, Flammia (2021)
- Chapman, Flammia, Kollár (2022)
- Chapman, Elman, Mann (2023)
- Fendley, Pozsgay (2023)
- Pozsgay (2024)

Integrability for general $\omega \in \mathbb{C}$

i.e: an infinite set of commuting charges

Commuting charges: $\boxed{\text{formed by the symmetrical } \ell\text{-products of generators that commutes among them.}}$

$$\mathcal{H}^{(1)} = -\mathcal{H} = h_1 + h_2 + \cdots + h_M$$

$$\mathcal{H}^{(2)} = \sum_{j_1, j_2 = 1,\ |j_2 - j_1| \geq p+1}^{M} h_{j_1} h_{j_2}$$

Ex. $p=1$, $M=4$: $\mathcal{H}^{(2)} = -h_1 h_3 - h_1 h_4 - h_2 h_4$

$\vdots$

$$\mathcal{H}^{(\ell)} = \sum_{j_1, j_2, \dots, j_\ell = 1}^{M} h_{j_1} h_{j_2} \cdots h_{j_\ell} \qquad |j_{i+1} - j_i| \geq p+1$$

The number of independent charges $\{\mathcal{H}^{(e)}\}$
is $\overline{M} = \mathrm{Int}\left(\frac{M+P}{P+1}\right)$

Theorem: $[\mathcal{H}^{(e)}, \mathcal{H}^{(e')}] = 0 \quad \forall\, e, e'$

$\Downarrow$

$$\underbrace{\{\mathcal{H}^{(e)}\} \xrightarrow[M\to\infty]{} \infty \text{ set of conserved charges}}$$

The model is exact integrable for arbitrary values of the parameter ω

exchange algebra $\Big\}$ $[h_i, h_j] = 0 \quad |i-j| > p \quad , h_i h_j = \omega h_j h_i \quad |j-i| \le p$

Representations of $\mathcal{H}$?

Examples of representations

N=2 (Ising generalizations)

$p=1 \rightarrow$ Quantum Ising chain

$$p=1 \rightarrow \mathcal{H}_2^{(1)} = -\lambda_1 \sigma_1^x - \lambda_2 \sigma_1^z \sigma_2^x - \lambda_3 \sigma_2^z \sigma_3^x \cdots - \lambda_M \sigma_{M-1}^z \sigma_M^x$$

$$p=2 \rightarrow \mathcal{H}_2^{(1)} = -\lambda_1 \sigma_1^z \sigma_2^z \sigma_3^x - \lambda_2 \sigma_2^z \sigma_3^z \sigma_4^x - \cdots - \lambda_M \sigma_{M-2}^z \sigma_{M-1}^z \sigma_M^x$$

(Fendley 3-spin model)

$$p=3 \rightarrow \mathcal{H}_2^{(3)} = -\lambda_1 \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^x - \lambda_2 \sigma_2^z \sigma_3^z \sigma_4^z \sigma_5^x - \cdots - \lambda_M \sigma_{M-3}^z \sigma_{M-2}^z \sigma_{M-1}^z \sigma_M^x$$

General N: $\sigma_i^z \rightarrow S_i$, $\sigma_i^x \rightarrow R_i$: $RS = \omega SR$

$$\omega = e^{i2\pi/N}$$

Multispin generalization of the

$Z(N)$ Baxter free-parafermionic chain.

How are their eigenspectra?

Generating function

$$G(u) = \sum_{\ell=0}^{M} (-u)^{\ell} \mathcal{H}^{(\ell)}$$

Since $[\mathcal{H}^{(\ell)}, \mathcal{H}^{(\ell')}] = 0 \Rightarrow [\mathcal{H}^{(\ell)}, G(u)] = 0$

$$[G(u), G(u')] = 0 \quad \forall\, u, u' \in \mathbb{C}$$

Remarkable property ("Inversion formula")

theorem:

$$G(u)\, G(-u) = \mathbb{1} \cdot P_M^{(p)}(u^2) \leftarrow$$ $Z(2)$ any p

$$G(u)\, G(\omega u)\, G(\omega^2 u) = \mathbb{1} \cdot P_M^{(p)}(u^3) \leftarrow$$ $Z(3)$ any p

$\vdots$

$$G(u)\, G(\omega u) \cdots G(\omega^{N-1} u) = \mathbb{1} \cdot P_M^{(p)}(u^N) \leftarrow$$ $Z(N)$ any p

Remarkable property ("Inversion formula")

theorem:

$$G(u)\,G(-u) = \mathbb{1}\cdot P_M^{(p)}(u^2) \leftarrow$$ $Z(2)$ any p

$$G(u)\,G(\omega u)\,G(\omega^2 u) = \mathbb{1}\cdot P_M^{(p)}(u^3) \leftarrow$$ $Z(3)$ any p

$$\vdots$$

$$G(u)\,G(\omega u)\cdots G(\omega^{N-1} u) = \mathbb{1}\cdot P_M^{(p)}(u^N) \leftarrow$$ $Z(N)$ any p

$$P_M^{(p)}(z) = P_{M-1}^{(p)}(z) - u^N \lambda_M^N\, P_{M-(p+1)}^{(p)}(z),$$

$$P_M^{(p)}(z) = 0\,, \quad M \leq 0$$

$\lambda = 1$

M	$P_M^{(1)}(z)$	$P_M^{(2)}(z)$	$P_M^{(3)}$
1	$1-z$	$1-z$	$1-z$
2	$1-2z$	$1-2z$	$1-2z$
3	$1-3z+z^2$	$1-3z$	$1-3z$
4	$1-4z+3z^2$	$1-4z+z^2$	$1-4z$
5	$1-5z+6z^2-z^3$	$1-5z+3z^2$	$1-5z+z^2$
6	$1-6z+10z^2-4z^3$	$1-6z+6z^2$	$1-6z+6z^2$
7	$1-7z+15z^2-10z^3+z^4$	$1-7z+10z^2-z^3$	$1-7z+6z^2$
8	$1-8z+21z^2-20z^3+5z^4$	$1-8z+15z^2-4z^3$	$1-8z+10z^2$
9	$1-9z+28z^2-35z^3+15z^4-z^5$	$1-9z+21z^2-10z^3$	$1-9z+15z^2-z^3$
10	$1-10z+36z^2-56z^3+35z^4-6z^5$	$1-10z+28z^2-20z^3+z^4$	$1-10z-21z^2-4z^3$
11	$1-11z+45z^2-84z^3+70z^4-21z^5+z^6$	$1-11z+36z^2-35z^3+5z^4$	$1-11z+28z^2-10z^3$

From the "inversion relation" for $G_M(u)$

$$G_M(u) = \sum_{\ell=0}^{\bar{M}} (-u)^\ell \, \mathcal{H}_M^{(\ell)}$$

eigenvalues

$$\Lambda_M(u) = \prod_{i=1}^{\bar{M}} \left(1 - \frac{u\,\omega^{s_i}}{z_i^{1/N}}\right), \quad P_M^{(p)}(z_i) = 0$$

$$s_i = 0, 1, \dots, N-1$$

we obtain the eigenvalues of all the $\mathcal{H}_M^{(\ell)}$ charges.

Ex. Hamiltonian: $-\mathcal{H}^{(1)}$

$$\underbrace{E_{\{s_1,\dots,s_{\bar{M}}\}} = \omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 + \dots + \omega^{s_{\bar{M}}}\epsilon_{\bar{M}}, \quad \epsilon_i = 1/z_i^{1/N}, \quad \{s_i = 0, 1, \dots, N-1\}}$$

free-parafermionic ($Z(N)$) or free-fermionic ($Z(2)$)
$N>2$
The eigenspectra ruled by the polynomial $P_M^{(p)}(z)$.

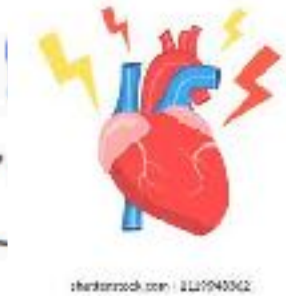
The eigenspectra are given in terms of the zeroes of polynomials

$$P^{(p)}_{M}(z_i)=0 \quad , \quad i=1,\dots,\bar{M}$$

Eigenenergies: $$E_{s_1,\dots,s_{\bar{M}}} = -\sum_{i=1}^{\bar{M}} \omega^{s_i}\epsilon_i \quad ; \quad \epsilon_i = z_i^{-1/N}$$

recursion relations:

$$P^{(p)}_{M}(z) = P^{(p)}_{M-1}(z) - z\lambda_M^N \, P^{(p)}_{M-(p+1)}(z) \;;$$

 em!

Inic. Cond. $P^{(p)}_{M}(z) = 1 \quad (M \le 0)$

"Circle-repulsion" × no "circle-repulsion"

Known Correspondence

Decoupled Ising quantum chains

$\leftarrow \sigma^z, \sigma^x$

$\leftarrow \tau^z, \tau^x$

$$\mathcal{H}^{\text{Ising}^2} = -\sum_i \left(\sigma_i^z \sigma_{i+1}^z + \sigma_i^x\right) - \sum_i \left(\tau_i^z \tau_{i+1}^z + \tau_i^x\right)$$

XY quantum chain (*XX* quantum chains)

$$\mathcal{H}^{XY} = -\sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right)$$

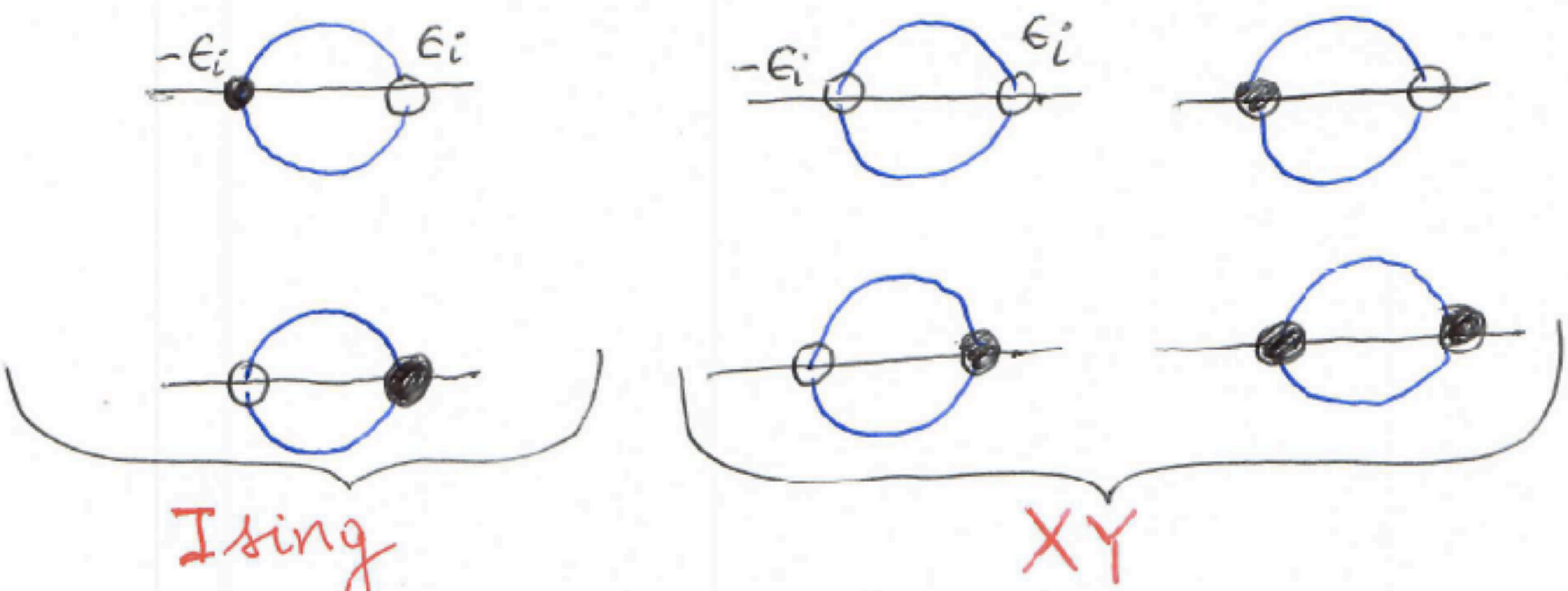
or

$$\mathcal{H}^{XY} = -\sum \left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+\right), \quad \sigma^\pm = \sigma^x \pm i\sigma^y$$

"Circle repulsion" X no "circle repulsion"

Ex. Quantum Ising $Z(2)$ x XY-quantum chain ($U(1)$)

For each quasi-energy ϵ_i



The spectrum of the XY model is the same as the one of two decoupled Ising chains.

(XXZ quantum chain x Ashkin-Teller chain)

Constructing models with no-"circle-repulsion"

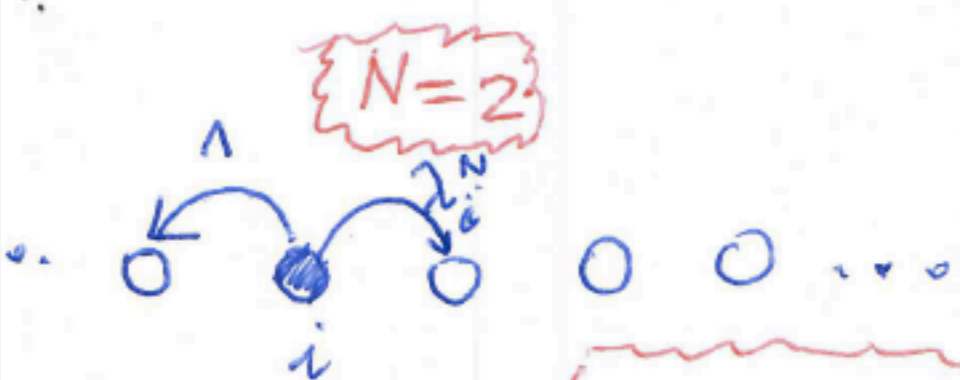
(same pseudo-energies as the $Z(N)$ free-parafermionic quantum chains

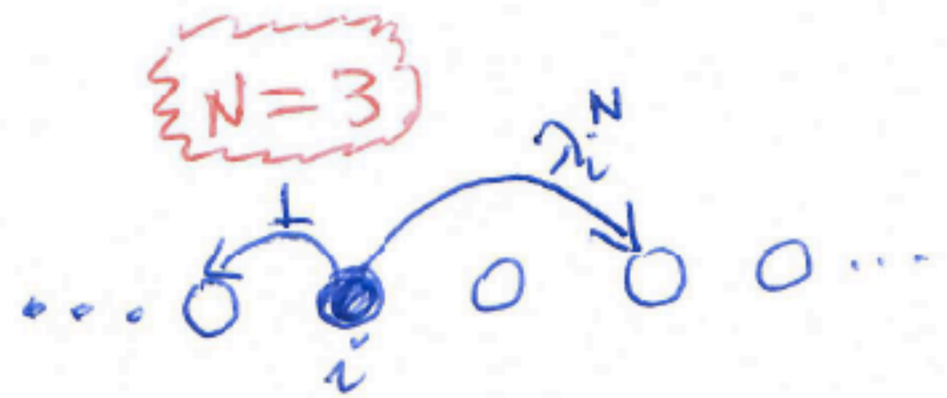
XY models with N-multispin interactions.

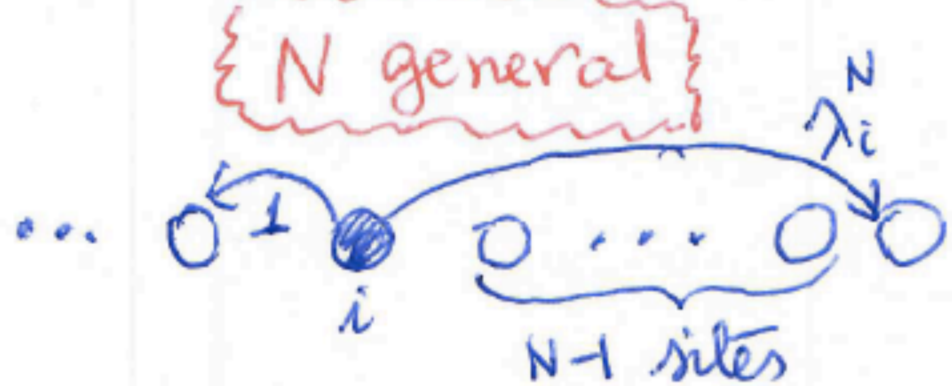
$$\mathcal{H}_M^{(N,XY)}(\{\lambda_i\}) = \sum_{i=1}^{M+N-2} \sigma_i^+ \sigma_{i+1}^- + \sum_{i=1}^{M} \lambda_i^N \sigma_i^- \left(\prod_{j=i+1}^{i+N-2} \sigma_j^z \right) \sigma_{i+N-1}^+$$

coupling constants

$\sigma^\pm = \sigma^x \pm i\sigma^y$ ($\sigma^x, \sigma^y \to$ Pauli matrices)







$U(1)$ symmetry

$$\left[\sum_\ell \sigma_\ell^z, \mathcal{H} \right] = 0 \,!$$

Introduce fermions (c_i, c_i^{+}) : Jordan-Wigner transformation : $\sigma_i^{\pm} \longrightarrow c_i^{\pm}$

$$\mathcal{H} = -\sum_{i,j=1}^{M+N-1} c_i^{+} A_{ij} C_j$$ ⇐ bilinear in fermions

$$A_{ij} = \delta_{j,i+1} + \lambda_j^N \delta_{j,i+1-N}$$ ⇐ (generalized tri-diagonal)

Diagonalization ("Bogoliubov")

Eigenspectrum $$E_{s_1,\dots,s_{M+N-1}} = -\sum_{k=1}^{M+N-1} s_k \Lambda_k \,, \quad s_k = 0,1$$

$\Lambda_k \rightarrow$ Eigenvalues of the connectivity matrix A

Define: $\underline{P}_M^{(N-1)}(u) = \mathrm{Det}(1 - Au)$,

$$\Lambda_k \Rightarrow P_M^{(N-1)}(u_k) = 0 \,, \quad \Lambda_k = \frac{1}{u_k}$$

(we need the zeroes of $\underline{P}_M^{(N-1)}(z_i) = 0$) !!

Laplace cofactor's rule of determinants

$$P^{(N-1)}_{M}(z) = P^{(N-1)}_{M-1}(z) - z\lambda^{N}_{M}\, P^{(N-1)}_{M-N}(z) \quad , \quad P^{(N-1)}_{M}(z)=1,\ M\leq 0$$

to compare with the Z(N) - p - polynomials.

$$P^{(p)}_{M}(z) = P^{(p)}_{M-1}(z) - z\lambda^{N}_{M}\, P^{(p)}_{M-(p+1)}(z) \ , \ P^{(p)}_{M}(z)=1;\ M\leq 0$$

If $\boxed{N=p+1}$ ⟵ Same pseudo-energies $\epsilon_i = \frac{1}{z_i^N}$

$P(z_i)=0$

$N=2$, $p=1$ (Ising) ⟷ $N=2$ XY model

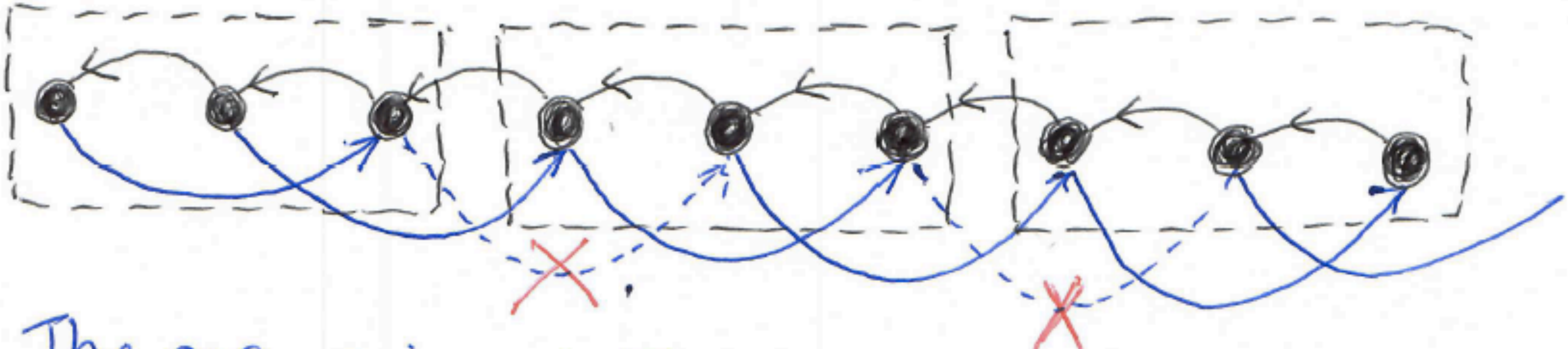
$N=3$, $p=2$ (parafermionic version of the Fendley's model) ⟷ $N=3$ XY model

⋮

How to generate the others $Z(N)$ - free parafermionic Baxter models ($Z(N)$ with $p=1$) ?

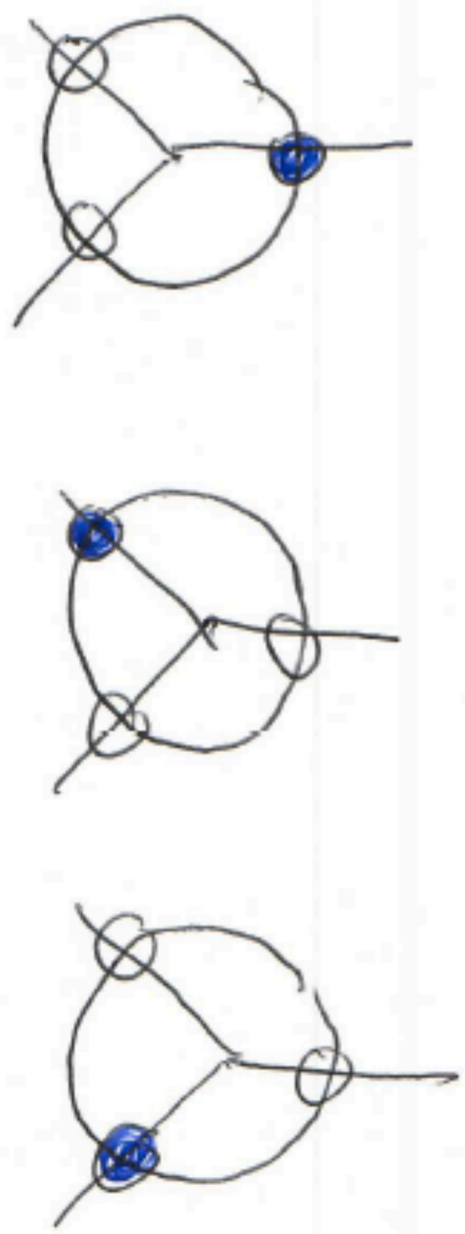
Split the M coupling constants $\{\lambda_i\}$ in cells of size N, as take as non-zero only the first two in each cell.

Ex. $N=3$

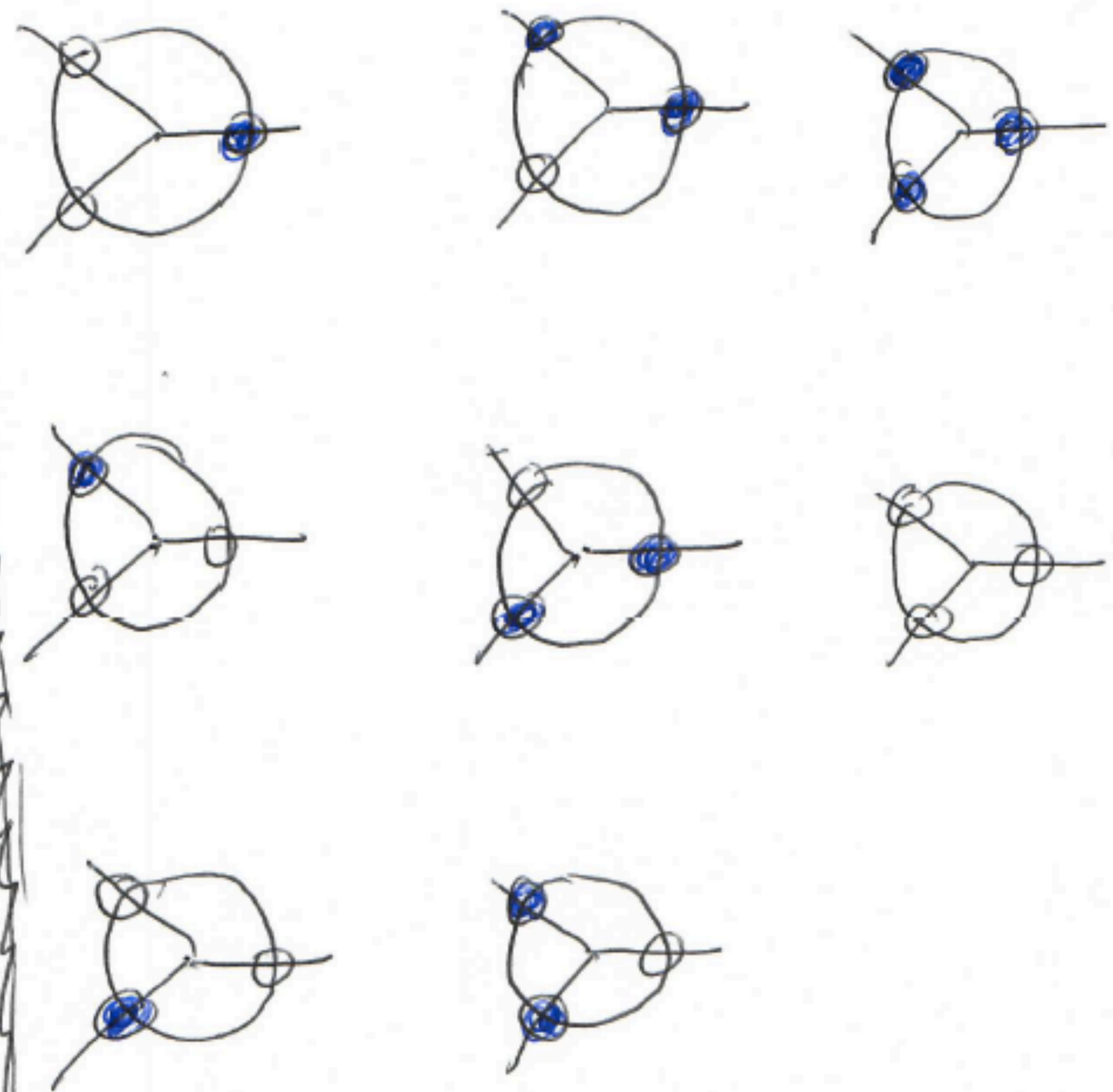


The recursion relations become the same as the $Z(N)$ - free parafermionic Baxter Chains ($p=1$).

Example : spectral comparison for each pseudo-energy ϵ_i



$Z(3)$ - Baxter
$(P=1)$



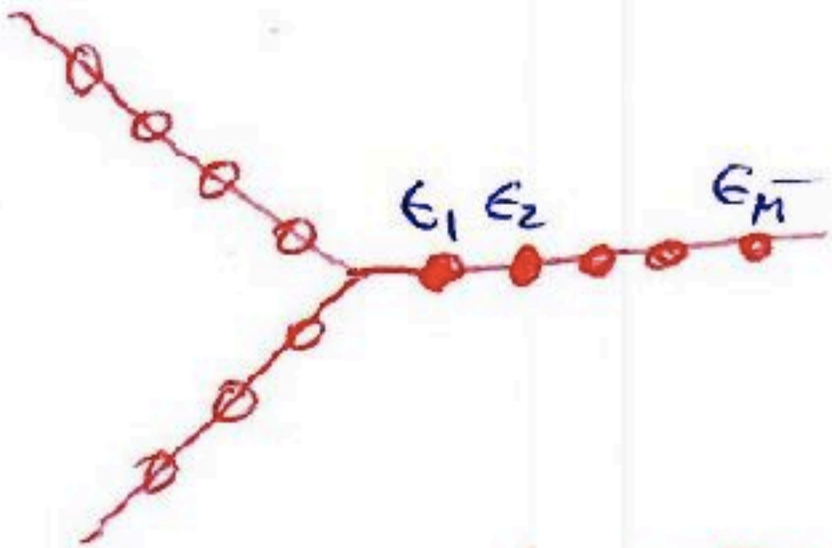
N=3 multispin XY

special critical point: $\lambda_1 = \lambda_2 = \dots = \lambda_{\bar{M}} = 1$

(symmetrical point)

Ground-state energy:

$$E_0 = -\sum_{i=1}^{\bar{M}} \epsilon_i$$



$\Downarrow$

In the bulk limit $M \to \infty$

$$e_\infty = \lim_{M\to\infty} E_0/M = -\int \varepsilon(p)\,dp \Rightarrow$$

Exactly given by integral representations of Lauricella series

Critical exponents: $G_{\Delta\beta} \propto 1/M^z$

Specific heat $$C_V = -\frac{\lambda^2}{M}\frac{\partial^2 E_0(\lambda)}{\partial \lambda^2} \sim \frac{1}{M^\alpha}$$

		$N=2$	$N=3$	$N=4$	$N=5$
$p=1$	z	1	$2/3$	$2/4$	$2/5$
	α	0	$1/3$	$2/4$	$3/5$
$p=2$	z	$3/2$	1	$3/4$	$2/5$
	α	0	0	$1/4$	$4/5$
$p=3$	z	2	$4/3$	1	$4/5$
	α	0	0	0	$1/5$

$$z = \frac{p+1}{N} \quad , \quad \alpha = \mathrm{Max}\left\{0,\ 1-(p+1)/N\right\}$$

New universatity class of critical behavior

$Z = \frac{p+1}{N} \neq 1$ in general not conformally invariant

For $Z = 1$ Are they conformally invariant???

[Non-Hermitian, complex spectrum, no chiral symmetry]

$$p = 1, N = 2 : H(\{\lambda_i\}) = -\sum_i \lambda_i S_{i-1} R_i, \quad RS = e^{i\frac{2\pi}{2}} SR = -SR$$

$$p = 2, N = 3 : H(\{\lambda_i\}) = -\sum_i \lambda_i S_{i-2} S_{i-1} R_i, \quad RS = e^{i\frac{2\pi}{3}} SR$$

$$p = 3, N = 4 : H(\{\lambda_i\}) = -\sum_i \lambda_i S_{i-3} S_{i-2} S_{i-1} R_i, \quad RS = e^{i\frac{2\pi}{4}} SR$$

Isotropic case : $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda$

Open Boundary Condition (OBC) : $\mathcal{H}_{OBC}(\lambda) = \lambda \mathcal{H}_{OBC}(1)$

Periodic Boundary Condition (PBC) : $\mathcal{H}_{PBC}(\lambda) \neq \lambda \mathcal{H}_{PBC}(1)$

Non Hermitean effect !

$\boxed{\text{OBC Case}}$

$Z(N)$-parafermionic: $E(\{s_i\}) = -\sum_{i=1}^{M} e^{i\frac{2\pi}{N}s_i} \epsilon_i \quad (s_i = 0, 1, \cdots, N-1)$

$Z(N)$ Circle exclusion

N-multispin XX : $E(\{t_i, s_j\}) = -\sum_{i=1}^{M} t_i \sum_{s_j=0}^{N-1} e^{i\frac{2\pi}{N}s_j} \epsilon_i$ $\quad (t_i = 0, 1)$

No circle repulsion

$\epsilon_i = \frac{1}{z_i^{1/N}}$ roots of polynomial $P_M(z)$

We can solve for the larger roots up to $M \sim 10^9$!!

We order: $\mathrm{Real}(E_0) < \mathrm{Real}(E_1) \leq \mathrm{Real}(E_2) \leq \cdots$

Conformal Invariance $\Rightarrow$ finite-size behavior $(M \to \infty)$

$$\boxed{\mathrm{Re}\left(E(M,r) - E_0(M,r)\right) = \frac{\pi v_s}{M}\left(X_s^{(N)} + r\right) \quad ; r = 0,1,2,\ldots}$$

sound velocity — surface critical exponent

Lattice sequences: $M_1, M_2, \ldots$; $\mathrm{Mod}(M_i, N) = k$ fixed $(k = 0,1,\ldots,N-1)$

We obtain numerically

$\boxed{Z(N) \text{ parafermionic models}}$

$\boxed{p=2, N=3}$ $v_s = 3\sqrt{3}$ [redacted] , $X_s^{(3)} = \frac{7}{6}, \frac{1}{6}, \frac{5}{6}$ $(k = 0,1,2)$

$\boxed{p=3, N=4}$ $v_s = 4\sqrt{2}$, $X_s^{(4)} = \frac{5}{4}, \frac{1}{2}, \frac{3}{4}, 1$ $(k=0,1,2,3)$

$\vdots$

N-multispin XX models

The same dimensions but with sound velocities v_s, that depend on the magnetization sector

PBC Case

Spectral relations

Z(N) parafermionic $\cancel{\Longleftrightarrow}$ N-multispin XX

Exact solution
[redacted]
(not a free particle)

Exact solution from
Jordan-Wigner transf. $\{c_i, c_i^+\}$
+ Fourier transform $\{\eta_k, \eta_k^+\}$

$$\mathcal{H}^{(N)}_{XY} = \sum_j \epsilon(k_j)\,\eta^\dagger_{k_j}\eta_{k_j} \qquad k_j = \begin{cases} \dfrac{2\pi j}{M+N-1} \\[2ex] \dfrac{2\pi(j+1/2)}{M+N-1} \end{cases}$$

The effective dispersion relation

$$\Lambda(k) = \mathrm{Re}(\epsilon(k)) = -(\cos k + \lambda^N \cos[(N-1)k])$$

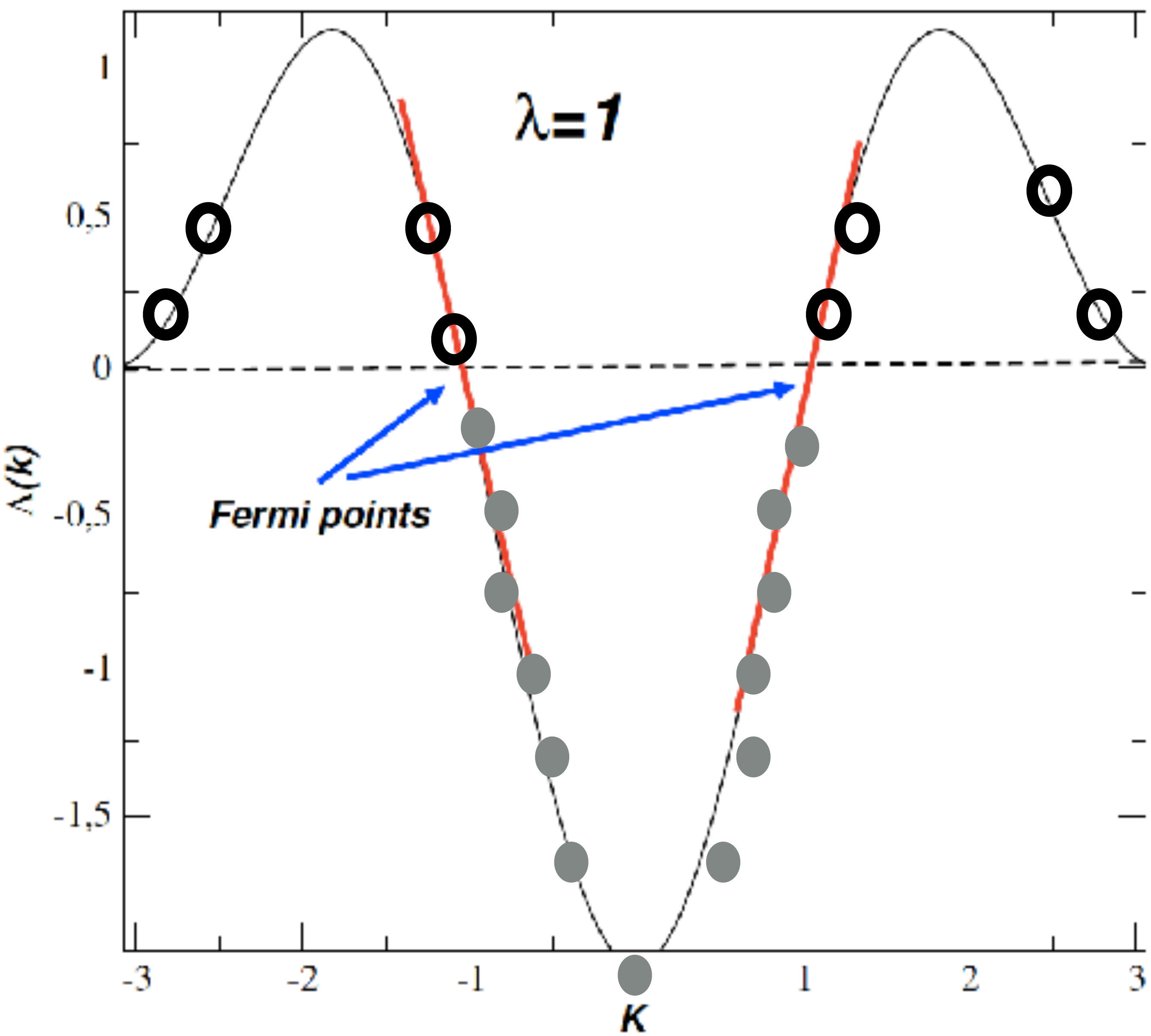
$$\Lambda(k) = -\left(\cos k + \lambda^N \cos[(N-1)k]\right)$$

For N=2 $\quad \Lambda(k) = -(1+\lambda^2)\cos k \neq \lambda\Lambda(1)$

$\boxed{\text{OBC relation } \cancel{E_0(\lambda) = \lambda E_0(1)}}$

$\lambda \neq 1$, OBC and PBC distinct E_0 $\Leftarrow$ anomalous behavior

For N=3



Ground State

$$\frac{E_0(M)}{M} = -\frac{\sqrt{3}}{4}\,\frac{1}{\sin\left(\frac{\pi}{M}\right)}\left(2 + \frac{1}{\cos(\pi/M)}\right)$$

$M \to \infty$

$$\frac{E_0(M)}{M} = -\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}\,\pi}{4M^2} + \mathcal{O}(M^{-4})$$

Conf. Invariance prediction

$$\frac{E_0}{M}(M) = e_\infty - \frac{\pi v_s c}{6M^2} + \mathcal{O}(M^{-2})$$

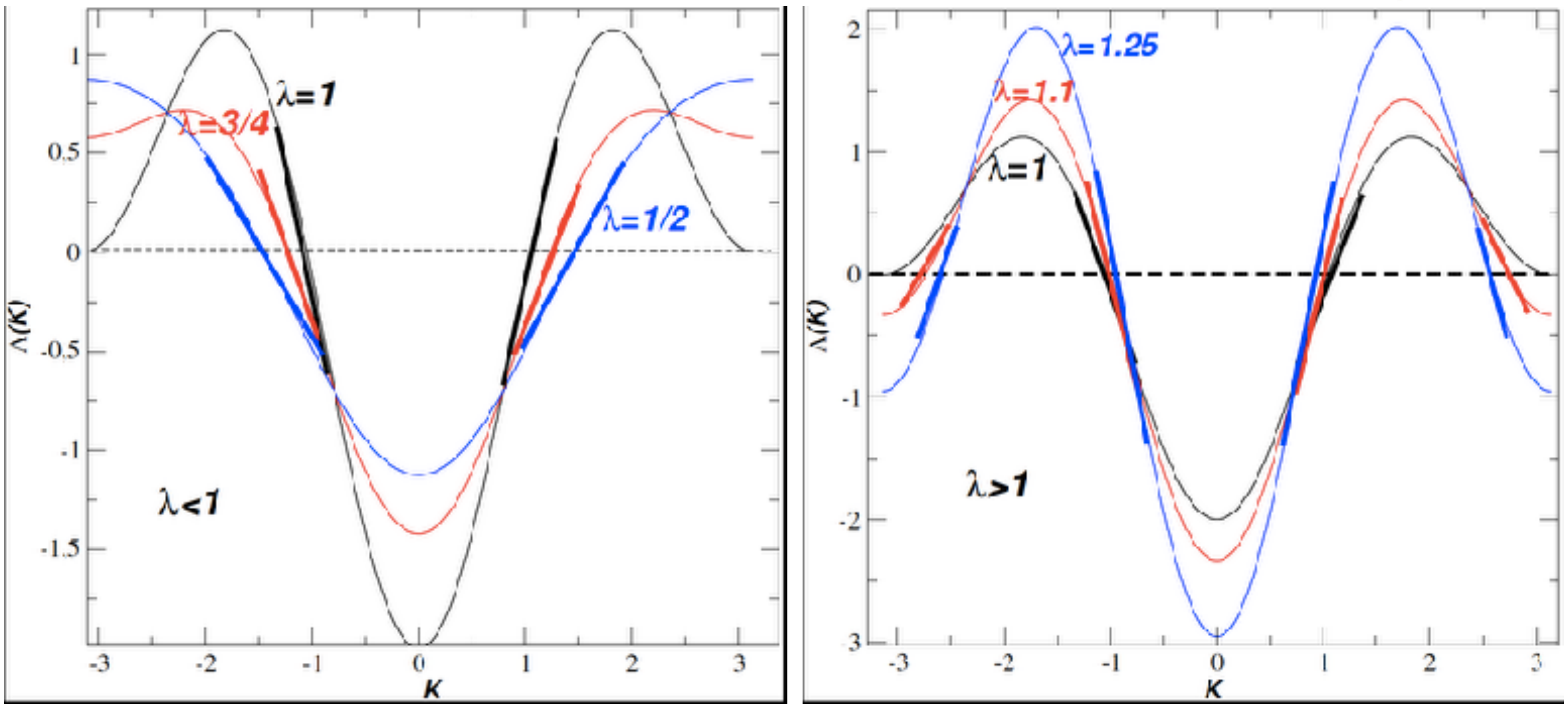
$$v_s = \left.\frac{dE_0(k)}{dk}\right|_{k=k_F} = \frac{3\sqrt{3}}{2}$$ [redacted] $c = 1$ (as standard XX chain)

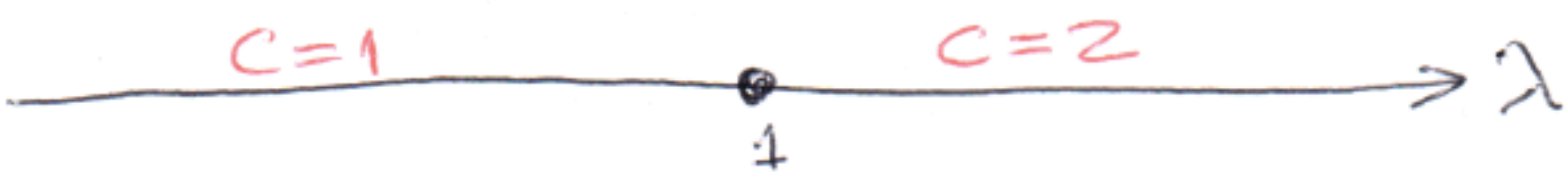
Analytically: Operator content of a Gaussian Conformal Field Theory with spin wave number Q and vorticity β:

$$x_{Q,\beta} = \left(Q\,x_p + \frac{\beta^2}{4x_p}\right), \quad \begin{array}{l} Q = 0, \pm 1, \pm 2, \dots \\ \beta = 0, \pm 1, \pm 2, \dots \end{array}$$

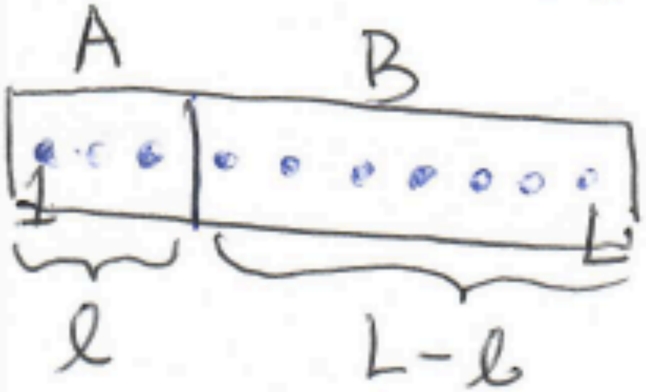
$x_p = 1/4$ (like standard XX model)

$\boxed{\lambda < 1}$ $\boxed{\lambda > 1}$





Central charge c, from entanglement entropy



$$\rho = |\Phi_L\rangle\langle\Phi_R|$$

left and right eigenvectors

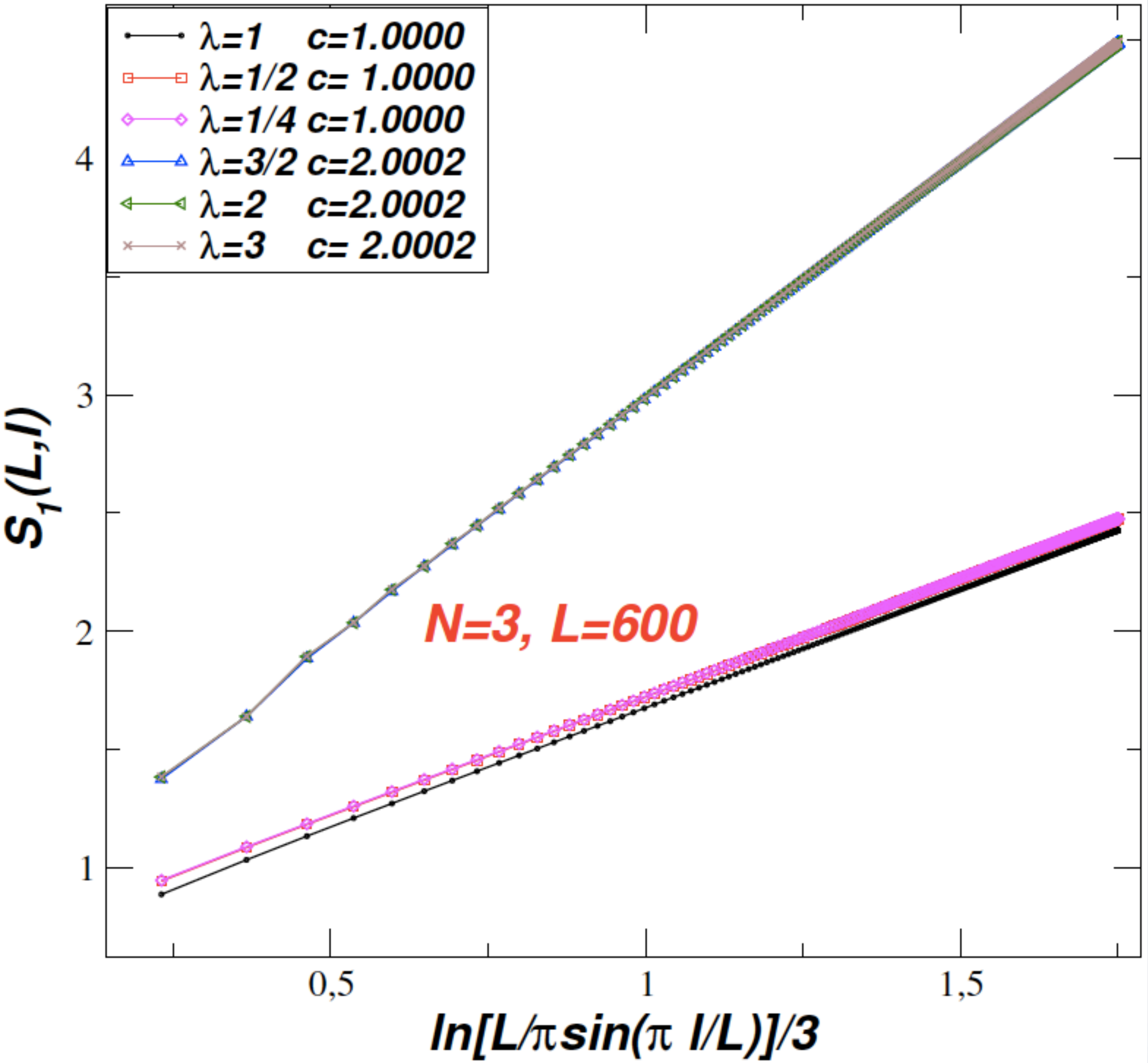
reduced matrices: $\rho_A = \mathrm{Tr}_B\,\rho$, $\rho_B = \mathrm{Tr}_A\,\rho$

von Neumann entropy: $S_1 = -\mathrm{Tr}\left(\rho_A \ln \rho_A\right)$

Rényi entanglement entropy: $S_\alpha(\ell, L) = \frac{1}{1-\alpha} \ln\left[\mathrm{Tr}(\rho_A)^\alpha\right]$

Prediction from Conformal Invariance

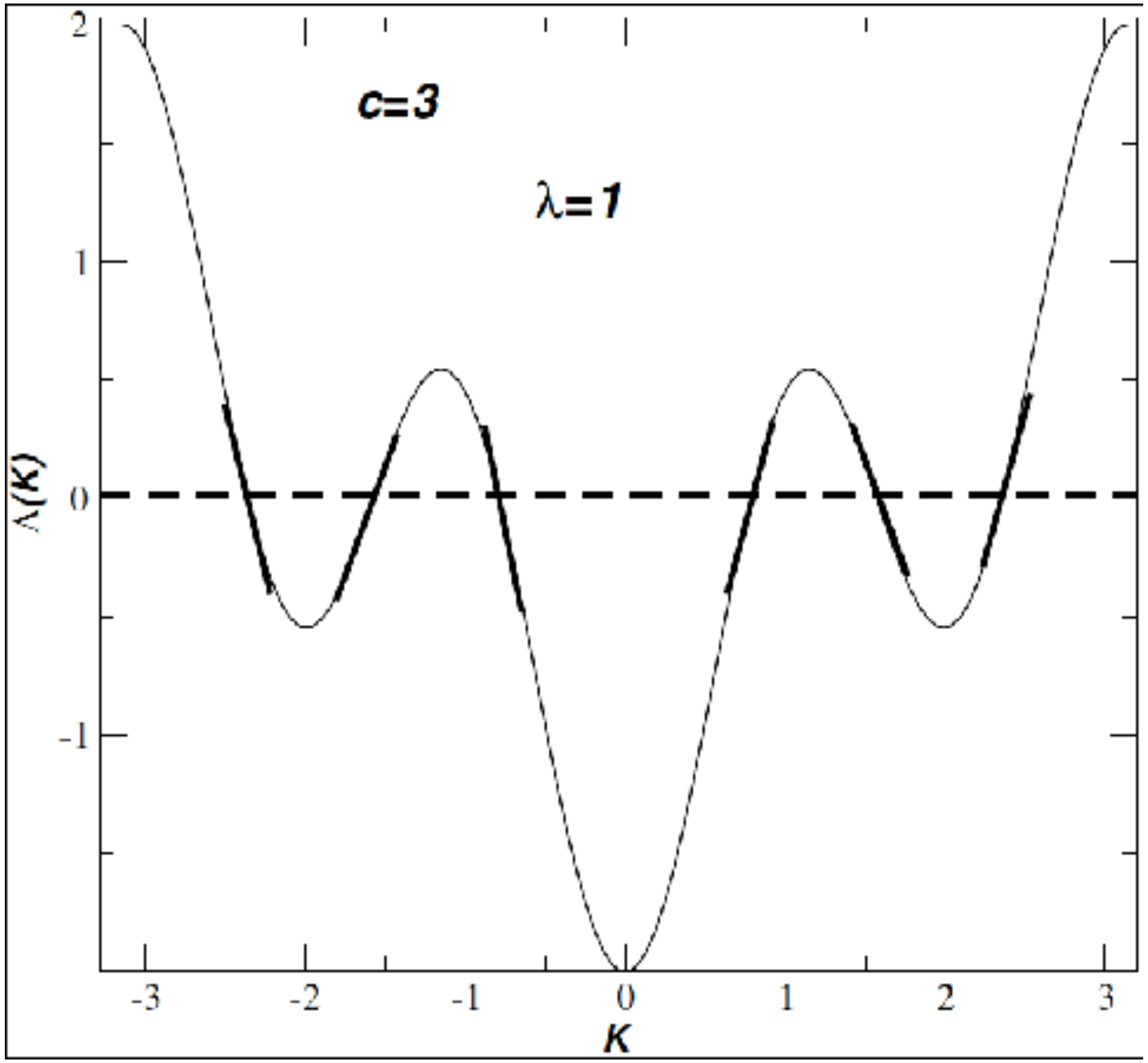
$$\boxed{S_\alpha(L,\alpha) = \frac{c}{6}\left(1+\frac{1}{\alpha}\right)\ln\left[\frac{L}{\pi}\sin\left(\frac{\pi \ell}{L}\right)\right] + a^{(\alpha)}}$$



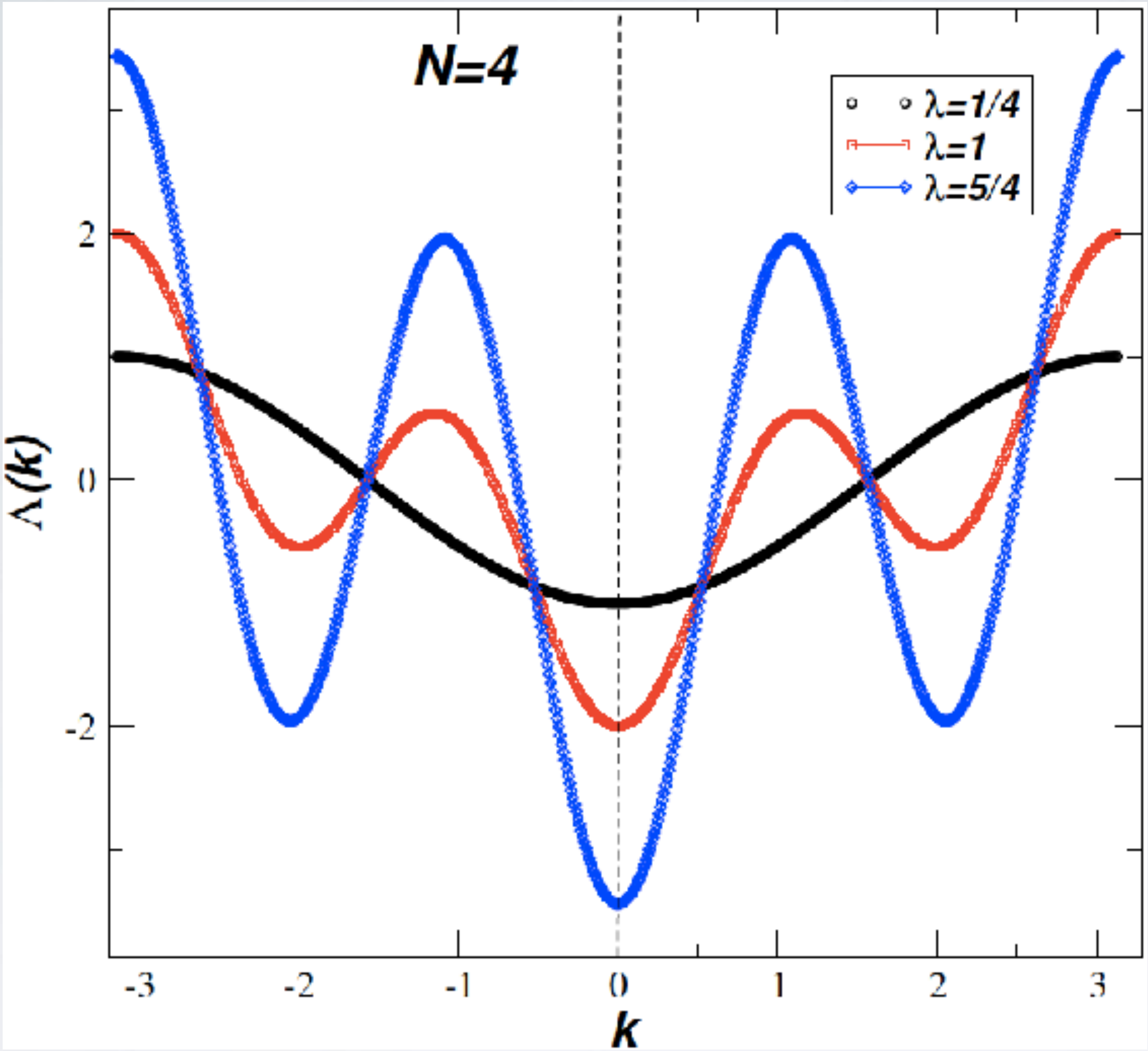
λ=1 c=1.0000
λ=1/2 c= 1.0000
λ=1/4 c=1.0000
λ=3/2 c=2.0002
λ=2 c=2.0002
λ=3 c= 2.0002
N=3, L=600
S1(L,l)
ln[L/π sin(π l/L)]/3
1
2
3
4
0,5
1
1,5

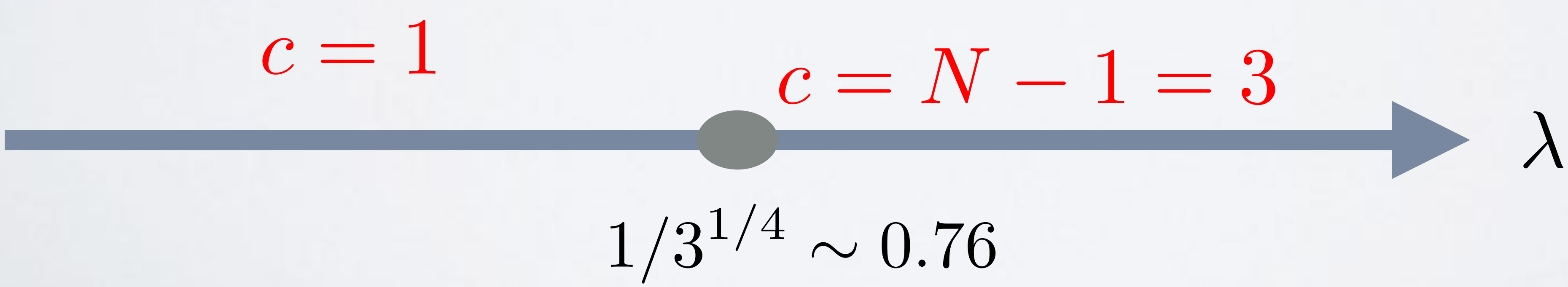
$N = 4$

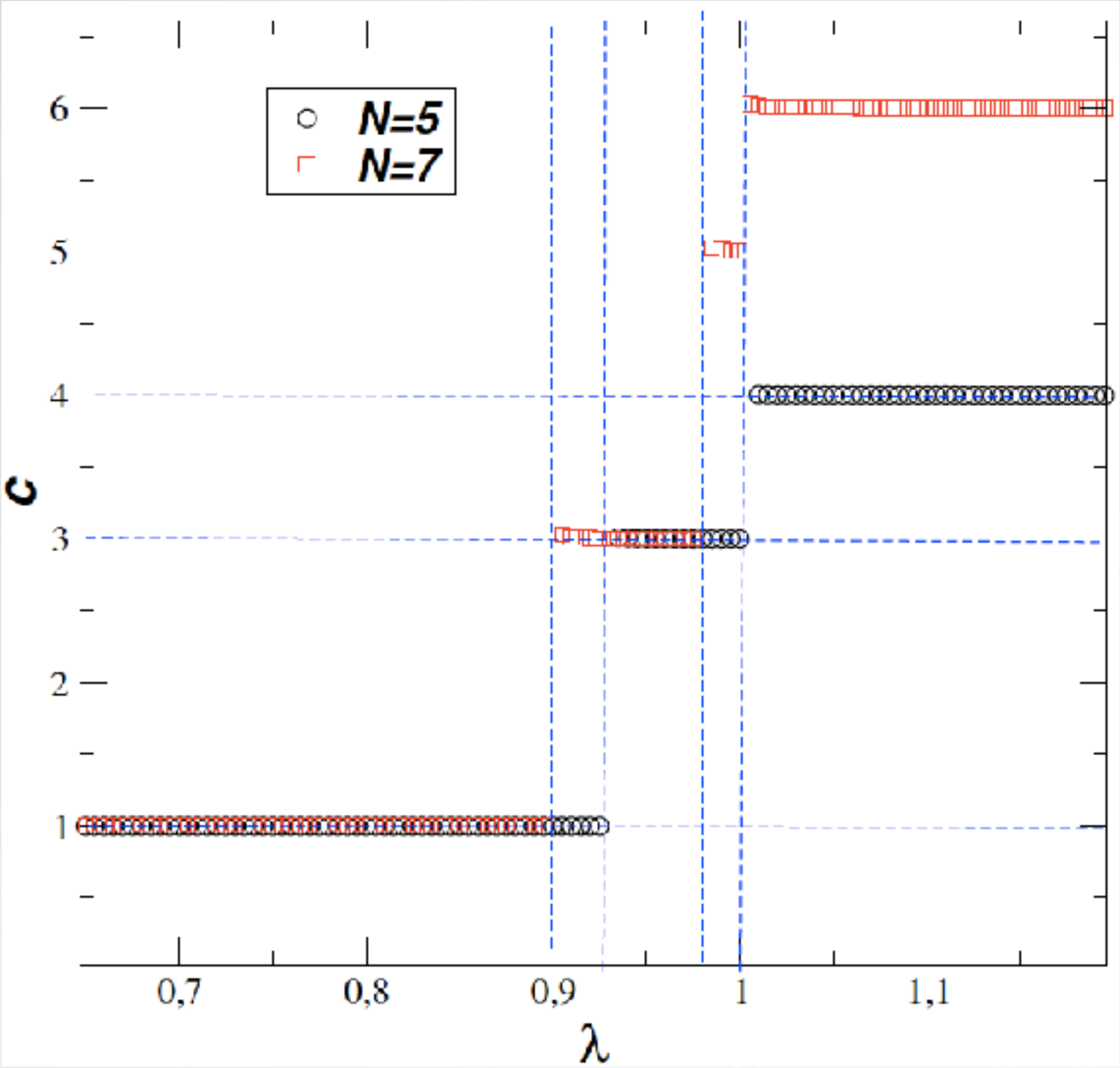
$\lambda = 1$



$c = 3$


N=4
λ=1/4
λ=1
λ=5/4
Λ(k)
2
0
-2
-3
-2
-1
0
1
2
3
k


$c = 1$
$c = N - 1 = 3$
λ
$1/3^{1/4} \sim 0.76$



$\lambda << 1 \rightarrow c = 1$

$\lambda << 1 \rightarrow c = N - 1$

Conclusions

- We introduce two new families of free-particle quantum chains
  - A family of free fermionic ($Z(2)$) and free-parafermionic ($Z(N)$) quantum chains with multispin interactions
  - A family of XY models with multispin interactions
- The eigenspectra of these models are related, generalizing the equivalence

  Ising quantum chain $\times$ XY model
- These models belong (when critical) to new universality classes of critical behavior.

- Our solution of the eigenspectra for the free-fermionic and free-parafermionic models give us a powerful method to evaluate the eigenspectra

$$P_M^{(N-1)}(z) = P_{M-1}^{(N-1)}(z) - z\,\lambda_M^N\, P_{M-N}^{(N-1)}(z),$$ 

$$P_M^{(N-1)}(z) = 1 \quad (M \leq 0)$$

For example:

Ising system $L = 50, 100, 600.000$.

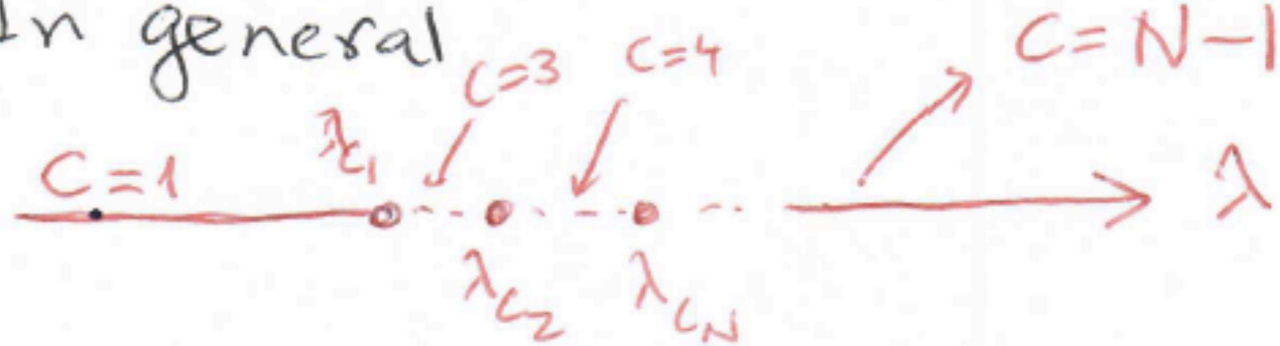
12 000 gaps in 2 seconds of CPU time.

Remarkable for random systems.

- The special families: $Z(N)$ - parafermionic with $p+1$ interactions and XX models with $N = p+1$ multispin interactions are
  Conformally invariant

- The periodic XX model show phase transitions, absent in the open boundary cases.

  In general



- Are these transitions present in the periodic $Z(N)$ parafermionic free model?
- Interacting cases?

Thank you