$T\bar{T}$ deformations and integrable models

MPI2024 - MATRIX -Creswick

Roberto Tateo

INFN TORINO
Istituto Nazionale di Fisica Nucleare
Sezione di Torino

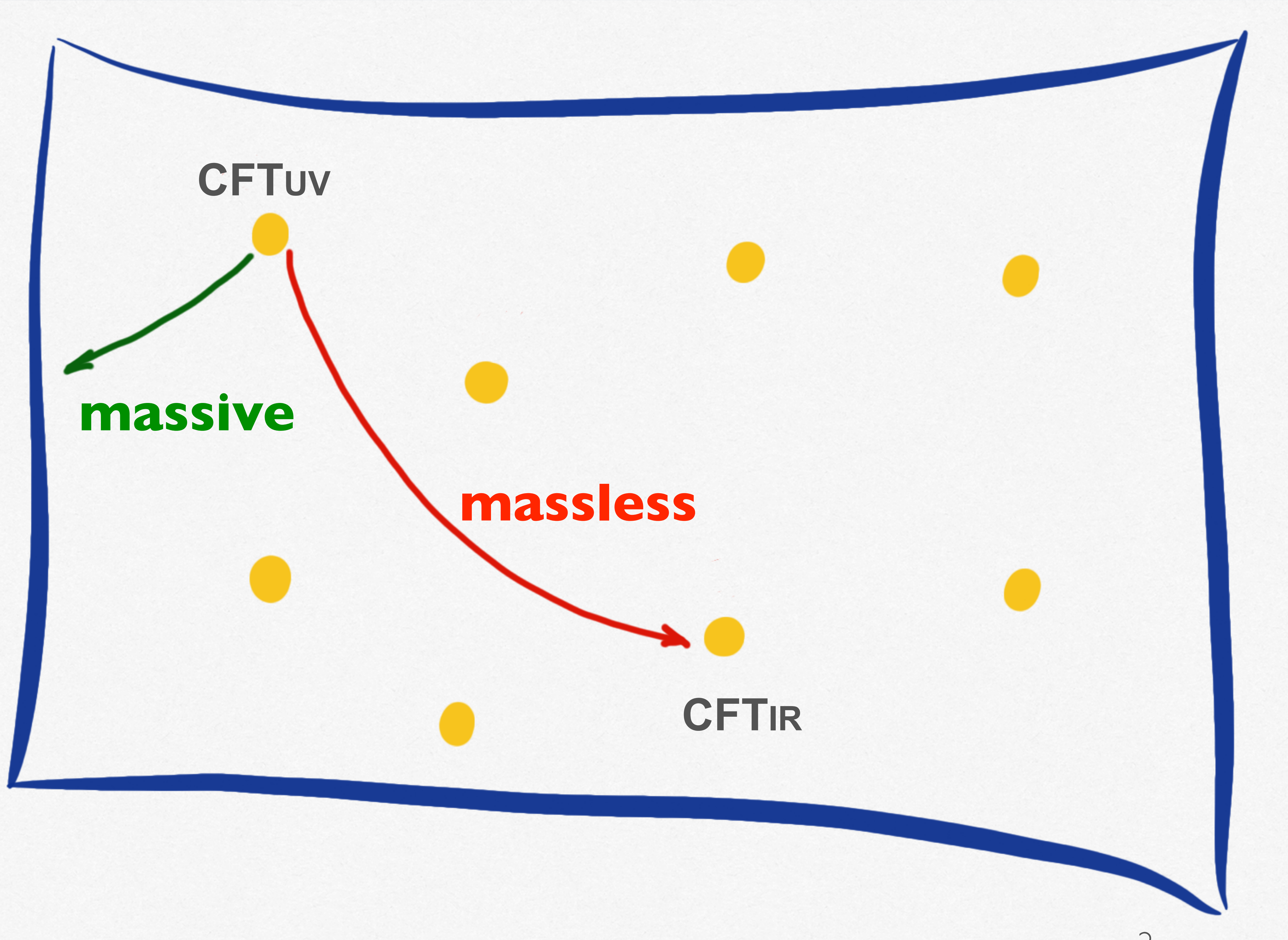
UNIVERSITÀ DEGLI STUDI DI TORINO
ALMA UNIVERSITAS TAURINENSIS

Collaborators

N. Brizio, A. Cavaglià, R. Conti, L. Iannella, T. Morone, S. Negro, J. Romano, P. Ceschin, I. Szécsényi

Initial motivation:

Understanding the Space of Quantum Field Theories



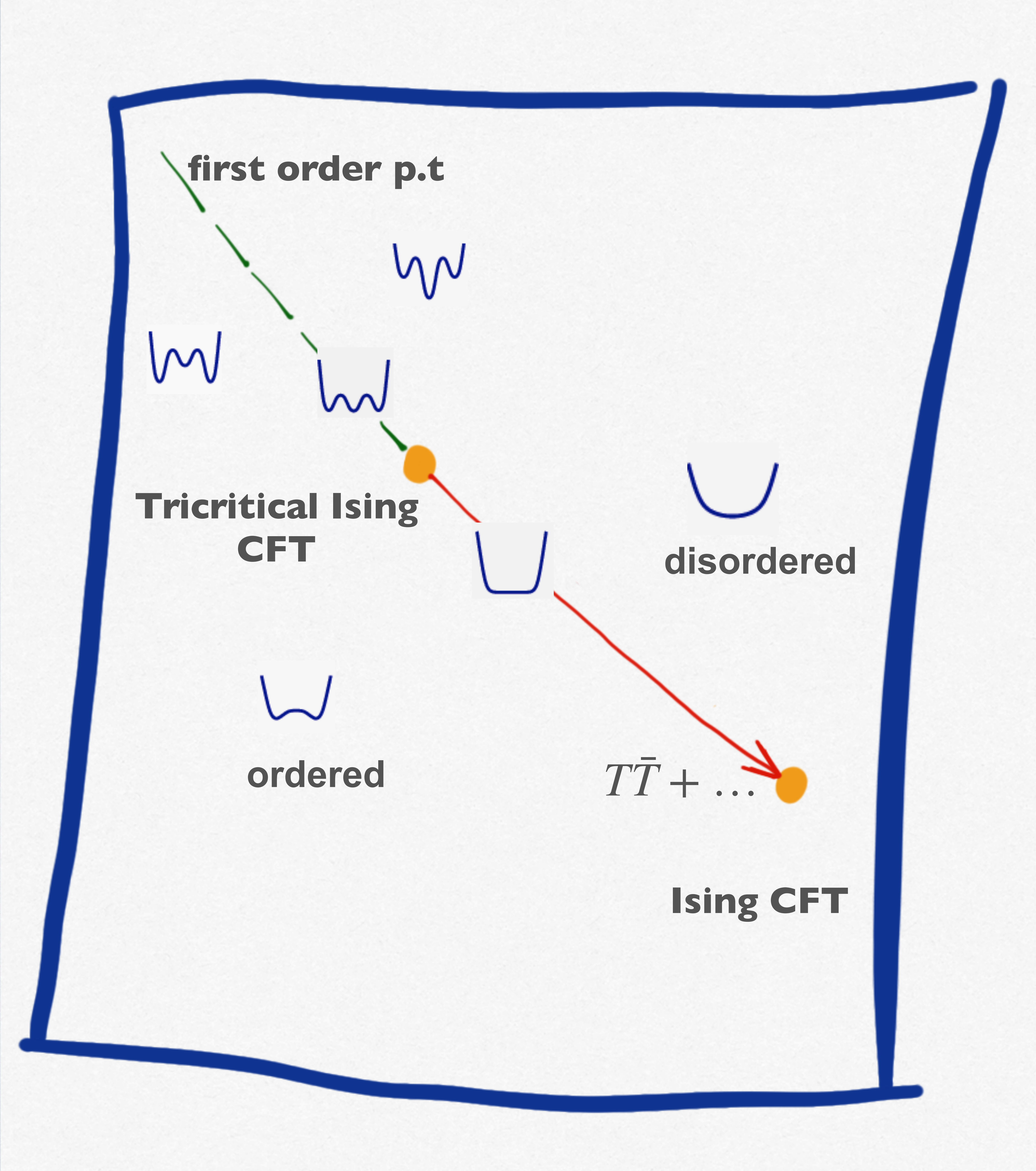
CFT:
Operator and state content

Critical exponents and correlation functions

Massive integrable CFT perturbations:
Exact S-matrix

Finite-Size spectrum
(Thermodynamic Bethe ansatz)

Correlation Functions
(Form-Factors)



Massless integrable CFT perturbations:

Exact S-matrix

Finite-Size spectrum
(Thermodynamic Bethe ansatz)

IR leading attracting operators

In a CFT

$$T_{xx} = -T_{yy} = -\frac{1}{2\pi}(\bar{T}+T)$$

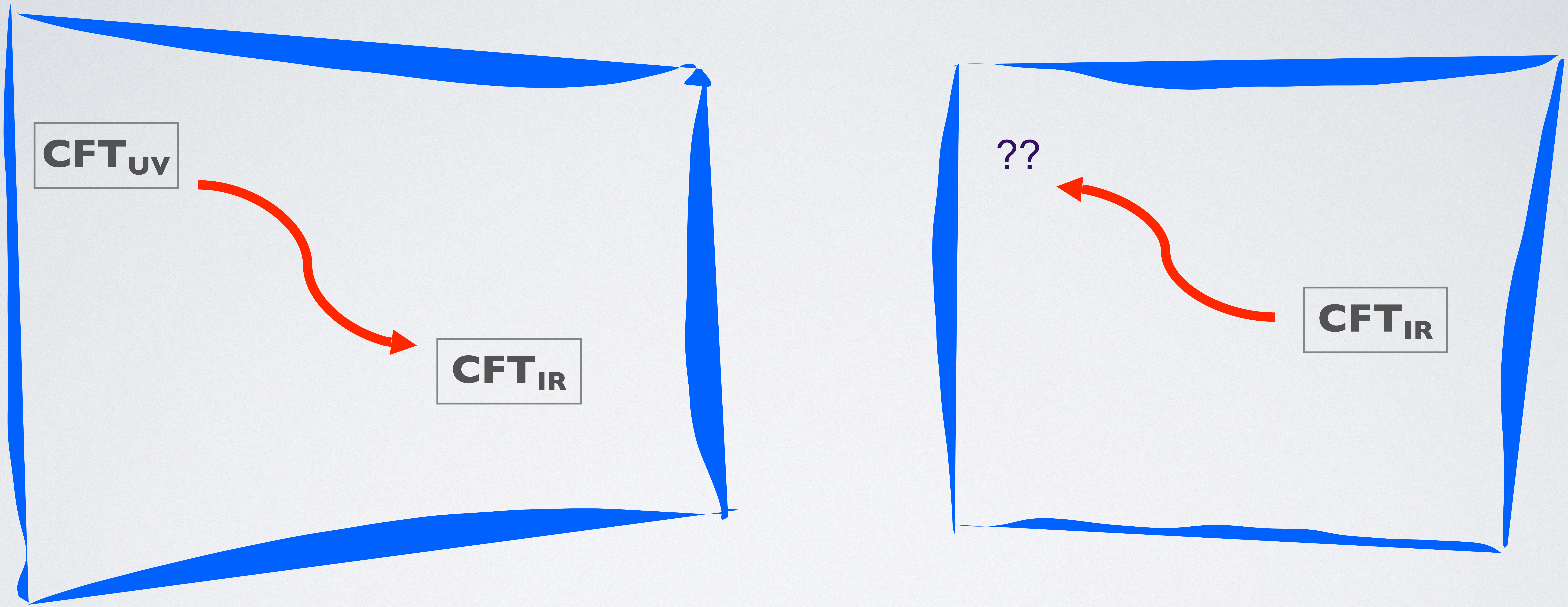
$$T_{yx} = T_{xy} = \frac{i}{2\pi}(\bar{T}-T)$$

and

$$T\bar{T}(z,\bar{z}) = T(z)\bar{T}(\bar{z})$$

$$(z = x+iy,\ \bar{z} = z-iy)$$

Can we reverse the renormalisation group trajectory?



Let us try with the $T\bar{T}$ perturbation …

We need the correct definition of $T\bar{T}$ outside a CFT fixed point:

$$T_{xx} = -\frac{1}{2\pi}(\bar{T} + T - 2\Theta)\,, \quad T_{yy} = \frac{1}{2\pi}(\bar{T} + T + 2\Theta)\,, \quad T_{xy} = \frac{i}{2\pi}(\bar{T} - T)\,.$$

Sasha Zamolodchikov (2004):

$$\mathrm{T\bar{T}}(z,\bar{z}) := \lim_{(z,\bar{z})\to(z',\bar{z}')} T(z,\bar{z})\bar{T}(z',\bar{z}') - \Theta(z,\bar{z})\Theta(z',\bar{z}') + \text{total derivatives}$$

“4. (L) CFT limit at short distances. I will assume that the short-distance behavior of the field theory is governed by a conformal field theory … Here I just mention that this assumption is needed in order to make definition of the composite field $T\bar{T}$ essentially unambiguous.”

Therefore, up to total derivatives

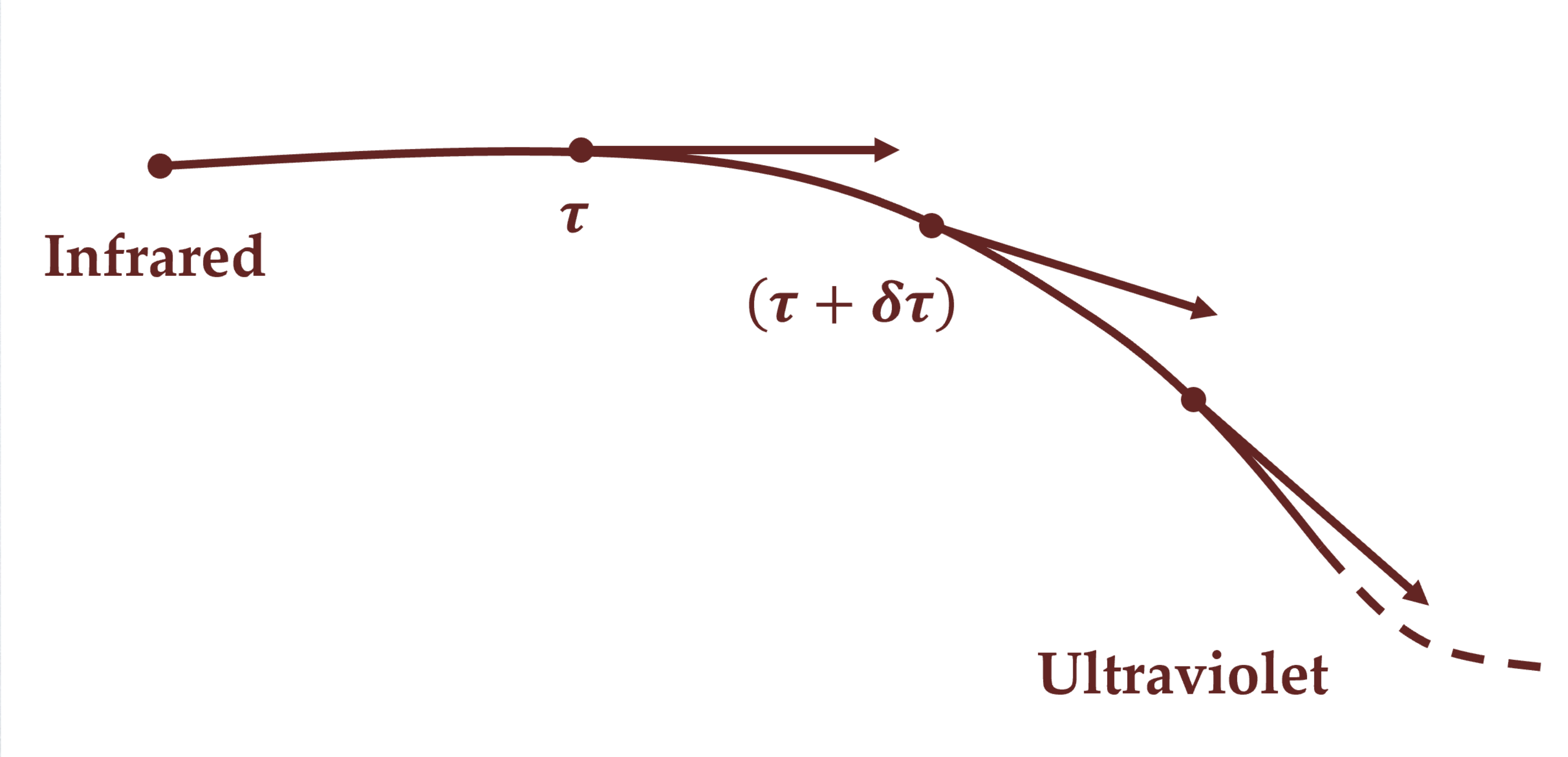
$$T\bar{T}(z,\bar{z}) := -\pi^2 \det(T_{\mu\nu}(z,\bar{z}))$$

The $T\bar{T}$ Lagrangian flow equation is:

$$\begin{cases} \partial_\tau \mathscr{L}(\tau) = \det(T_{\mu\nu}(\tau))\,, \\ T_{\mu\nu}(\tau) = \frac{-2}{\sqrt{g}}\frac{\partial \mathscr{L}(\tau)}{\partial g^{\mu\nu}}\,, \end{cases}$$

$$\partial_\tau \mathscr{H}(\tau) = \det(T^{\mu\nu}(\tau))$$

(Euclidean space-time)



Example: bosons with generic potential

$$\mathscr{L}^V(0) = \mathscr{L}(0) - V \qquad \text{with} \qquad \mathscr{L}(0) = \partial\overrightarrow{\phi}\cdot\bar{\partial}\overrightarrow{\phi}\,,\; V = V(\overrightarrow{\phi})$$

$$\mathscr{L}^V(\tau) = \frac{-V}{1+\tau V} + \frac{1}{2\bar{\tau}}\left(-1 + \sqrt{1 + 4\bar{\tau}\mathscr{L}(0) - 4\bar{\tau}^2\,\mathscr{B}}\right)$$

with $\bar{\tau} = \tau\,(1 + \tau V)$ and $\mathcal{B} = |\partial\vec{\phi}\times\bar{\partial}\vec{\phi}|^2$

A local change of coordinates

$$\mathcal{J}^{-1} = \begin{pmatrix} \partial_w z & \partial_w \bar{z} \\ \partial_{\bar{w}} z & \partial_{\bar{w}} \bar{z} \end{pmatrix} = \begin{pmatrix} 1+\tau V & -\tau \left(\frac{\partial\phi}{\partial w}\right)^2 \\ -\tau \left(\frac{\partial\phi}{\partial \bar{w}}\right)^2 & 1+\tau V \end{pmatrix}$$

$(z = x^1 + ix^2, \bar{z} = x^1 - ix^2)$

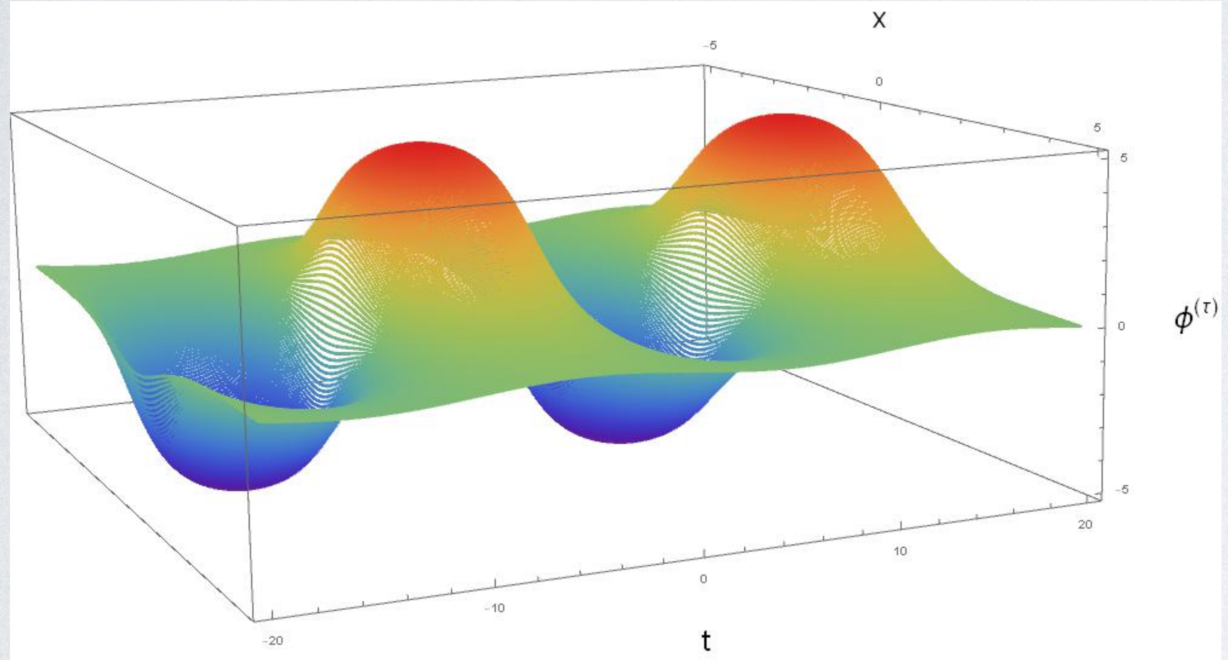
$(w = y^1 + iy^2, \bar{w} = y^1 - iy^2)$

$$\phi^{(\tau)}(\mathbf{z}) = \phi^{(0)}\left(\mathbf{w}(\mathbf{z})\right) \ , \quad \mathbf{z} = (z, \bar{z}) \, , \quad \mathbf{w} = (w, \bar{w})$$

$$\partial\left(\frac{\bar{\partial}\phi}{S}\right) + \bar{\partial}\left(\frac{\partial\phi}{S}\right) = -\frac{V'}{4S}\left(\frac{S+1}{1+\tau V}\right)^2$$

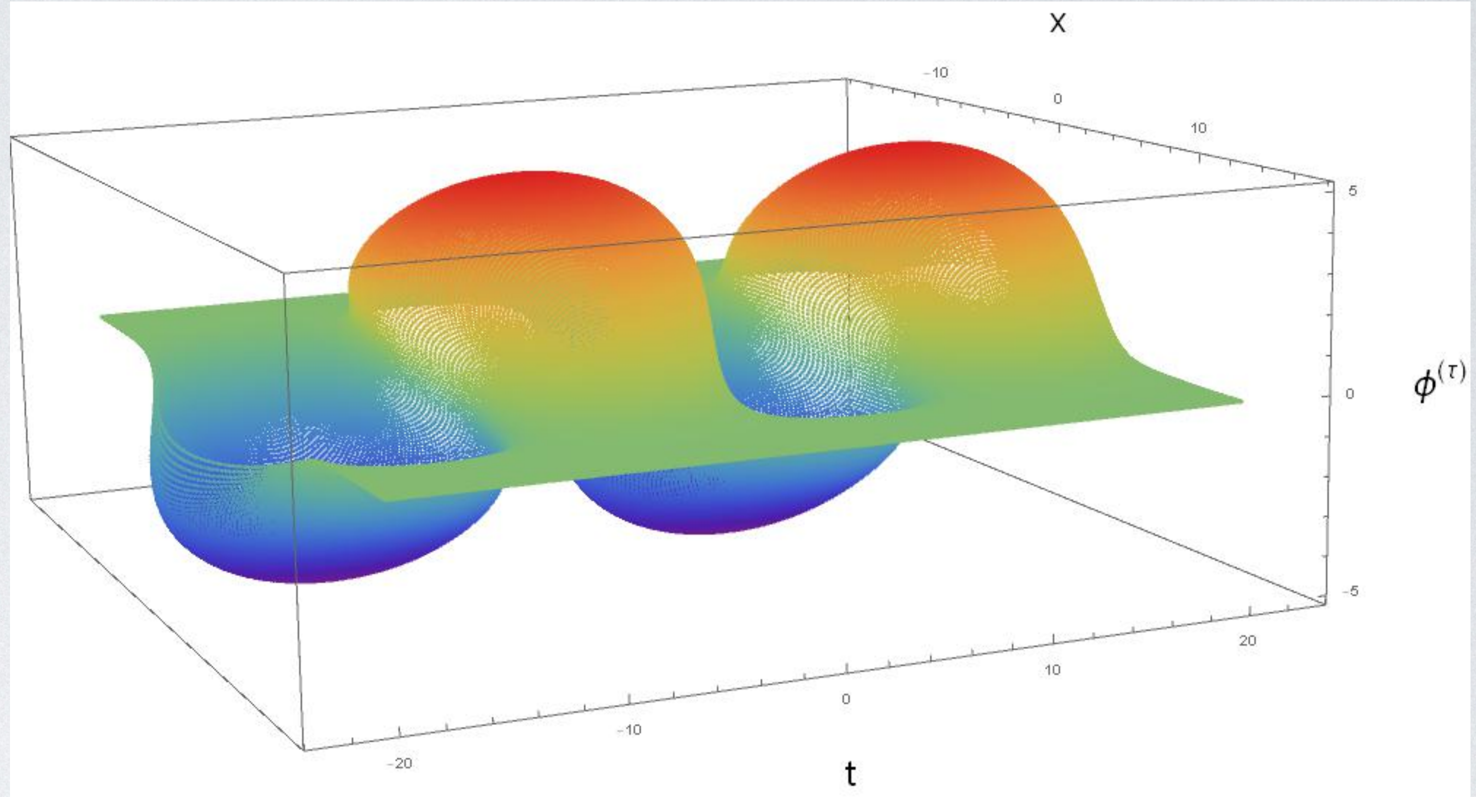
$$2\partial_w\partial_{\bar{w}}\phi = -V'$$

$$S = \sqrt{1 + 4\tau(1+\tau V)\partial\phi\bar{\partial}\phi}$$

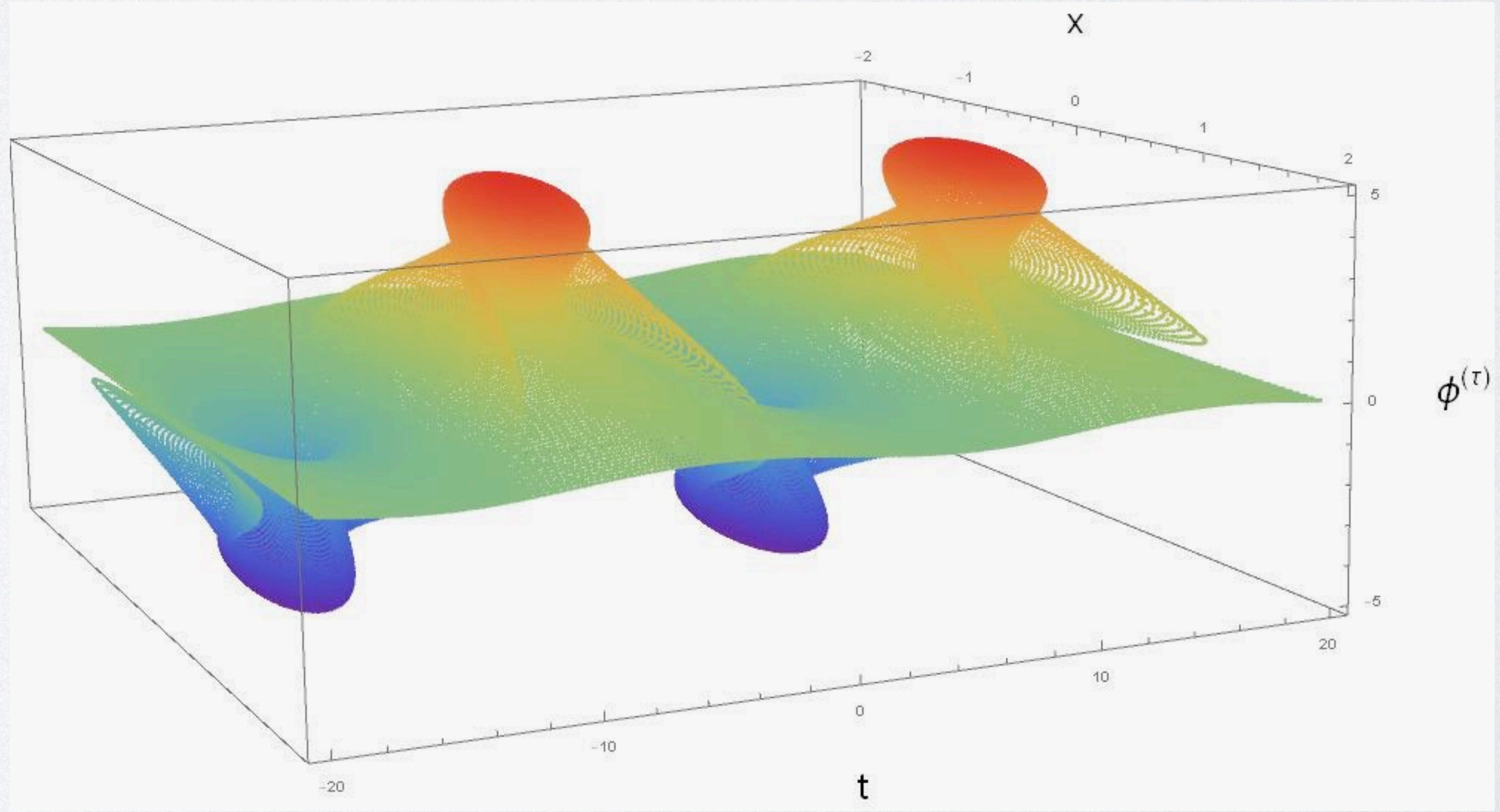


The deformed
sine-Gordon breather

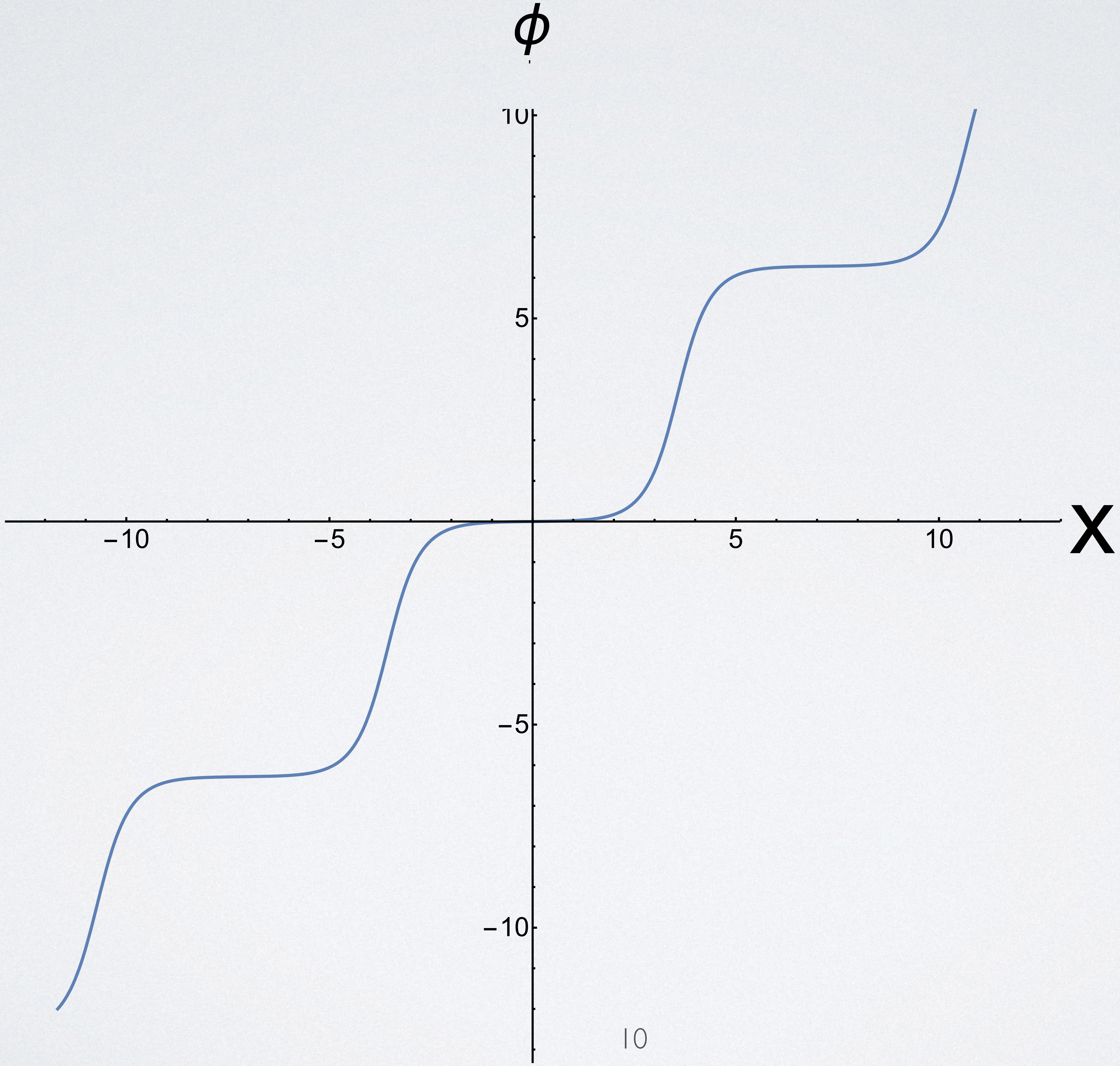
$\tau > 0$



$\tau < 0$



The shock-wave phenomenon and the Hagedorn-type critical point



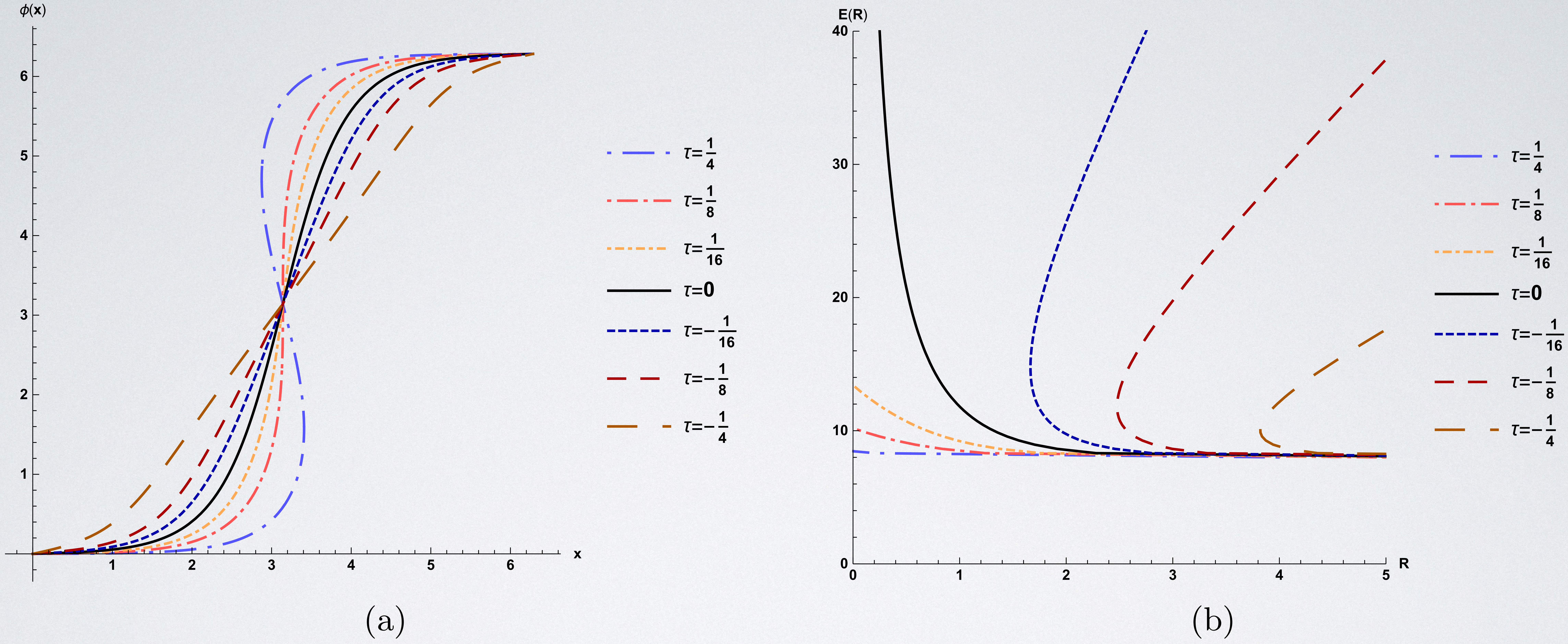


Figure 5. The kink solution to the T$\bar{\text{T}}$-deformed sG model on a cylinder of radius R (a) and the corresponding energies as functions of R (b).

Generic $T\bar{T}$-deformed models

$$\mathcal{J}^{-1} = \begin{pmatrix} \partial_w z & \partial_w \bar{z} \\ \partial_{\bar{w}} z & \partial_{\bar{w}} \bar{z} \end{pmatrix} = \begin{pmatrix} 1+\tau V & -\tau\left(\frac{\partial\phi}{\partial w}\right)^2 \\ -\tau\left(\frac{\partial\phi}{\partial \bar{w}}\right)^2 & 1+\tau V \end{pmatrix} \longrightarrow \begin{pmatrix} 1-\tau\Theta(\mathbf{w}) & -\tau\bar{T}(\mathbf{w}) \\ -\tau T(\mathbf{w}) & 1-\tau\Theta(\mathbf{w}) \end{pmatrix}$$

Notice that:

$$\frac{\partial^2 x^\mu}{\partial y^\rho \partial y^\sigma} = \frac{\partial^2 x^\mu}{\partial y^\sigma \partial y^\rho} \quad \Longleftrightarrow \quad \partial_\mu \mathbf{T}^\mu{}_\nu = 0$$

$$\mathbf{g}'_{\mu\nu} = \delta_{\mu\nu} - \tau\epsilon_{\mu\rho}\epsilon^\sigma{}_\nu\left(2T+\tau T^2\right)^\rho{}_\sigma$$

$$\begin{aligned}\mathcal{A}[\phi] &= \int dz\, d\bar{z}\, \mathcal{L}^{(\tau)}(\mathbf{z}) = \int dw\, d\bar{w}\, \left|\det\left(\mathcal{J}^{-1}\right)\right| \mathcal{L}^{(\tau)}(\mathbf{z}(\mathbf{w})) \\ &= \int dw\, d\bar{w}\, \left(\mathcal{L}^{(0)}(\mathbf{w}) + \tau\, \mathrm{T}\bar{\mathrm{T}}^{(0)}(\mathbf{w})\right)\end{aligned}$$

Further alternative geometric intrepretations

1) There exists a random geometry interpretation of the $T\bar{T}$ deformation of quantum field theory [Cardy]

$$e^{2\delta t \int_{\mathcal{D}} \epsilon_{ik}\epsilon_{jl}T^{ij}T^{kl}d^2x} \propto \int [dh] e^{-(1/8\delta t)\int\int_{\mathcal{D}} \epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl}d^2x + \int_{\mathcal{D}} h_{ij}T^{ij}d^2x}$$

(Hubbard-Stratonovich transformation)

2) Any $T\bar{T}$-deformed field theory is dynamically equivalent to its associated unperturbed theory coupled to (flat) Jackiw-Teitelboim gravity [Dubovsky-Gorbenko-Mirbabayi].

$$S_{\mathrm{M},\tau} \simeq S_{\mathrm{M}} + \int \mathrm{d}^2\mathbf{x}\sqrt{-g}\,(\varphi R - \Lambda_2) \qquad \tau \propto \Lambda_2^{-1}$$

3) The $T\bar{T}$ deformation of a generic field theory is equivalent to coupling the undeformed field theory to 2D ‘ghost-free massive gravity’ [Tolley].

$$S_{T\bar{T}}[\varphi, f, e] = \int \mathrm{d}^2x\, \frac{1}{2\lambda}\epsilon^{\mu\nu}\epsilon_{ab}(e^a_\mu - f^a_\mu)(e^b_\nu - f^b_\nu) + S_0[\varphi, e] \qquad \lambda \propto \tau$$

Quantum $T\bar{T}$-deformations on infinite cylinder of circumference R

$$\partial_\tau \mathscr{H}(\tau) = \det(T_{\mu\nu}(\tau)) \to \partial_\tau \langle n | \mathscr{H}(\tau) | n \rangle = \langle n | \det(T_{\mu\nu}(\tau)) | n \rangle$$

Using Zamolodchikov factorisation property:

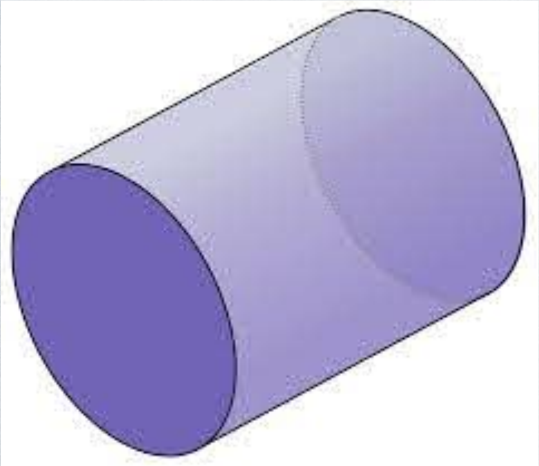
$$\langle n | \det(T_{\mu\nu}(\tau)) | n \rangle = \langle n | T_{11} | n \rangle \langle n | T_{22} | n \rangle - \langle n | T_{12} | n \rangle \langle n | T_{21} | n \rangle$$

with

$$E_n(R,\tau) = -R \langle n | T_{22} | n \rangle \ , \ \partial_R E_n(R,\tau) = -\langle n | T_{11} | n \rangle \ , \ P_n(R) = -\mathbb{i} R \langle n | T_{12} | n \rangle$$

and

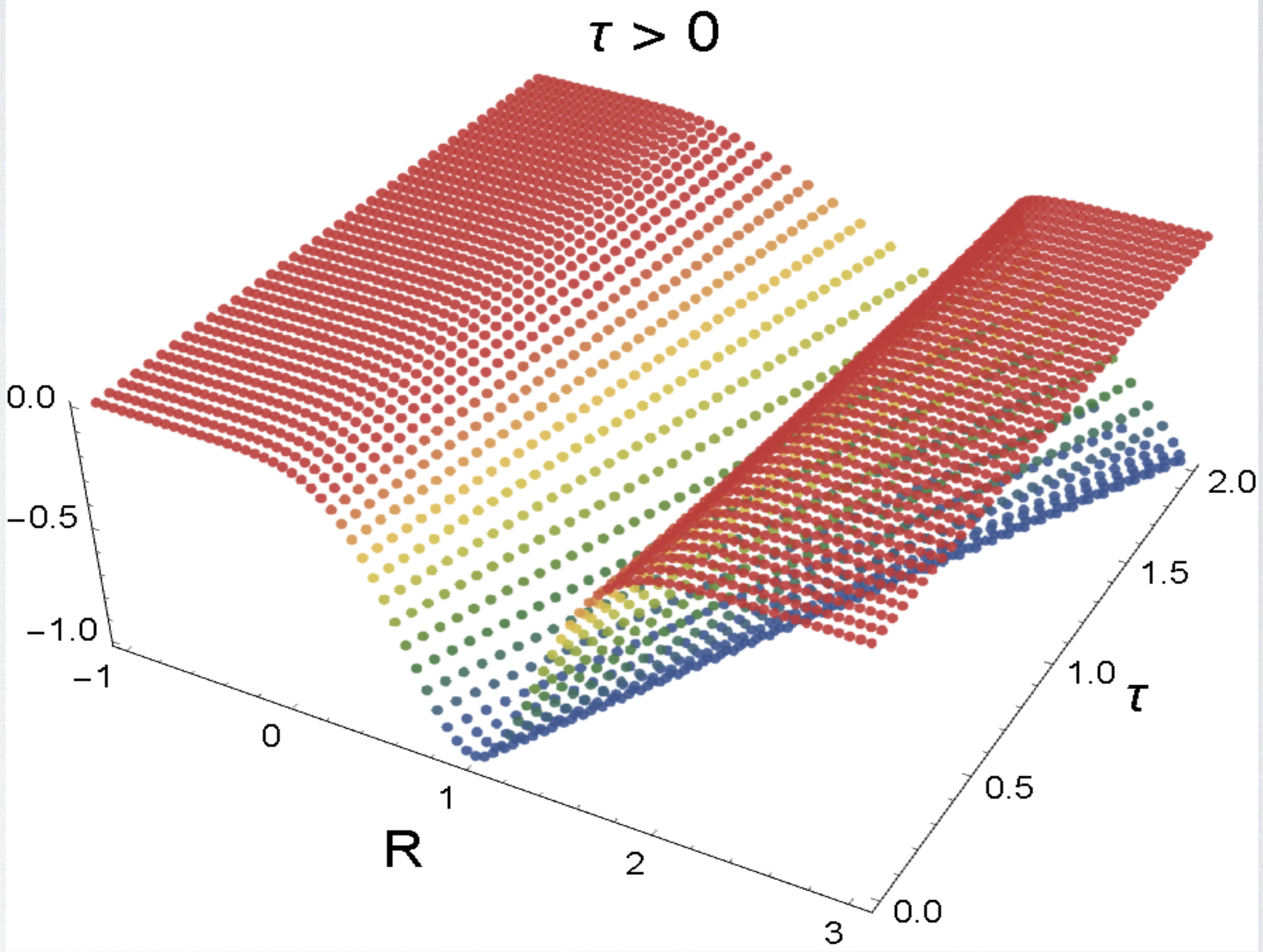
$$P(R,\tau) = P(R) = \frac{2\pi k}{R}, \quad k \in \mathbb{Z}.$$



The inviscid Burgers equation for the quantum spectrum

$$\partial_\tau E_n(R,\tau) = E_n(R,\tau)\partial_R E_n(R,\tau) + \frac{P_n^2(R)}{R}$$

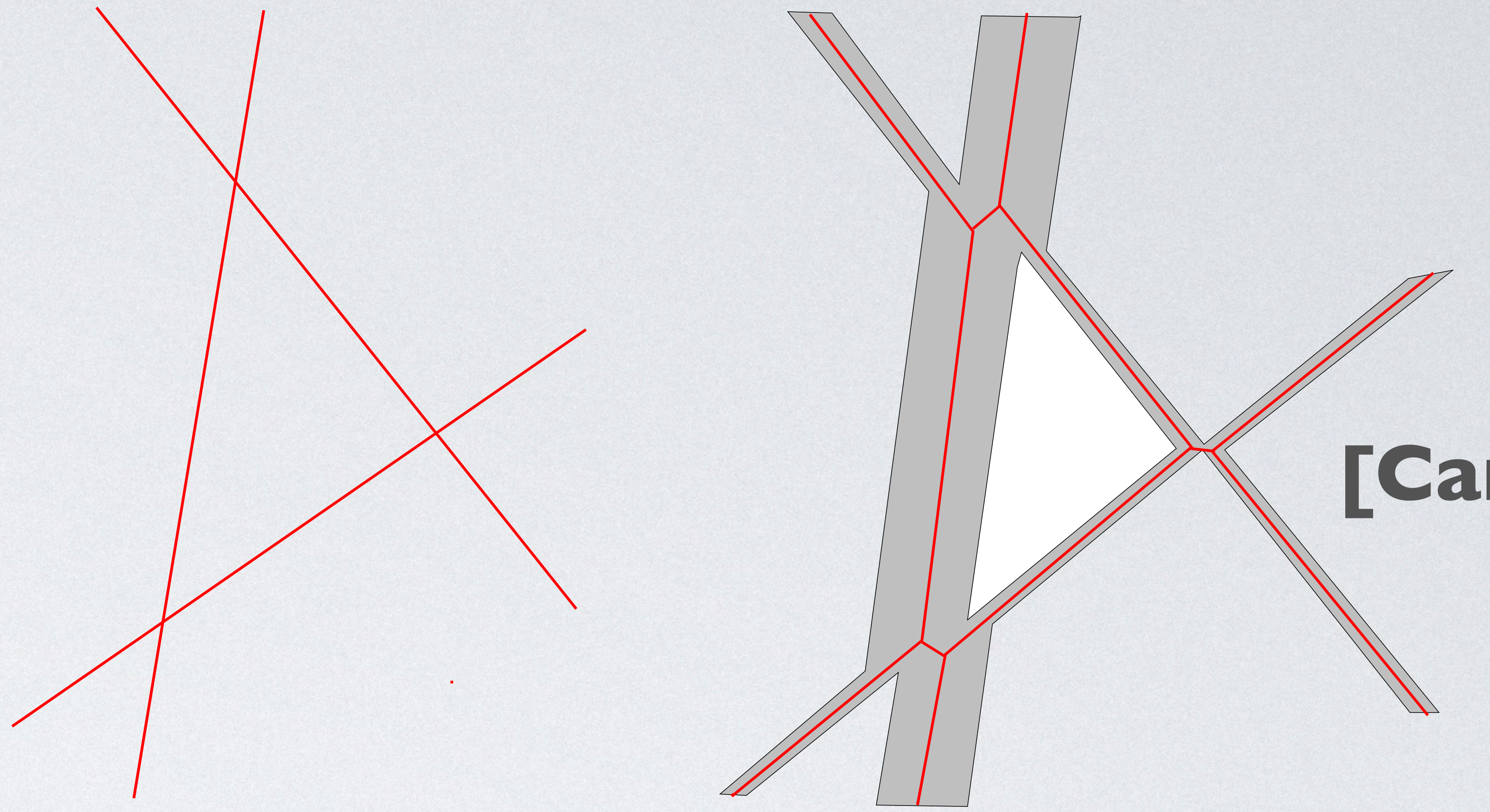
source term



$$P_n = 0 \to E_n(R,\tau) = E_n(R + \tau E_n(R,\tau),0)$$

A possible quantum interpretation:

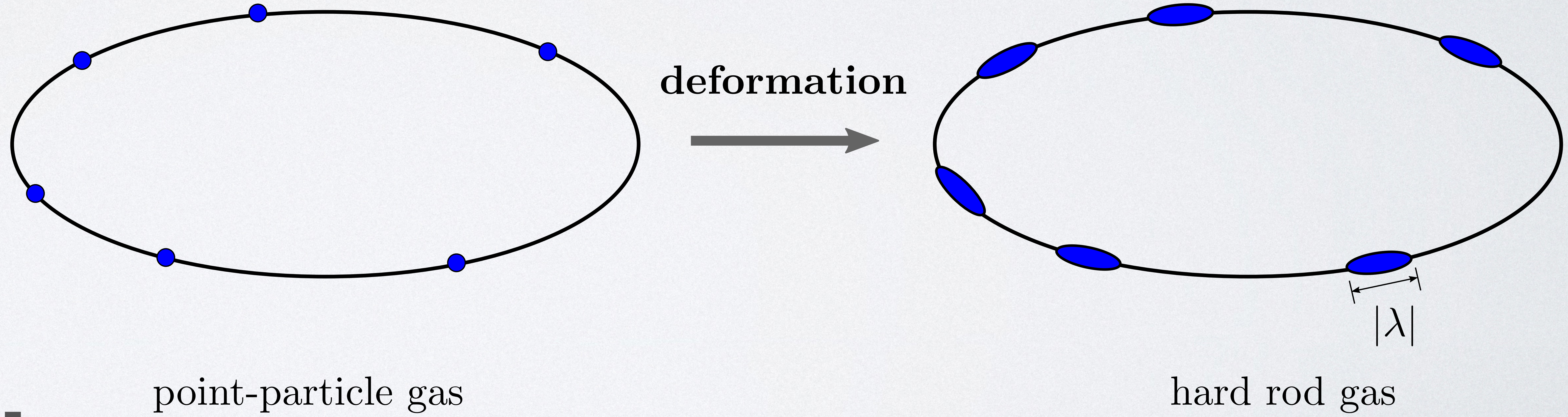
Point particles → **Finite size particles**



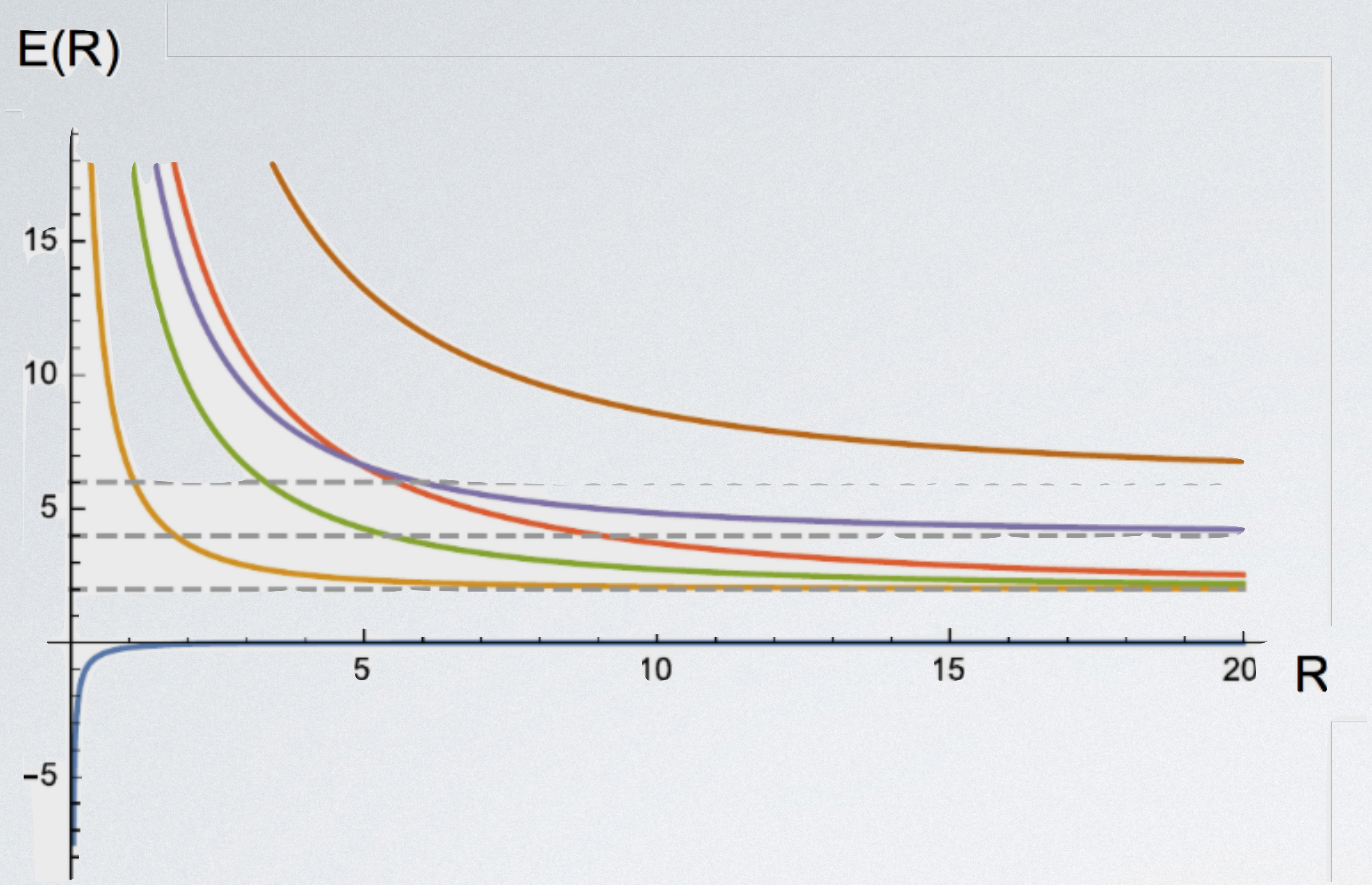
[Cardy-Doyon]

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i<j}^{N} v(x_i - x_j)$$

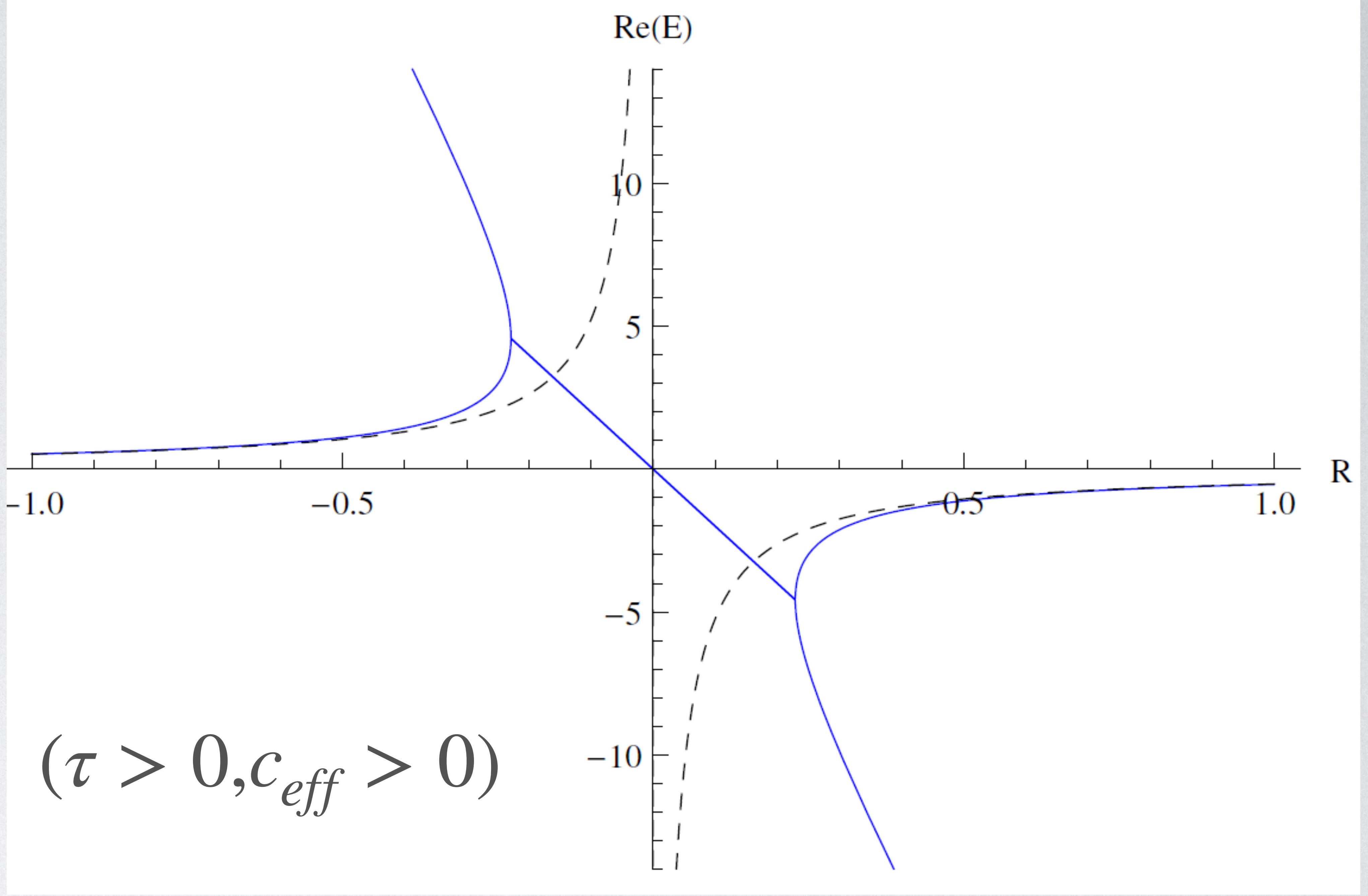
$$v(x) = \begin{cases} \infty, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$



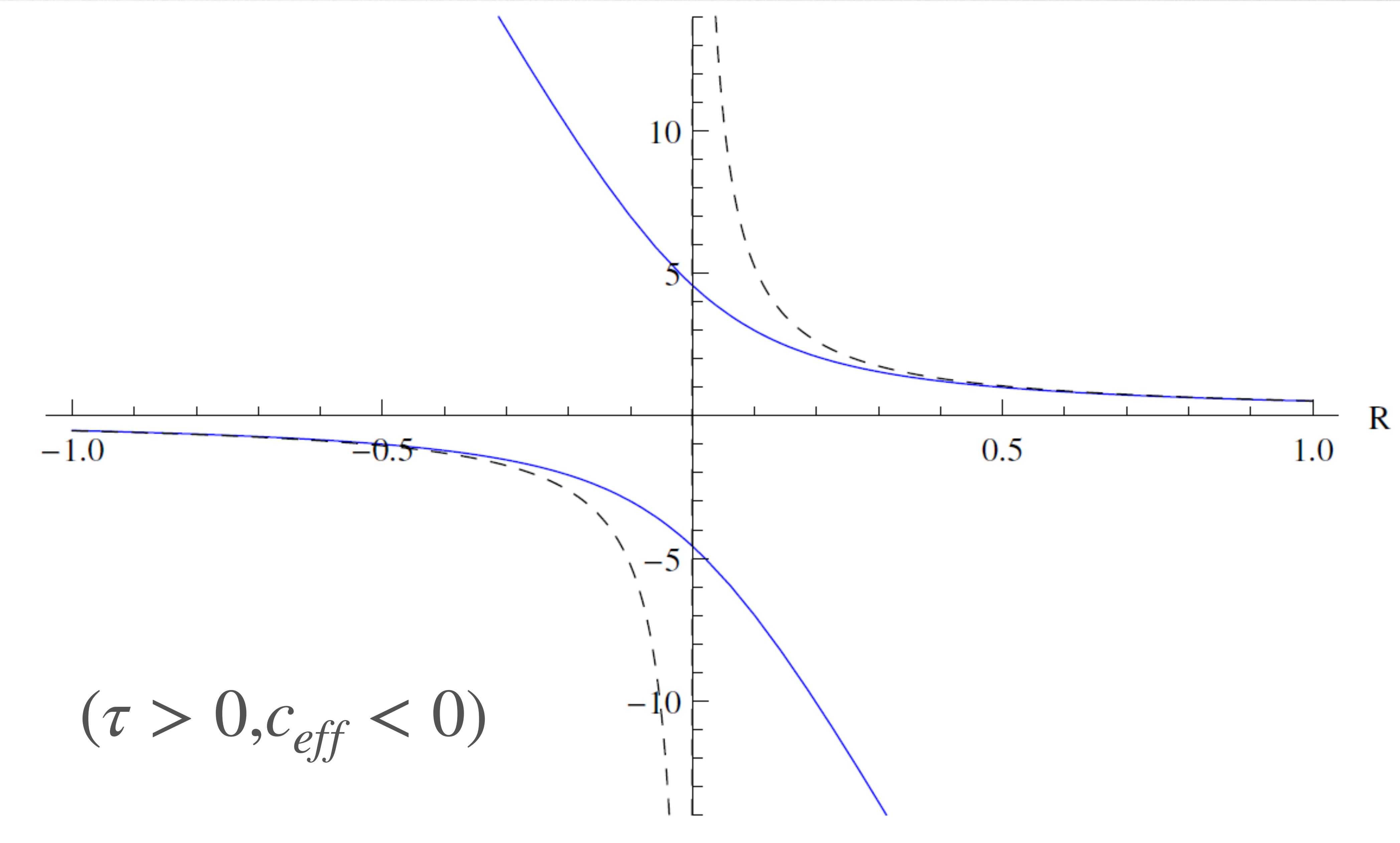
[Y. Jiang]



(Typical $\tau = 0$ finite-volume spectrum)



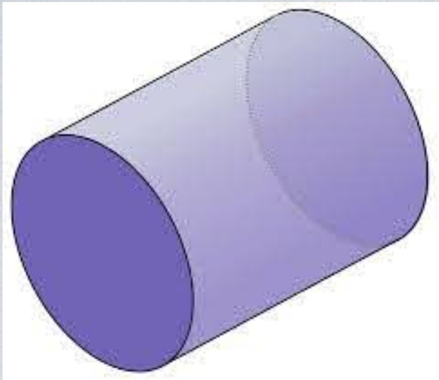
$(\tau > 0, c_{eff} > 0)$



$(\tau > 0, c_{eff} < 0)$

$$E(R,0) \sim -\pi \frac{c_{\rm eff}}{6\,R}, \quad R \sim 0,$$

$$c_{eff} = c - 24\Delta$$



The Conformal Field Theory case

The total energy is:

**[Dubovsky-Flauger-Gorbenko 2012,
Caselle-Gliozzi-Fioravanti-Tateo 2013]**

$$E(R,\tau) = E^{(+)}(R,\tau) + E^{(-)}(R,\tau)$$

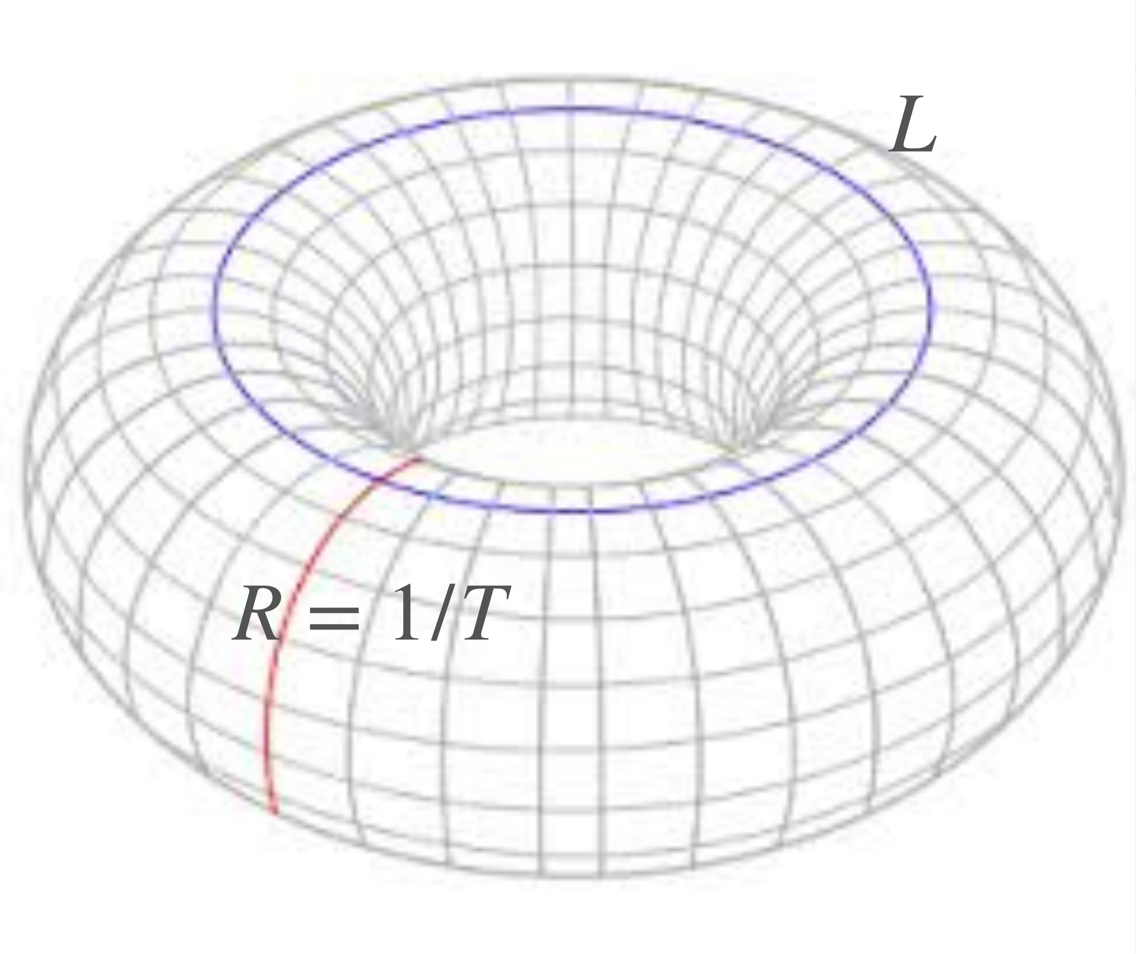
$$= -\frac{R}{2\tau} + \sqrt{\frac{R^2}{4\tau^2} + \frac{2\pi}{\tau}\left(n_0 + \bar{n}_0 - \frac{c_{\rm eff}}{12}\right) + \left(\frac{2\pi(n_0 - \bar{n}_0)}{R}\right)^2}$$

$$c_{eff} = c - 24\Delta \quad \text{(primary)}$$

which matches the form of the (D=26, $c_{eff} = 24$) Nambu Goto spectrum, for a generic CFT.

Notice that there are spectral singularities connecting the two branches.
The most evident being the tachyonic critical point at

$$R_{cr} = \sqrt{\frac{2\pi c\tau}{3}}$$



From the point of view of a QFT at finite temperature $T = 1/R$, this critical point is consequence of an exponential growth of the degeneracy of the energy levels at large energy E

Consider the degeneracy of a free (massless) fermionic system on a circle, with $c = 1/2$ and circumference $L \to \infty$

The asymptotic behaviour of the level degeneracy for large $n_0 = \bar{n}_0 = n$ is

$$\rho(n) = \frac{1}{16\sqrt{3n^3}} e^{2\pi\sqrt{n/3}} = \rho(E)\frac{dE}{dn} = 3\left(\frac{\pi T_H}{3E}\right)^3 e^{E/T_H} \quad \text{we used:} \quad T_H = \sqrt{\frac{3}{\pi\tau}},\ E(n) \simeq \sqrt{4\pi n/\tau}$$

$$\delta S(E) = \frac{\delta E}{T} \to T(E) = \frac{1}{\partial_E S(E)}, \quad S = \log \rho(E)$$

indeed, T_H coincides with the upper limit temperature of the system:

$$T_H = \mathrm{sup}(T(E))$$

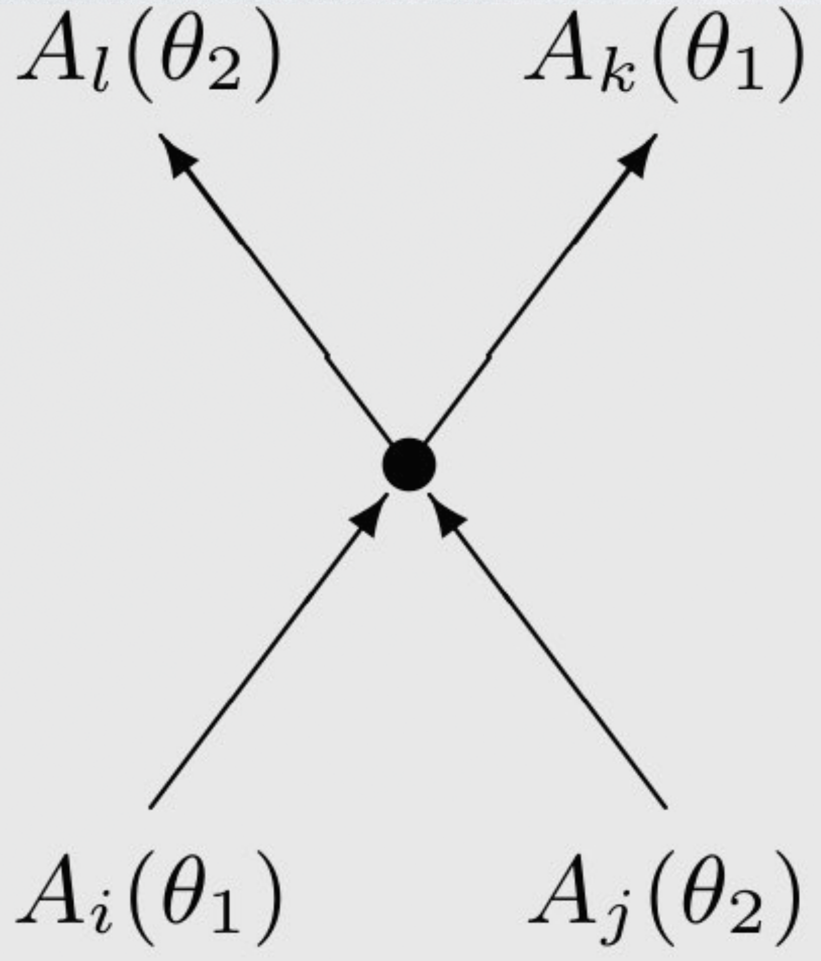
Comparing this result with the tachyonic singularity at R_{cr} we obtain:

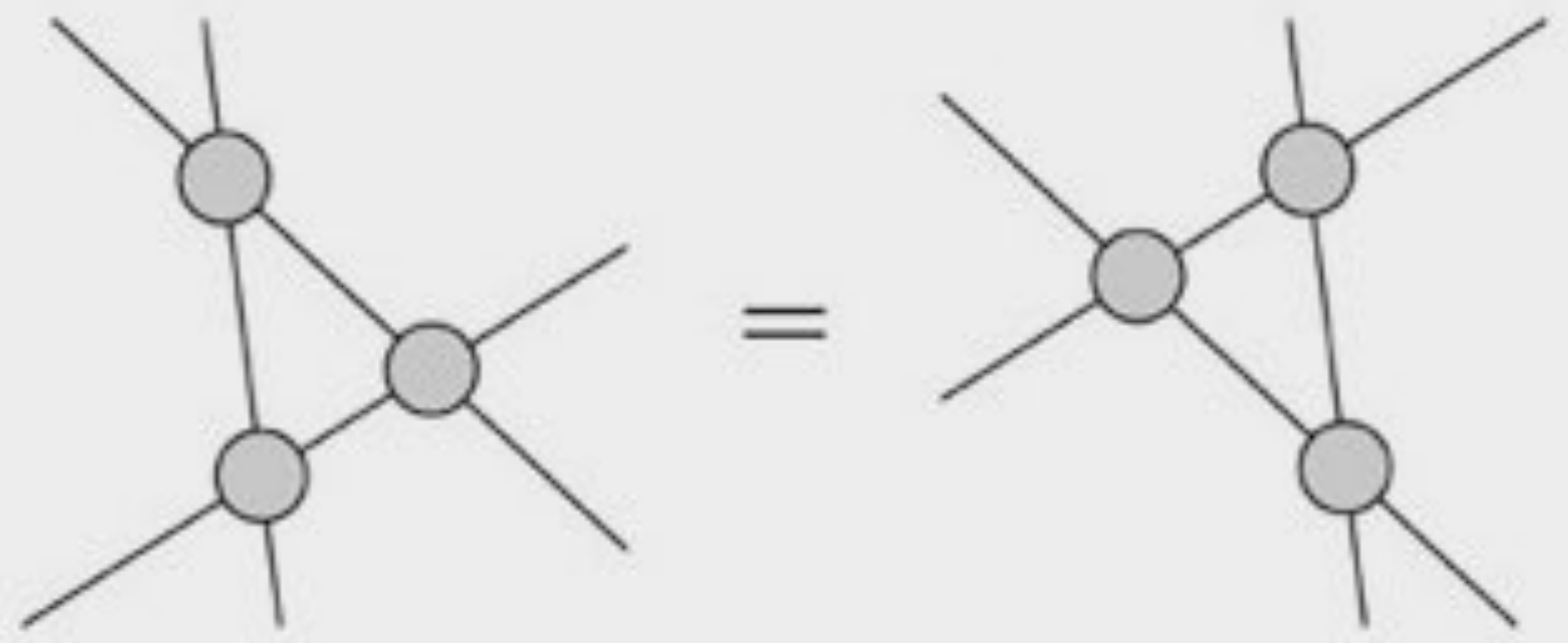
$$R_{cr} = 1/T_H .$$

Exact S-matrix and CDD ambiguity

Consider a relativistic integrable field theory with factorised scattering:

$$S_{ij}^{kl}(\theta_1 - \theta_2) \quad =$$





Castillejo-Dalitz-Dyson ambiguity:

$$S_{ij}^{kl}(\theta) \to S_{ij}^{kl}(\theta)\, e^{i\delta_{ij}^{(\tau)}(\theta)}$$

The simplest possibility, consistent with the crossing and unitarity relations is:

$$\delta_{ij}^{(\tau)}(\theta) = \delta^{(\tau)}(m_i, m_j, \theta) = \tau\, m_i m_j \,\sinh(\theta)$$

Burgers equation from integrability

[Klümper-Batchelor-Pearce, 1991][Destri-DeVega, 1992]

The finite-size properties of the sine-Gordon model are encoded in the single counting function f(θ), solution to the following nonlinear integral equation:

$$f(\theta) = -imR\sinh(\theta) + i\alpha - \int_{\mathcal{C}_1} dy\,\mathcal{K}(\theta-y)\,\ln\left(1+e^{-f(y)}\right) + \int_{\mathcal{C}_2} dy\,\mathcal{K}(\theta-y)\,\ln\left(1+e^{f(y)}\right)$$

where

$$\mathcal{K}(\theta) = \frac{1}{2\pi i}\partial_\theta \ln S_{sG}(\theta)$$

$$E(R) = m\left[\int_{\mathcal{C}_1}\frac{dy}{2\pi i}\sinh(y)\ln\left(1+e^{-f(y)}\right) - \int_{\mathcal{C}_2}\frac{dy}{2\pi i}\sinh(y)\ln\left(1+e^{f(y)}\right)\right]$$

$$P(R) = m\left[\int_{\mathcal{C}_1}\frac{dy}{2\pi i}\cosh(y)\ln\left(1+e^{-f(y)}\right) - \int_{\mathcal{C}_2}\frac{dy}{2\pi i}\cosh(y)\ln\left(1+e^{f(y)}\right)\right]$$

replacing

$$\mathcal{K}(\theta) \to \mathcal{K}(\theta) - \frac{1}{2\pi}\partial_\theta \delta_{CDD}(\theta) = \mathcal{K}(\theta) - \tau \frac{m^2}{2\pi}\cosh(\theta)$$

we get

$$\begin{aligned} f(\theta) = & -i\, m\, \mathcal{R}_0 \, \sinh(\theta - \theta_0) + i\alpha \\ & - \int_{\mathcal{C}_1} dy\, \mathcal{K}(\theta - y) \, \ln\left(1 + e^{-f(y)}\right) + \int_{\mathcal{C}_2} dy\, \mathcal{K}(\theta - y) \, \ln\left(1 + e^{f(y)}\right) \end{aligned}$$

with

$$\sinh\theta_0 = \frac{\tau\, P(R)}{\mathcal{R}_0} = \frac{\tau\, P(\mathcal{R}_0)}{R}\,, \ \cosh\theta_0 = \frac{R + \tau\, E(R,\tau)}{\mathcal{R}_0} = \frac{\mathcal{R}_0 - \tau\, E(\mathcal{R}_0, 0)}{R}$$

$$P(R,\tau) = P(R) = \frac{2\pi k}{R}, \quad k \in \mathbb{Z}.$$

Then

$$f(\theta|R,\tau) = f(\theta - \theta_0|\mathcal{R}_0, 0)$$

which allows to compute the exact form of the τ-deformed energy level once its R-dependence is known at τ = 0. The result is:

$$\begin{pmatrix} E(R,\tau) \\ P(R) \end{pmatrix} = \begin{pmatrix} \cosh(\theta_0) & \sinh(\theta_0) \\ \sinh(\theta_0) & \cosh(\theta_0) \end{pmatrix} \begin{pmatrix} E(\mathcal{R}_0,0) \\ P(\mathcal{R}_0) \end{pmatrix}$$

therefore

$$E^2(R,\tau) - P^2(R) = E^2(\mathcal{R}_0,0) - P^2(\mathcal{R}_0,0)$$

with

$$\mathcal{R}_0^2 = (R + \tau\, E(R,\tau))^2 - \tau^2 P^2(R)\,,\ R^2 = (\mathcal{R}_0 - \tau\, E(\mathcal{R}_0,0))^2 - \tau^2 P^2(\mathcal{R}_0)$$

It is then possible to prove that this sets of constraints are equivalent to the Burgers equation!

ModMax and the $\sqrt{T\bar{T}}$ deforming operators

Consider the recent results on the Modified Maxwell Theory [Bandos-Lechner-Sorokin-Townsend 2020]

$$\mathcal{L}_\gamma^{\mathrm{MM}} = \cosh(\gamma)\, S - \sinh(\gamma)\sqrt{S^2 - P^2}$$

where

$$S := \frac{1}{4}F_{ab}F^{ab}\,, \quad P := \frac{1}{4}\widetilde{F}_{ab}F^{ab} = \sqrt{\det[\mathbf{F}]}$$

—The unique nonlinear extension of the source-free Maxwell theory preserving both the electromagnetic duality invariance and conformal invariance—

$$\frac{\partial \mathcal{L}_\gamma^{\mathrm{MM}}}{\partial \gamma} = \frac{1}{2}\sqrt{\mathrm{tr}[(\mathbf{T}_\gamma^{\mathrm{MM}})^2] - \frac{1}{4}\mathrm{tr}[\mathbf{T}_\gamma^{\mathrm{MM}}]^2}$$

[Babaei-Aghbolagh, Velni, Yekta, Mohammadzadeh]

[Conti-Negro-RT, Ferko-Sfondrini-Smith-Tartaglino Mazzucchelli, Babaei Aghbolagh, Babaei Velni, Mahdavian Yekta, Mohammadzadeh].

A novel classically marginal deformation in 2d, was recently introduced, and denoted root-$T\bar{T}$

$$\frac{\partial \mathcal{L}_\gamma}{\partial \gamma} = -\frac{1}{\sqrt{2}}\sqrt{\mathrm{tr}[(\mathbf{T}_\gamma)^2] - \frac{1}{2}\mathrm{tr}[\mathbf{T}_\gamma]^2}$$

it commutes with the $T\bar{T}$

$$\frac{\partial^2 \mathcal{L}_{\tau,\gamma}}{\partial\tau\partial\gamma} = \frac{\partial^2 \mathcal{L}_{\tau,\gamma}}{\partial\gamma\partial\tau}$$

Finally, it corresponds to a change metric, but not to a global change of coordinates

Thank you for your attention!