Correlation functions in integrable supersymmetric four dimensional gauge theories

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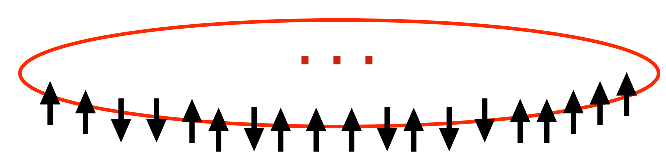
Correlation functions in planar $\mathcal{N}$=4 SYM from integrability

- **Integrability of planar N=4 SYM theory** opened a window into exploring in a detailed and precise manner the gauge/string, or **AdS/CFT correspondence**

- Local, **gauge-invariant operators** with definite conformal dimension correspond to **eigenstates** of a long-range interacting (super)spin chain - and to a **string propagating** in the AdS5 x S5 background (described by a sigma model)

$$\mathrm{Tr}_{\,\mathrm{SU(N)}} ZZXXZZXZZZ...(x) \quad \longleftrightarrow$$



- Correlation functions of such operators can be computed with techniques inspired from 2d integrable field theories (**non-local form factors**)

- A practical motivation is to **replace the tedious Feynman graph computations** in gauge theories with **more powerful techniques based on symmetries**, and eventually to get a non-perturbative description for the gauge theory observables

AdS/CFT spin chain

- At weak coupling the spin chain is a supersymmetric, **long range** generalisation of the **Heisenberg** spin chain with a continuous parameter ('t Hooft coupling constant) $g = \frac{\sqrt{\lambda}}{4\pi}$
- The **spin states** correspond to the **fundamental fields** of the gauge theory: SU(Nc) gauge fields, six real bosons and four complex fermions

$$A_\mu,\ \Phi_i \text{ and } \psi_a \qquad i = 1, ..., 6, \quad a = 1, ..., 4$$

- **Bosonic symmetry** = so(6) x so(4,2) ⊂ psu(2,2|4); isometry of S5 x AdS5

e.g. $Z = \Phi_1 + i\Phi_2$, $X = \Phi_3 + i\Phi_4$ form a su(2) subsector of so(6) ~ su(4)

- **Vacuum** state ⟷ **BPS operator** *e.g.* $\mathrm{Tr}\ Z^L$
- **Magnons** have $\mathrm{psu}(2|2)_\mathrm{L} \times \mathrm{psu}(2|2)_\mathrm{R}$ flavours and scatter with Beisert's scattering matrix^2
- **Sutherland solutions** of the spin chain ⟷ rotating **string solution**

  finite gap solution of the sigma model

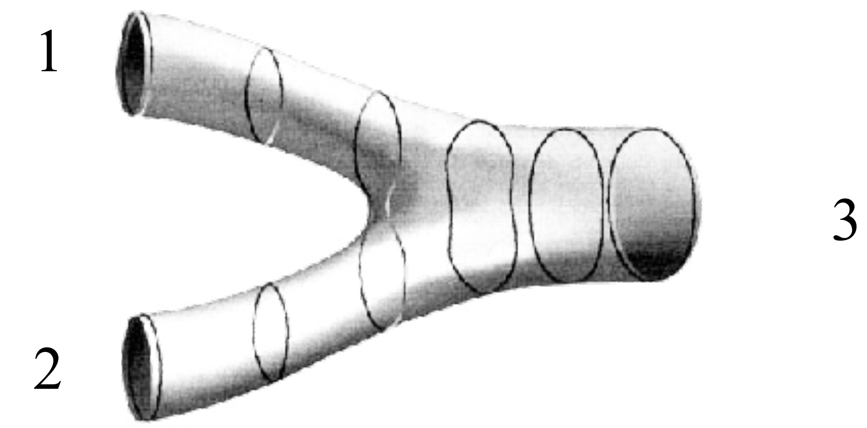
- **Finite size corrections** obtained via a reformulation of the TBA, **Quantum Spectral Curve**

Correlation functions in N=4 SYM

- two and three point functions are determined by **conformal invariance**

$$\langle \mathcal{O}_A(x)\mathcal{O}_B(y)\rangle = \frac{\delta_{AB}}{|x-y|^{2\Delta_A(g)}}$$ ← spin chain energy

the three point function
dual to three-string interaction
is the **basic building block** for
higher point correlation function



$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{C_{123}(g)}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|x-z|^{\Delta_1+\Delta_3-\Delta_2}|y-z|^{\Delta_2+\Delta_3-\Delta_1}}$$

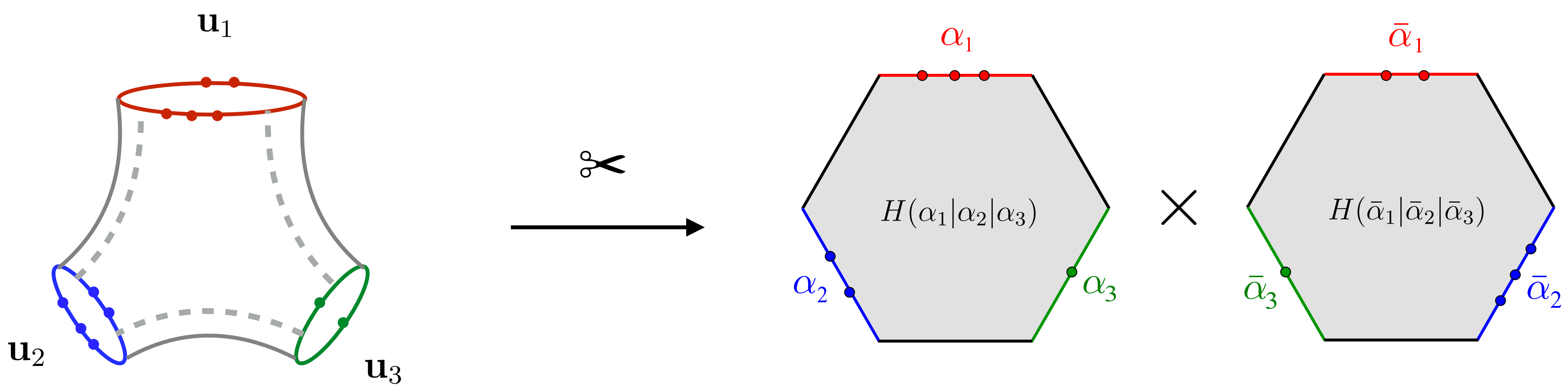
initial data: three states with definite conformal dimensions and psu(2,2|4) charges

$$\mathcal{O}_\alpha(x)\ , \quad \alpha = 1,2,3$$

The hexagon decomposition of correlation functions

[Basso, Komatsu, Vieira, 15]

- the **asymptotic part** of the three point function can be written as a sum over partitions for the three groups of rapidities $\mathbf{u}_1 = \alpha_1 \cup \bar{\alpha}_1, \mathbf{u}_2 = \alpha_2 \cup \bar{\alpha}_2, \mathbf{u}_3 = \alpha_3 \cup \bar{\alpha}_3$



$$[\mathcal{C}_{123}^{\bullet\bullet\bullet}]^{\text{asympt}} = \sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} (-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|} \, w_{\ell_{31}}(\alpha_1, \bar{\alpha}_1) \, w_{\ell_{12}}(\alpha_2, \bar{\alpha}_2) \, w_{\ell_{23}}(\alpha_3, \bar{\alpha}_3)$$
$$\times \quad \mathrm{H}(\alpha_1|\alpha_3|\alpha_2)\mathrm{H}(\bar{\alpha}_2|\bar{\alpha}_3|\bar{\alpha}_1) \, .$$

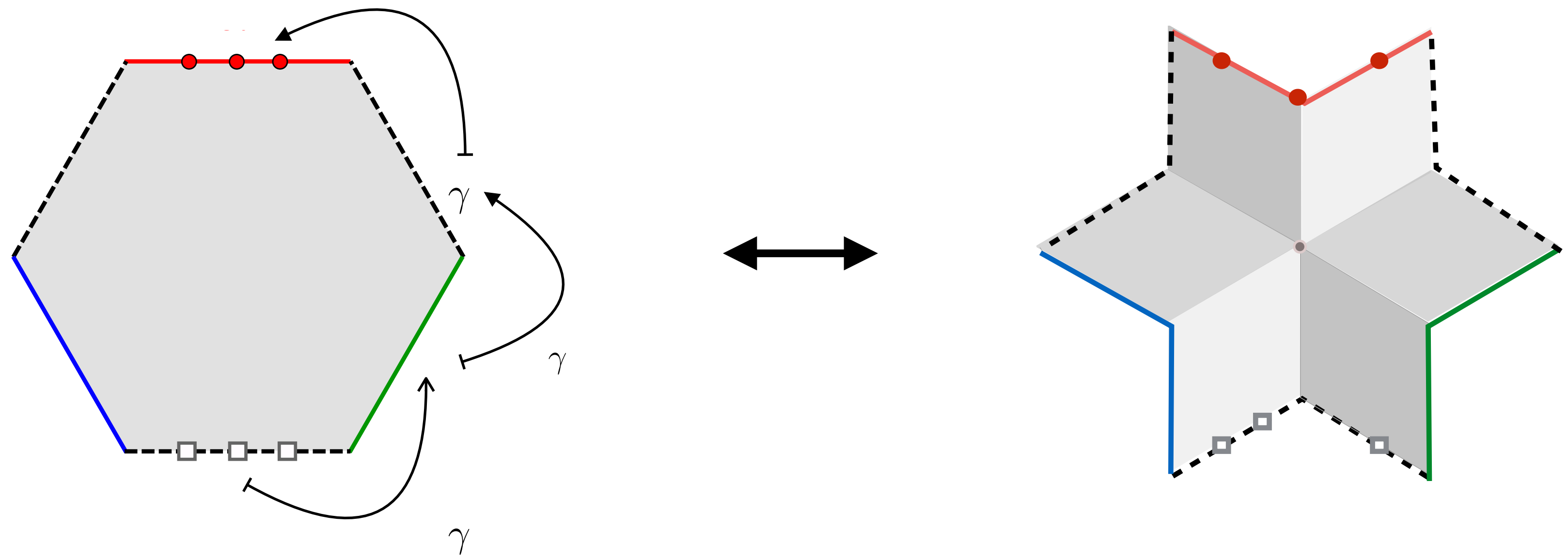
- contribution of virtual particles **exponentially suppressed** if the bridges $\ell_{12}, \ \ell_{23}, \ \ell_{31} >> 0$

$$\ell_{ij} = \frac{1}{2}(L_i + L_j - L_k)$$

- sewing back over the black lines: **insertion of an arbitrary number of virtual particles**

The hexagon as a non-local form factor

- the hexagon can be seen as the infinite-volume **form factor of a twist-like operator** inducing a curvature excess of 180 degrees **[Cardy, Castro-Alvaredo, Doyon, 06]**



γ "**mirror transformation**", equivalent to $\theta \to \theta + \frac{i\pi}{2}$ in relativistic theories

- solution from **bootstrap** (form factor axioms)

$$\mathrm{H}^{\,A_1 \dot{A}_1 \cdots} = \prod_{i<j} h_{ij} \quad \langle \chi_N^{\dot{A}_N} \ldots \chi_1^{\dot{A}_1} | \mathcal{S} | \chi_1^{A_1} \ldots \chi_N^{A_N} \rangle$$

dynamical part

matrix part
[Beisert, 06]

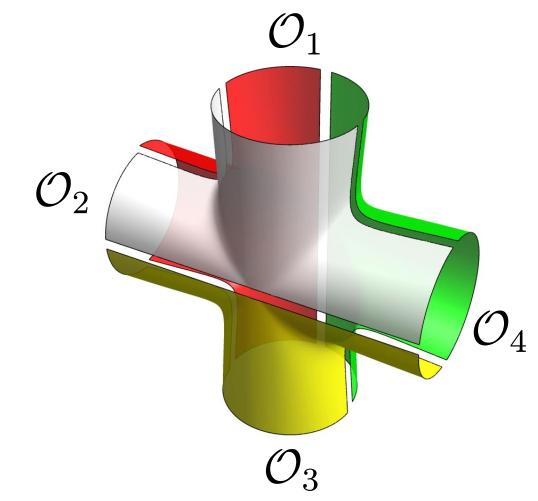
$|\chi_i^{A_i}\rangle$: $\mathrm{psu}(2|2)_{\mathrm{L}}$ state

$|\chi_i^{\dot{A}_i}\rangle$: $\mathrm{psu}(2|2)_{\mathrm{R}}$ state

The hexagon as building block for correlation functions

- **four point** function by hexagon decomposition: [Fleury, Komatsu, 16; also Eden, Sfondrini, 16]

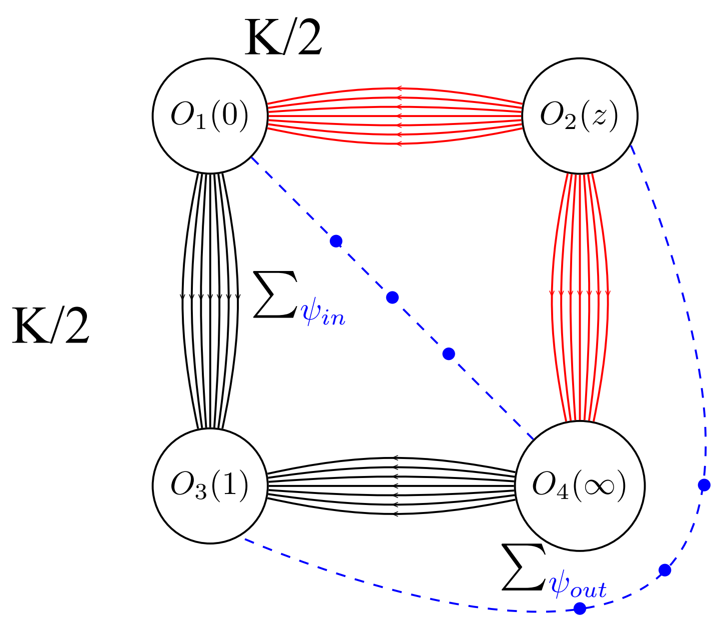
$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle =$$



- **sewing back** hexagons imply insertion of an arbitrary number of virtual particles - in general the sum over virtual particles is not easy to perform, except in the case of the **octagon,** see below

- when a leg is formed by sewing different hexagons, **divergences** appear; a **systematic regularisation** was not yet achieved, but **important results were conjectured** in [Basso, Georgoudis, Klemenchuk-Sueiro, 22]

Four point functions: the "simplest" correlator



- four point function: dependence on **two cross ratios**:

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- for BPS operators with **large R-charges** and particular polarisations: **factorisation into two octagons** **[Coronado, 18]**

$$\langle O_1 O_2 O_3 O_4 \rangle = \left[\frac{1}{x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2}\right]^{\frac{K}{2}} \times \mathbb{O}^2(z,\bar{z}) \qquad K \to \infty$$

- **compute the octagon analytically** by summing up the virtual particle contribution

⟶ Fredholm determinant **[Kostov, Petkova, D.S., 19]**

- analysis of the Fredholm determinant in various regimes, including resurgent analysis

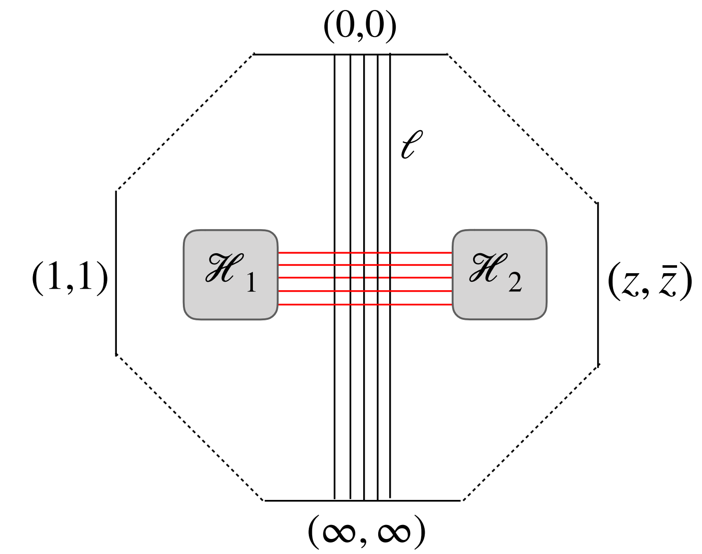
[Belitsky, Korchemsky, 19-21; Bajnok, Boldis, Korchemsky, 24]

Four point functions: the "simplest" correlator

$$\mathbb{O}_l(z,\bar z,\alpha,\bar\alpha) = 1 + \sum_{n=1}^{\infty} \mathcal{X}_n(z,\bar z,\alpha,\bar\alpha) \times \mathcal{I}_{n,l}(z,\bar z)$$

simple kinematical factor

more general setting: octagon with a bridge of length l



multiple integral over **virtual particles**:

$$\mathcal{I}_{n,l}(z,\bar z) = \frac{1}{n!} \sum_{a_1=1}^{\infty} \cdots \sum_{a_n=1}^{\infty} \int du_1 \cdots \int du_n \prod_{j=1}^{n} \bar\mu_{a_j}(u_j,l,z,\bar z) \times \prod_{j<k}^{n} P_{a_j a_k}(u_j,u_k)$$

one-particle measure: $\bar\mu_a(u,l,z,\bar z) = \frac{1}{\sqrt{z\bar z}} \frac{\sin a\phi}{\sin\phi} \times \mu_a(u) \times e^{-E_a(u)\,l} \times (z\bar z)^{-i\,p_a(u)}$

two-particle interaction: $P_{ab}(u,v) = \mathcal{K}_{ab}^{++}(u,v)\mathcal{K}_{ab}^{+-}(u,v)\mathcal{K}_{ab}^{-+}(u,v)\,\mathcal{K}_{ab}^{--}(u,v)$

$$\mathcal{K}_{ab}^{\pm\pm}(u,v) = \frac{x^{[\pm a]}(u) - x^{[\pm b]}(v)}{1 - x^{[\pm a]}(u)\,x^{[\pm b]}(v)} \qquad x^{[\pm a]} + \frac{1}{x^{[\pm a]}} = \frac{u \pm ia/2}{g}$$

Exact results for the octagon

convenient parametrisation for the cross ratios:

$$z = e^{-\xi+i\phi}, \qquad \bar{z} = e^{-\xi-i\phi},$$
$$\alpha = e^{\varphi-\xi+i\theta}, \quad \bar{\alpha} = e^{\varphi-\xi-i\theta}.$$

$$\mathbb{O}_\ell(z,\bar{z},\alpha,\bar{\alpha}) = \tfrac{1}{2}\sum_\pm \ \mathrm{Det}\left(\mathbf{I} - \lambda_\pm \mathbf{K}_\ell^{\mathrm{oct}}\right)$$

[Kostov, Petkova, D.S., 19] simplified by **[Belitsky, Korchemsky, 19]**

$$(\mathrm{K}_\ell^{\mathrm{oct}})_{mn} = -2\sqrt{(2m+\ell+1)(2n+\ell+1)}\int_0^\infty \frac{dt}{t}\,\chi(t)\; J_{2m+\ell+1}(2gt)\; J_{2n+\ell+1}(2gt)$$

$$m,n \geq 0$$

$$\chi(t) = \frac{\cos\phi - \cosh\xi}{\cos\phi - \cosh\sqrt{t^2+\xi^2}}$$

Bessel kernel

- the octagon kernel is a rather universal object showing up in other instances, *e.g.*
  - circular Wilson loop in N=4 SYM
  - sphere partition function in N=2 Z2 orbifold SYM
  - two-and three-point function of twisted BPS operators in the above theory

N=2 Z2 orbifold theory

- a version of N=4 SYM where the sphere part is orbifolded by a Z2 twist $\tau = \begin{pmatrix} \mathbf{1}_{N_c} & 0 \\ 0 & -\mathbf{1}_{N_c} \end{pmatrix}$
- the gauge group is $SU(N_c) \times SU(N_c)$ and the fields are $2N_c \times 2N_c$ matrices
- same field content as N=4 SYM, with definite action of the twist:

$$A_\mu = \tau\, A_\mu\, \tau\,, \qquad \{Z, \bar{Z}\} = \tau\, \{Z, \bar{Z}\}\, \tau\,, \qquad \{X, Y, \bar{X}, \bar{Y}\} = -\tau\, \{X, Y, \bar{X}, \bar{Y}\}\, \tau \qquad \cdots$$

- symmetry reduced from $\mathrm{psu}(2,2|4) \to \mathrm{su}(2,2|2) \times \mathrm{su}(2)$

- expected to be **integrable**:
  - Bethe Ansatz equations **[Beisert, Roiban, 05];**
  - twisted magnons **[Gadde, Rastelli, 10]**

- results from **localisation**:
  - sphere partition function $\longrightarrow$ matrix model **[Pestun 07,…, Beccaria, Korchemsky, Tseytlin, 22]**
  - two point functions of twisted BPS ops **[Beccaria, Billo, Galvagno, Hasan, Lerda, 20,…]**
  - three point functions of (twisted) BPS **[Billo, Frau, Lerda, Pini, Vallarino, 22,…]**

$$\longrightarrow \quad \mathbf{K}_\ell^{\mathrm{oct}}$$

N=2 Z2 orbifold theory

$$\tau = \begin{pmatrix} \mathbf{1}_{N_c} & 0 \\ 0 & -\mathbf{1}_{N_c} \end{pmatrix}$$

- BPS (vacuum) sector

$$U_k(x) = \frac{1}{\sqrt{2}} \mathrm{Tr}\, Z^k(x) = \frac{1}{\sqrt{2}} \mathrm{Tr}\, \left(Z_0^L + Z_1^L\right) \qquad \text{untwisted}$$

$$T_k(x) = \frac{1}{\sqrt{2}} \mathrm{Tr}\, \tau Z^k(x) = \frac{1}{\sqrt{2}} \mathrm{Tr}\, \left(Z_0^L - Z_1^L\right) \qquad \text{twisted}$$

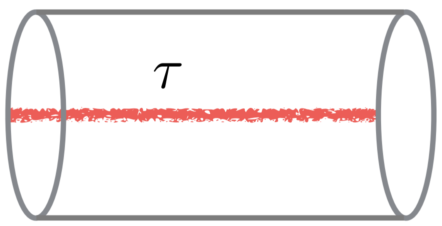
$$\Delta_{U_k} = \Delta_{T_k} = k.$$

- two-point functions (from localisation + perturbative checks **[Galvagno, Preti, 20]**)

$$\langle U_k(x) \bar{U}_k(y) \rangle = \frac{G_{U_k}}{|x-y|^{2k}}$$

$$G_{U_k} = k N^k \equiv \mathcal{G}_k$$

$$\langle T_k(x) \bar{T}_k(y) \rangle = \frac{G_{T_k}}{|x-y|^{2k}}$$



$$G_{T_k} = \mathcal{G}_k \frac{\det(1 - K_{k+2})}{\det(1 - K_k)}$$

$$K_{\ell+1} \equiv \mathbf{K}_\ell^{\mathrm{oct}} \qquad \text{with} \qquad \chi(t) = \frac{e^t}{(e^t - 1)^2}$$

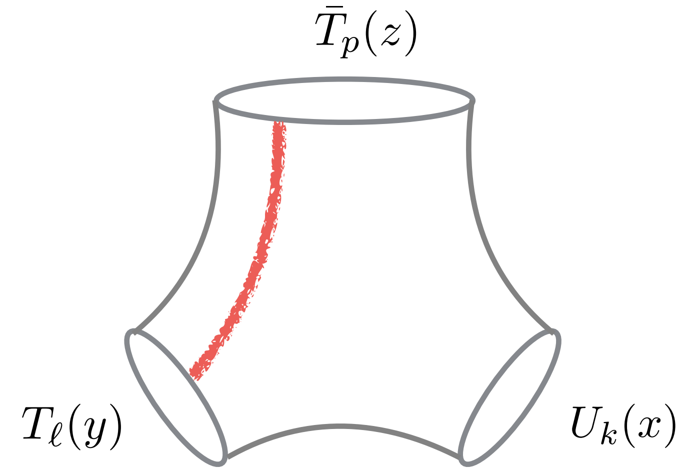
N=2 Z2 orbifold theory

- three-point functions (extremal) **[Billo, Frau, Lerda, Pini, Vallarino, 22]**

$$p = k + \ell$$

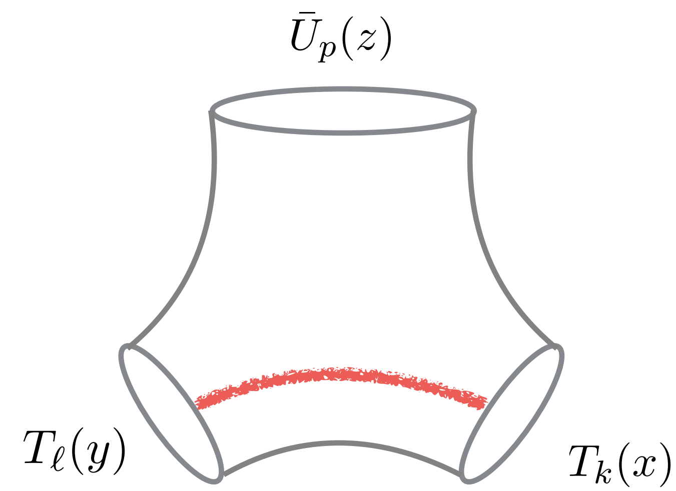
$$\langle U_k(x) T_\ell(y) \bar{T}_p(z) \rangle = \frac{G_{U_k, T_\ell, \bar{T}_p}}{|x-z|^{2k} |y-z|^{2\ell}}$$

$$C_{U_k, T_\ell, \bar{T}_p} = \frac{G_{U_k, T_\ell, \bar{T}_p}}{\sqrt{G_{U_k} G_{T_\ell} G_{T_p}}} = \frac{\sqrt{k\ell p}}{\sqrt{2}N} \sqrt{1 + \frac{1}{2\ell} g \partial_g \ln G_{T_\ell}} \sqrt{1 + \frac{1}{2p} g \partial_g \ln G_{T_p}}$$



$$\langle T_k(x) T_\ell(y) \bar{U}_p(z) \rangle = \frac{G_{T_k, T_\ell, \bar{U}_p}}{|x-z|^{2k} |y-z|^{2\ell}}$$

$$C_{T_k, T_\ell, \bar{U}_p} = \frac{G_{T_k, T_\ell, \bar{U}_p}}{\sqrt{G_{T_k} G_{T_\ell} G_{U_p}}} = \frac{\sqrt{k\ell p}}{\sqrt{2}N} \sqrt{1 + \frac{1}{2\ell} g \partial_g \ln G_{T_\ell}} \sqrt{1 + \frac{1}{2k} g \partial_g \ln G_{T_k}}$$



$$\sqrt{1 + \frac{1}{2\ell} g \partial_g \ln G_{T_\ell}} = \frac{\det(1 - K_{\ell+1})}{\sqrt{\det(1 - K_\ell) \det(1 - K_{\ell+2})}}$$

[Ferrando, Komatsu, Lefundes, D.S.; Korchemsky, 24]

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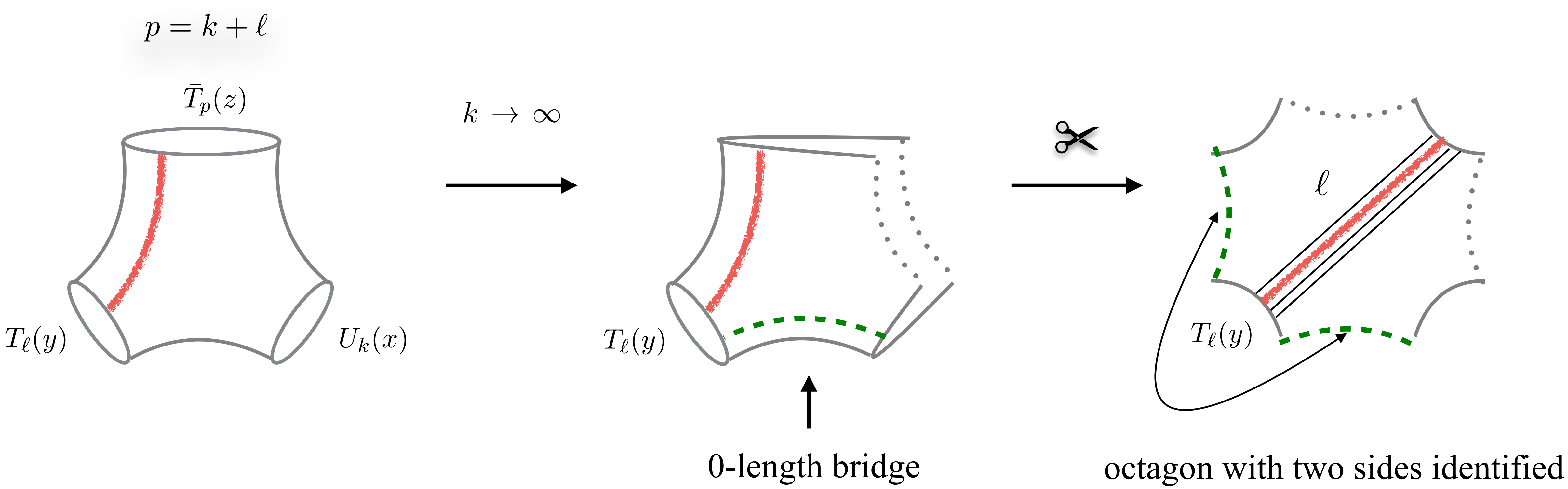
N=2 Z2 orbifold theory

- compute the three-point function using integrability ⟶ hexagon decomposition

[Ferrando, Komatsu, Lefundes, D.S.]



= insertion of a twist $\quad \tau \to \mathbf{1}_L \times \mathrm{diag}(\mathbf{1}_F, -\mathbf{1}_B)_R$

breaks $\quad \mathrm{psu}(2|2)_L \times \mathrm{psu}(2|2)_R \to \mathrm{psu}(2|2)_L \times [\mathrm{su}(2) \times \mathrm{su}(2)]_R$

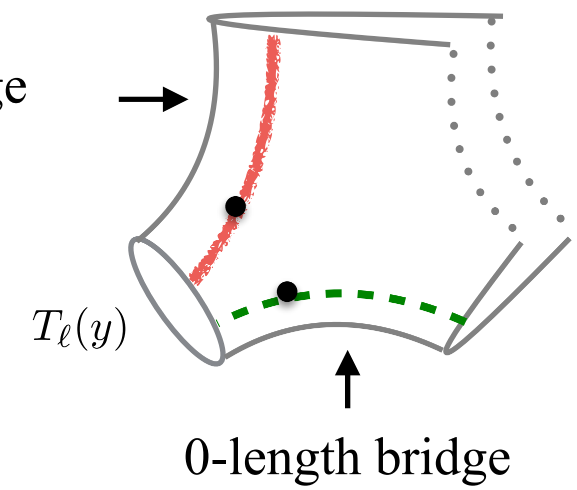
$$\sqrt{1 + \frac{1}{2\ell} g\partial_g \ln G_{T_\ell}} = \frac{\det(1 - K_{\ell+1})}{\sqrt{\det(1 - K_\ell)\det(1 - K_{\ell+2})}}$$

⟵ octagon contribution $\quad K_{\ell+1} \equiv \mathbf{K}_\ell^{\mathrm{oct}}$

⟵ wrapping magnons

N=2 Z2 orbifold theory

- **one-magnon wrapping**: from **contact terms** of the two magnons in the two bridges



singularities in the hexagon weights

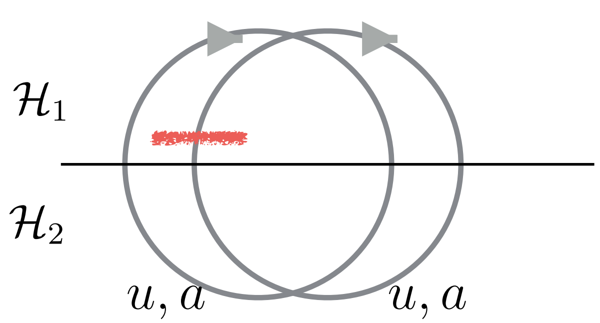
[Basso, Gonçalves, Komatsu, 17]

$$\frac{1}{(u-v-i\epsilon)(u-v+i\epsilon)} \sim \frac{\pi}{\epsilon}\delta(u-v)$$

$$\epsilon \to 0$$

- contribution of the MATRIX part to the contact term

$\frac{1}{\epsilon}$



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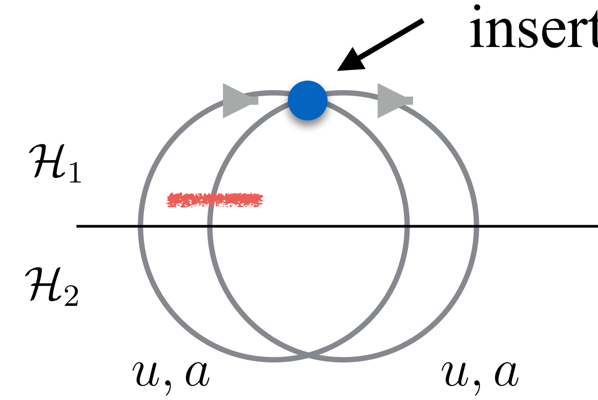
insertion of a derivative



$$\mathrm{STr}_a\, \mathbf{1}_a \times \mathrm{STr}_a\, \tau_a = 0$$

$$\widetilde{\mathcal{K}}_{aa}(u,u) = -i\, \mathrm{STr}_{a\otimes b}\{\mathcal{S}_{ba}(v^\gamma, u^\gamma)\, \tau_a\, \partial_u \mathcal{S}_{ab}(u^\gamma, v^\gamma)\}|_{v\to u;\ b\to a}$$

= insertion of a twist $\quad \tau \to \mathbf{1}_L \times \mathrm{diag}(\mathbf{1}_F, -\mathbf{1}_B)_R \equiv \mathbf{1}_L \times \tau_R$

N=2 Z2 orbifold theory

- **one-magnon wrapping**

$$W_1 = \sum_{a\geq 1} \int_{-\infty}^{\infty} \frac{du}{2\pi}\, e^{-\tilde{E}_a(u)\ell}\, \tilde{\mathcal{K}}_{aa}(u,u) \qquad e^{-\widetilde{E}_a(u)} = \frac{1}{x^{[+a]}x^{[-a]}}$$

$$\widetilde{\mathcal{K}}_{aa}(u,u) = -i\, \mathrm{STr}_{a\otimes b}\{\mathcal{S}_{ba}(v^\gamma, u^\gamma)\, \tau_a\, \partial_u \mathcal{S}_{ab}(u^\gamma, v^\gamma)\}|_{v\to u;\ b\to a}$$

Mathematica-suported conjecture for the functional dependence

using Beisert's S matrix in the ath antisymmetric representation

$$\frac{\det(1-K_{\ell+1})}{\sqrt{\det(1-K_\ell)\det(1-K_{\ell+2})}} = 1 - \mathrm{Tr}\, K_{\ell+1} + \frac{1}{2}\mathrm{Tr}\,(K_\ell + K_{\ell+2}) + \ldots \equiv 1 + B_1 + W_1 + \ldots$$

N=2 Z2 orbifold theory

- **two-magnon contributions?**

- the regularisation procedure is more subtle $\longrightarrow$ start with a four point function and perform an OPE limit **[Basso, IGST 21; Basso, Georgoudis, Klemenchuk-Sueiro, 22]**

- the combinatorics of diagrams is more involved, but we obtained the necessary building blocks

- draw inspiration from the fishnet case **[Ferrando, Olivucci, unpublished]**

- factorisation of the bridge and wrapping contributions

Summary and outlook

- Some correlation functions of local gauge invariant operators can be computed exactly in terms of Fredholm determinants, either by **integrability techniques or by localisation**

- The cases accessible by both are a good laboratory to study the **interplay between the two approaches**

- We can hope to develop a more systematic understanding of the structure constants via from the **analysis of wrapping (TBA-like) corrections** and the connection with the **SoV methods** **[Bercini, Homrich, Vieira, 22 & various groups, in progress]**

- **Five point functions** have a richer structure, being associated to more complicated Feynman diagrams - but their analysis is more complicated **[Fleury, Komatsu, 17; Fleury, Gonçalves 20 & various groups, in progress]**