Representations of fractional-level WZW models and W-algebras

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Mathematics and Physics of Integrability, MATRIX

Outline

Motivation

I want to understand conformal field theory (CFT)...

rational — **factorisable** — non-rational

[compactified bosons]
[free fermions]
[bc-ghosts]
[minimal models]
[compact WZW models]
[a few W-algebras]

[non-compact bosons]
[Liouville/Toda]
[non-compact WZW?]

discrete - - - - - - **continuous**

[symplectic fermions]
[triplet models]
[polymers, percolation?]
[log minimal models?]
[SLE/CLE?]

[$\beta\gamma$-ghosts]
[supergroup WZW]
[Nappi–Witten]
[fractional-level WZW]
[most W-algebras]
[spin chains?]

log-rational — **logarithmic** — generic

CFTs are built from reps of its chiral algebra, *aka*. vertex operator algebra (VOA).

A rational CFT has VOA-modules that are

- completely reducible: they're all direct sums of irreducibles,
- finite: there are finitely many irreducibles (up to $\cong$),
- q-finite: modules have q-characters ($\operatorname{tr} \mathsf{q}^{L_0 - \mathsf{c}/24}$).

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Log-rational means not completely reducible, but still finite and q-finite.
[But few accessible examples.]

Non-rational means completely reducible but not finite (and may be q-finite). [Usually notoriously difficult.]

Generically, we lose all three conditions. But here we have surprisingly many accessible (and important!) examples... this is log CFT.

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The logarithmic ones have proven crucial in our (presently limited) understanding of general logarithmic CFTs. And they are beautiful.[1]

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But good news! There's still an awful lot we don't yet understand...

As usual, we expect to make progress by working out the details of families of examples.

ie., this is an opportunity for 21st century mathematical physics.

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If the constraints are sufficiently strong, aim to understand the rep theory and build consistent CFTs (without additional physical input).

This goal is still a bit lofty at present! But models with these properties may be easier to analyse while exhibiting new features.

These fractional-level models are expected to act as stepping stones to a deeper understanding of physically interesting theories...

Fractional-level WZW models and W-algebras

Input: simple Lie algebra $\mathfrak{g}$, complex number $k \neq -h^\vee$.

Construction: induce the trivial $\mathfrak{g}$-module to a level-k $\widehat{\mathfrak{g}}$-module.

Result: the universal affine VOA $V^k(\mathfrak{g})$.

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Theorem [Gorelik–Kac'06]: $V^k(\mathfrak{g})$ is not simple iff

$$k + h^\vee = \frac{u}{v}, \quad u \in \mathbb{Z}_{\geqslant 2},\ v \in \mathbb{Z}_{\geqslant 1},\ \gcd\{u, v\} = 1.$$

The simple quotient VOA is denoted by $L_k(\mathfrak{g})$. This is the VOA of the (fractional-level) WZW model.

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Rep theory of $V^k(\mathfrak{g})$ is essentially unconstrained:

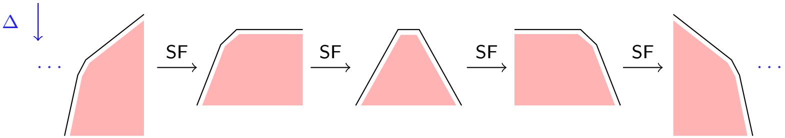
$$V^k(\mathfrak{g})\text{-module} \equiv \text{“smooth” level-}k\ \widehat{\mathfrak{g}}\text{-module.}$$

That of $L_k(\mathfrak{g})$ is much more interesting.

Weight modules

A weight module for $\widehat{\mathfrak{g}}$ is one on which the h_0, $h \in \mathfrak{h}$, act diagonalisably and L_0 acts with finite-rank Jordan blocks.

Every irreducible weight $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$-module is the "spectral flow" of a lower-bounded one. [Futorny–Tsylke'01, Adamović–Kawasetsu–DR'23]



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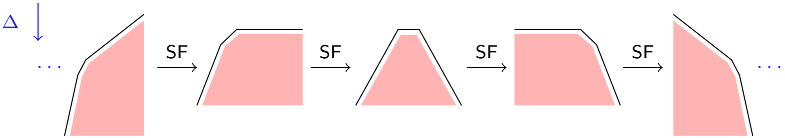
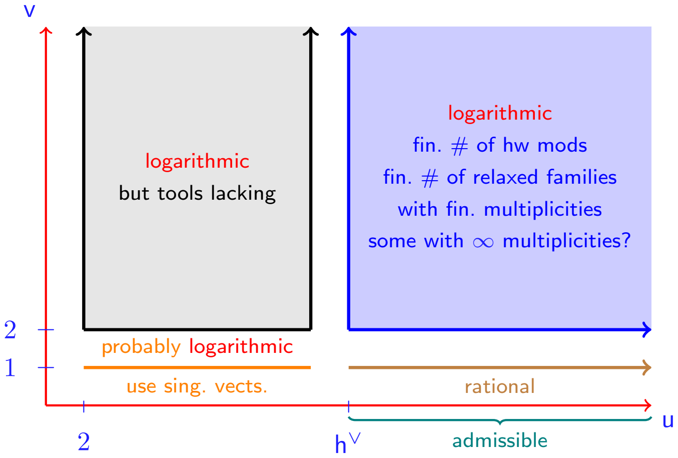


A lower-bounded irreducible is a relaxed highest-weight module [Feigin–Semikhatov–Tipunin'97, DR–Wood'15].
Relaxed means generated by a single weight vector of minimal Δ.

The weight category is modular *wrt.* generalised characters and closed under fusion. It's a good candidate for building consistent CFTs.

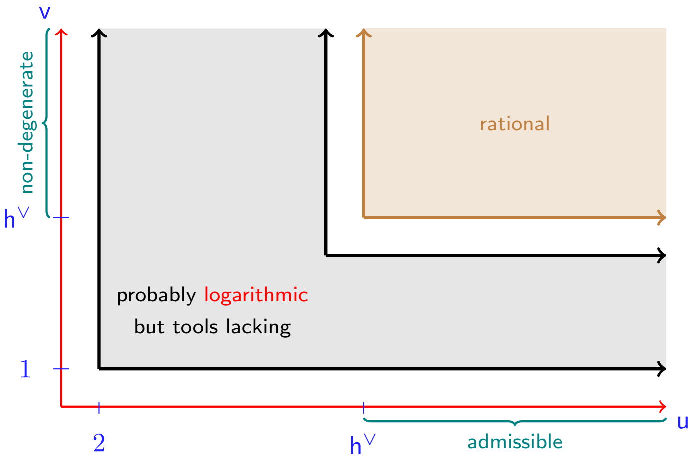
Weight modules for fractional-level WZW models

Given $k + \mathsf{h}^\vee = \frac{\mathsf{u}}{\mathsf{v}}$:

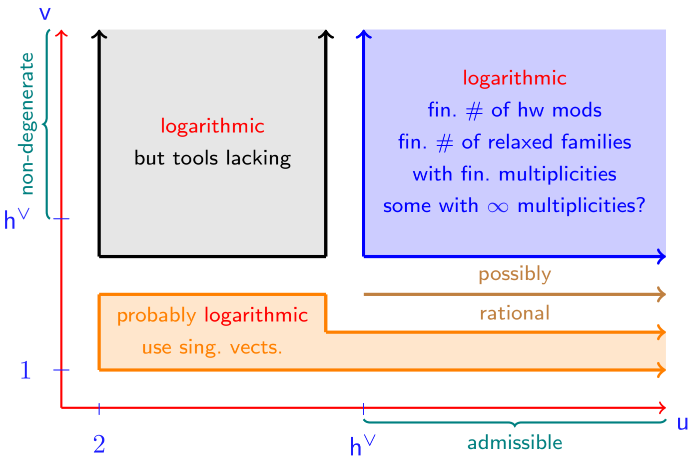


[I'll assume throughout that $\mathfrak{g}$ is simply laced for simplicity.]

Weight modules for principal W-algebras



Weight modules for other W-algebras



[This picture is mostly plausible speculation so don't hold me to it...]

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Using coherent families [Mathieu'00], this was lifted to an algorithmic classification of irreducible relaxed highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules with finite multiplicities in [Kawasetsu–DR'19].

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In general, there also exist irreducible weight modules with infinite multiplicities, *eg.* when $\mathfrak{g} = \mathfrak{sl}_n$, $n > 2$, and $\mathsf{v} > 2$, some of which admit generalised characters.

The theory of these modules is currently poorly developed...

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But, indecomposability occurs at finitely many parameter values in each family. The (finite-multiplicity) weight modules are almost always completely reducible *wrt.* a natural measure on parameter space.

cf. log-rational VOAs like the triplet algebra. Here, the parameter space is finite so the measure of the non-completely reducible cases is positive [Gaberdiel–Kausch'96].

Modules with parameters corresponding to complete reducibility are typical. The other modules are atypical.

Example: $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, $\mathsf{u},\mathsf{v} \geqslant 2$

Let $K_{\mathsf{u},\mathsf{v}} = \{1,\dots,\mathsf{u}-1\} \times \{1,\dots,\mathsf{v}-1\}$ and let $\mathbb{Z}_2$ be generated by $(r,s) \to (\mathsf{u}-r,\mathsf{v}-s)$. Up to spectral flow, there are:
[Adamović–Milas'95, DR–Wood'15, Kawasetsu–DR'19, Adamović–Kawasetsu–DR'23]

- Irreducible highest-weight modules $\mathcal{H}_{r,s}$, for $(r,s) \in K_{\mathsf{u},\mathsf{v}}$;
- Irreducible relaxed highest-weight modules $\mathcal{R}_{[\lambda];r,s}$, for $(r,s) \in K_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_2$ and $[\lambda] \in (\mathbb{C}/2\mathbb{Z}) - \{[\lambda_{r,s}],[\lambda_{\mathsf{u}-r,\mathsf{v}-s}]\}$;
- Reducible relaxed highest-weight modules $\mathcal{R}_{[\lambda_{r,s}];r,s}$ and $\mathcal{R}_{[\lambda_{\mathsf{u}-r,\mathsf{v}-s}];r,s}$, for $(r,s) \in K_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_2$.

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Moreover: [Creutzig–DR'13, Creutzig–Kanade–Liu–DR'18, Arakawa–Creutzig–Kawasetsu'23]

- The irreducible $\mathcal{R}_{[\lambda];r,s}$ are projective and injective (typical);
- The projective covers/injective hulls of the $\mathcal{H}_{r,s}$ are glueings of spectral flows of 2 reducible $\mathcal{R}_{[\lambda];r,s}$ (atypical).

The measure space is (roughly speaking) a countably infinite product of copies of $\mathbb{C}/2\mathbb{Z}$ with the product Haar measure.

Example: $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$

Up to spectral flow, there are: [Arakawa–Futorny–Ramirez'16, Kawasetsu–DR'19]

- Irreducible highest-weight modules $\mathcal{H}_0$ and $\mathcal{H}_{-\rho/2}$;
- Irreducible "semirelaxed" highest-weight modules $\mathcal{S}_{[\mu]}$, for $[\mu] \in (-\frac{3}{2}\Lambda_1 + \mathbb{C}\alpha_1)/\mathbb{Z}\alpha_1 - \{[-\frac{3}{2}\Lambda_1], [-\frac{1}{2}\rho]\}$;
- Reducible semirelaxed highest-weight modules $\mathcal{S}_{[-3\Lambda_1/2]}$ and $\mathcal{S}_{[-\rho/2]}$;
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Conjecture: [Creutzig–DR–Rupert'21]

- The irreducible $\mathcal{R}_{[\mu]}$ are projective and injective (typical);
- The projective covers/injective hulls of the irreducible $\mathcal{S}_{[\mu]}$, $\mathcal{H}_{-\rho/2}$ and $\mathcal{H}_0$ are explicitly known glueings of 2, 3 and 6 reducible $\mathcal{R}_{[\mu]}$ (atypical of degrees 1, 2 and 2), respectively.

The measure space is a product of countably many copies of $\mathfrak{h}^*/\mathsf{Q}$.

W-algebra weight modules

A few W-algebras may be constructed from other VOAs, *eg.* affine ones, as cosets (commutants) or ...

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In general, they are defined using quantum hamiltonian reduction.

This converts an affine VOA $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ into a W-algebra $\mathsf{W}^{\mathsf{k}}_f(\mathfrak{g})$, $f \in \mathfrak{g}^{\text{nil.}}$:

- Complete f to an $\mathfrak{sl}_2$-triple $\{f, h, e\}$.
- Tensor $\widehat{\mathfrak{g}}_k$ with pairs of bc-ghosts, one for each positive root, and pairs of $\beta\gamma$-ghosts, one for each root with $\alpha(h) = 1$.
- Construct a fermionic field with conformal weight 1 and (fermionic) ghost number 1:

$$d(z) = \sum_{\alpha>0} \big[e^{\alpha}(z) - \langle f|e^{\alpha}\rangle\big] c^{\alpha}(z) + \text{[terms in } b^{\alpha}, c^{\alpha}, \beta^{\alpha}, \gamma^{\alpha}\text{]}.$$

- d_0 is a differential and subspaces of $\mathsf{V}^{\mathsf{k}}(\mathfrak{g}) \otimes (bc)^{\#_1} \otimes (\beta\gamma)^{\#_2}$ define a differential complex on which the non-zero cohomology vanishes (?)
- The W-algebra $\mathsf{W}^{\mathsf{k}}_f(\mathfrak{g})$ is $H^{(0)}$. Its simple quotient is $\mathsf{W}^f_{\mathsf{k}}(\mathfrak{g})$.

Examples

- Taking $f = 0$ results in $\mathsf{W}^{\mathsf{k}}_f(\mathfrak{g}) = \mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ (reduction does nothing).
- Taking $f = \sum_{\alpha \text{ simple}} f^{\alpha}$ gives the regular W-algebra: $\mathsf{W}^{\mathsf{k}}_{\mathsf{reg.}}(\mathfrak{g})$.
- Taking $f = f^{\theta}$ gives the minimal W-algebra $\mathsf{W}^{\mathsf{k}}_{\mathsf{min.}}(\mathfrak{g})$.
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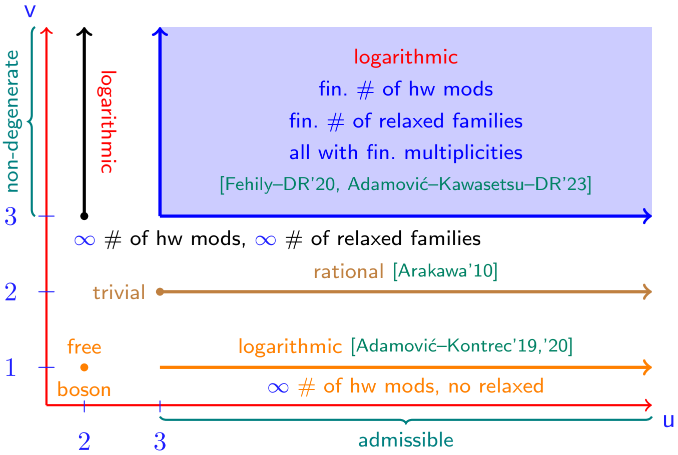
$\mathsf{W}^{\mathsf{k}}_{\text{reg.}}(\mathfrak{sl}_3)$	Zamolodchikov $\mathsf{W}^{\mathsf{k}}_3$
$\mathsf{W}^{\mathsf{k}}_{\text{reg.}}(\mathfrak{sl}_n)$	Casimir of type $(2, 3, 4, \dots, n)$
$\mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{sl}_3) = \mathsf{W}^{\mathsf{k}}_{\text{sub.}}(\mathfrak{sl}_3)$	Bershadsky–Polyakov $\mathsf{W}^{(2),\mathsf{k}}_3$
$\mathsf{W}^{\mathsf{k}}_{\text{reg.}}(\mathfrak{osp}_{1\vert 2}) = \mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{osp}_{1\vert 2})$	$N = 1$
$\mathsf{W}^{\mathsf{k}}_{\text{reg.}}(\mathfrak{sl}_{2\vert 1}) = \mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{sl}_{2\vert 1})$	$N = 2$
$\mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{osp}_{3\vert 2}) = \mathsf{W}^{\mathsf{k}}_{\text{sub.}}(\mathfrak{osp}_{3\vert 2})$	small $N = 3$
$\mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{psl}_{2\vert 2}) = \mathsf{W}^{\mathsf{k}}_{\text{sub.}}(\mathfrak{psl}_{2\vert 2})$	small $N = 4$
$\mathsf{W}^{\mathsf{k}}_{\text{min.}}(\mathfrak{d}_{2\vert 1;\alpha})$	big $N = 4$

Example: $\mathsf{W}^{\min.}_{\mathsf{k}}(\mathfrak{sl}_3)$ (Bershadsky–Polyakov)

For $\mathsf{k} + 3 = \frac{\mathsf{u}}{\mathsf{v}}$, the weight modules behave as follows.



Inverse quantum hamiltonian reduction

Take $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2) \xrightarrow{\text{QHR}} \mathsf{W}_\mathsf{k}^{\text{reg.}}(\mathfrak{sl}_2) \equiv \mathsf{Vir}_\mathsf{k}$ for fractional k:

$$\mathsf{k} + 2 = \frac{\mathsf{u}}{\mathsf{v}}, \quad \mathsf{u}, \mathsf{v} \in \mathbb{Z}_{\geqslant 2}, \ \gcd\{\mathsf{u}, \mathsf{v}\} = 1.$$

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What can we learn about their representations?

[q-finite highest-weight] ⟵ QHR ⟵ { [q-finite highest-weight], [highest-weight] }

[conjugate highest-weight]

[relaxed highest-weight]

[projective/injective]

[spectral flows]

[Whittaker]

Vir_k | $L_k(\mathfrak{sl}_2)$

Free-field realisations suggest a path:

- Feigin–Fuchs say $\mathsf{Vir}^{\mathsf{k}} \hookrightarrow \mathsf{H}$.
- Wakimoto says $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{H} \otimes \beta\gamma$.
- Bosonise the ghosts: $\beta\gamma \hookrightarrow \Pi$. [Friedan–Martinec–Shenker'86]
- Trade FF for FMS: $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{Vir}^{\mathsf{k}} \otimes \Pi$. [Semikhatov'94]
- Prove that $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{Vir}_{\mathsf{k}} \otimes \Pi$ iff $k \notin \mathbb{N}$. [Adamović'17]

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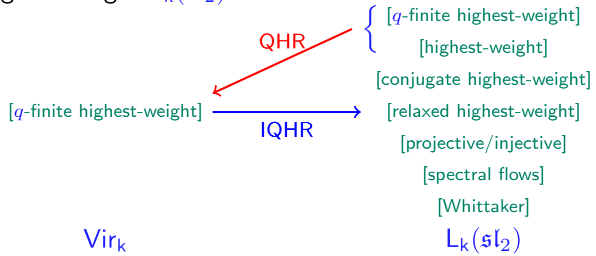
Thus, every $\mathsf{Vir}_{\mathsf{k}}$-module $\mathcal{M}$ and Π-module $\mathcal{N}$ yields an $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-module $\mathcal{M} \otimes \mathcal{N}$, by restriction (for $k \notin \mathbb{N}$).

$\mathsf{Vir}_{\mathsf{k}}$ only has q-finite highest-weight modules but weight Π-modules are always relaxed (up to spectral flows), so we always get spectral flows of relaxed highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules!



We call this tensoring and restricting an Adamović functor

$$\mathbb{A}^{\ell}_{[\lambda]} : \mathsf{Vir}_{\mathsf{k}}\text{-mod} \to \mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)\text{-mod}, \\ \mathcal{H} \mapsto \left(\mathcal{H} \otimes \Pi^{\ell}_{[\lambda]}\right)\downarrow, \quad \ell \in \mathbb{Z},\ [\lambda] \in \mathbb{C}/2\mathbb{Z}.$$

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Theorem [Adamović–Kawasetsu–DR'20]:

- If $\mathcal{H}$ is irreducible, then its image under $\mathbb{A}^{\ell}_{[\lambda]}$ is almost irreducible. [*cf.*, de Sole–Kac'05]

This means that there is an easy test for irreducibility.

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Theorem [Adamović–Kawasetsu–DR'23]:

- Every irreducible relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-module is the image of an irreducible $\mathsf{Vir}_{\mathsf{k}}$-module under some $\mathbb{A}^{\ell}_{[\lambda]}$.
- Every irreducible weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-module is a quotient of one in the image of some $\mathbb{A}^{\ell}_{[\lambda]}$.

Beyond $\mathfrak{sl}_2$

Adamović functors for other simple affine W-(super)algebras are known:

- $(\mathsf{N}=1)_\mathsf{k} \to \mathsf{L}_\mathsf{k}(\mathfrak{osp}_{1|2})$ for k admissible but non-integral.
  [Adamović'17, Kawasetsu–DR'18, Creutzig–Kanade–Liu–DR'19]
- $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2) \to \mathsf{L}_\mathsf{k}(\mathfrak{sl}_{2|1})$ and $(\mathsf{N}=2)_\mathsf{k} \to \mathsf{L}_\mathsf{k}(\mathfrak{sl}_{2|1})$ for $\mathsf{k}+1=\frac{\mathsf{u}}{\mathsf{v}}$ admissible with $\begin{cases} \mathsf{u}\neq 1 \\ \mathsf{v}\neq 1 \end{cases}$.
  [Creutzig–Fasquel–Genra–DR'24]
- $\mathsf{W}^{\text{reg.}}_\mathsf{k}(\mathfrak{sl}_3) \to \mathsf{W}^{\text{min.}}_\mathsf{k}(\mathfrak{sl}_3)$ iff $k+3=\frac{\mathsf{u}}{\mathsf{v}}$ with $\mathsf{v}\geqslant 3$.
  [Adamović–Kawasetsu–DR'20]
- $\mathsf{W}^{\text{min.}}_\mathsf{k}(\mathfrak{sl}_3) \to \mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$ iff $k+3=\frac{\mathsf{u}}{\mathsf{v}}$ with $\mathsf{v}\geqslant 2$. [Adamović–Creutzig–Genra'21]
- $\mathsf{W}^{\text{reg.}}_\mathsf{k}(\mathfrak{sp}_4) \to \mathsf{W}^{\text{sub.}}_\mathsf{k}(\mathfrak{sp}_4)$ iff $k+3=\frac{\mathsf{u}}{\mathsf{v}}$ with $\mathsf{v}\geqslant 3$. [Fasquel–Fehily–DR'24]
- $\mathsf{W}^{\text{reg.}}_\mathsf{k}(\mathfrak{sl}_n) \to \mathsf{W}^{\text{sub.}}_\mathsf{k}(\mathfrak{sl}_n)$ iff $k+n=\frac{\mathsf{u}}{\mathsf{v}}$ with $\mathsf{v}\geqslant n$. [Fehily'21]

There are also many universal examples being worked out, *eg.*
[Fehily'23, Fasquel–Nakatsuka'23, Creutzig–Fasquel–Linshaw–Nakatsuka'24, Fasquel–Fehily–Nakatsuka'24, ...].

There is clearly a lot still to do...

Conclusions

It seems that the right way to analyse W-algebra CFTs is:

- Start with the regular W-algebra at an admissible but non-degenerate level. These are rational with known representation theories!
- Use inverse reduction to construct the relaxed modules of the subregular W-algebra. Get the other irreducibles as quotients.
- Repeat, working your way up the lattice of nilpotents until the representation theory of the desired W-algebra is known!

Conclusions

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If the level is admissible but degenerate, don't despair: start instead with a rational exceptional W-algebra. [Arakawa–van Ekeren'19, McRae'21]

- When $\mathsf{v} = 1$, $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ is exceptional.
- For $\mathfrak{g} = \mathfrak{sl}_3$, $\mathsf{u} \geqslant 3$ and $\mathsf{v} = 2$, Bershadsky–Polyakov is exceptional.
- For $\mathfrak{g} = \mathfrak{sl}_n$, $\mathsf{u} \geqslant n$ and $\mathsf{v} = n - 1$, the subregular is exceptional.

[This needs generalising to the super case...]

Outlook

- Inverse quantum hamiltonian reduction lets us analyse logarithmic CFTs with W-algebra symmetries.
- It allows us to classify irreducible weight modules, compute modular transformations and (Grothendieck) fusion rules.
- These ideas are also relevant to the construction of projective and injective modules, needed for the CFT state space, (genuine) fusion rules, correlation functions, *etc.*.
- In principle, Adamović functors reduce the investigation of W-algebra rep theory to (admittedly very intricate) combinatorics.
- However, the big problem looming on the horizon is that we need to understand weight modules with infinite multiplicities. [Raymond–DR'24?]
- It is said that WZW models are the building blocks of rational CFT. If the same is true for admissible-level WZW models and log CFT, then we can expect these methods to generalise widely!
- Either way, the future of these CFTs is looking good...

"*Only those who attempt the absurd will achieve the impossible.*"

— Miguel de Unamuno