Reducible but indecomposable

[Representation theory for 21st century mathematical physics.]

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Symmetry and reducibility

Weyl and Wigner introduced representation theory into quantum physics. According to wikipedia:

> *"... the different quantum states of an elementary particle give rise to an irreducible representation of the Poincaré group."*

In other words, a quantum particle is modelled by an irreducible representation (= module) of some appropriate symmetry group/algebra.

The irreducibility here captures the fundamental indivisibility of an elementary particle while allowing intrinsic quantum states, *eg.* spin.

But what if we want our particle to have some internal structure, *eg.* a model that allows a one-way transition from an unstable to a stable state?

We'd then need a more refined version of irreducibility...

Recall that an irreducible module is one with exactly two submodules: the zero one 0 and the module itself.

Completely reducible modules then decompose as a direct sum of irreducible ones: $V = \bigoplus_i V_i$.

Often, all (interesting) modules are completely reducible.[1] *eg.*, finite groups, compact Lie groups, semisimple Lie algebras.

But, this is rare. Normally, one also has indecomposable modules: such a module is not a direct sum of two non-zero modules.

These might have a non-zero proper submodule $0 \subset W \subset V$ with no complement: $V \neq W \oplus W'$ for any submodule W'.

Such reducible but indecomposable modules are more complicated mathematically (sometimes much more so!), but may be necessary to construct a good model of a physical system.

[1] I'm going to assume throughout that my field is $\mathbb{C}$, because I'm being quantum.

Ex. 1: Lie superalgebras

Unlike simple Lie algebras, all simple Lie superalgebras except $\mathfrak{osp}(1|2n)$, $n \in \mathbb{Z}_{>0}$, have reducible but indecomposable finite-dimensional modules.

A (non-simple but typical) example is $\mathfrak{gl}(1|1) = \text{span}\{E, N \mid \psi^+, \psi^-\}$:

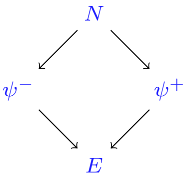
$$[N, \psi^{\pm}] = \pm\psi^{\pm}, \quad \{\psi^+, \psi^-\} = E.$$

With Cartan subalgebra $\text{span}\{E, N\}$, Verma modules are 2-dimensional:

A Verma module is always indecomposable. It is reducible (but indecomposable) if the E-eigenvalue is $e = 0$ and irreducible otherwise.

These Vermas coincide with the Kac modules of $\mathfrak{gl}(1|1)$. The irreducible ones are typical and the reducible but indecomposable ones are atypical.

The adjoint module of $\mathfrak{gl}(1|1)$ is also reducible but indecomposable:



This is not a Kac module. Instead, we may regard it as either:

- The projective cover of the atypical irreducible $\mathrm{span}\{N\}$.
- The injective hull of the atypical irreducible $\mathrm{span}\{E\}$.

We'll come back to what this means and why it may be physically relevant later.

Of course, Lie superalgebras are essential to much of modern mathematical physics, especially when supersymmetry is involved.

Ex. 2: Quantum groups

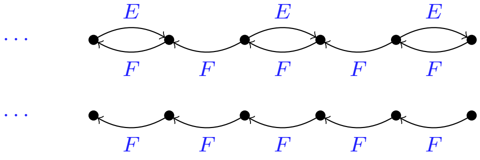
Drinfeld–Jimbo quantum groups are Hopf algebra deformations of the universal enveloping algebra of a simple Lie (super)algebra.

A neat example is $\mathsf{U}_q(\mathfrak{sl}_2) = \langle E, F, K \rangle$:

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Reducible but indecomposable modules arise when q is a root of unity.

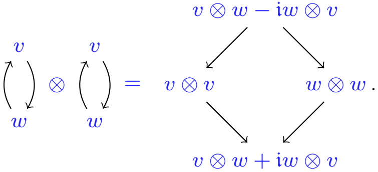
eg., $q = \mathfrak{i} \implies E^2$, F^2 and K^2 are central, so Vermas are always reducible but indecomposable:



(typical)

(atypical)

Even if we restrict attention to irreducible quotients, reducible but indecomposable modules arise again in tensor products, *eg.*



$(q = \mathrm{i})$

Interestingly, such tensor products are good news for physicists!

Quantum groups are well known to govern many examples of quantum integrable systems. The TT, TQ and QQ relations are reflections of reducible but indecomposable structure in tensor products of irreducibles.

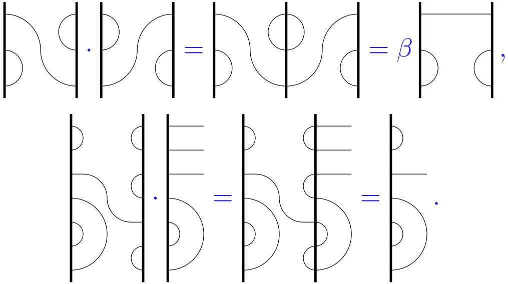
Ex. 3: Temperley–Lieb algebras

The Temperley–Lieb algebra is a ubiquitous ingredient for constructing 2D integrable lattice models. It's closely related to $\mathsf{U}_q(\mathfrak{sl}_2)$ and also arises in knot theory and subfactors.

It is the unital associative algebra $TL_n(\beta) = \langle u_1, \ldots, u_{n-1} \rangle$ satisfying

$$u_i^2 = \beta u_i, \qquad u_i u_{i\pm1} u_i = u_i, \qquad u_i u_j = u_j u_i \quad \text{if } |i-j| \geqslant 2.$$

$TL_n(\beta)$ also admits a faithful diagrammatic presentation which extends to its standard modules:



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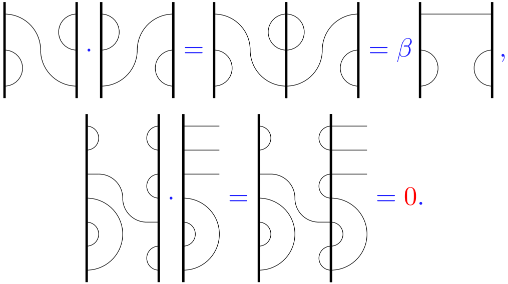
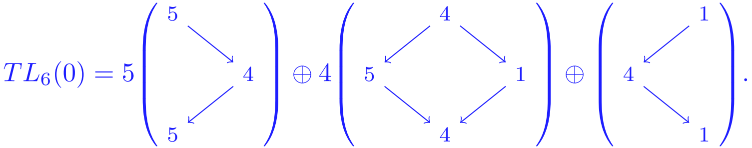


Some standard $TL_n(\beta)$-modules are reducible but indecomposable when $\beta = q + q^{-1}$ and q is a root of unity (and n is sufficiently large).

These standard modules have an irreducible maximal proper submodule, *ie.* they consist of two irreducibles glued together.

More complicated reducible but indecomposable $TL_n(\beta)$-modules may be obtained by decomposing the algebra itself into direct summands, *eg.*



These algebras underlie Q-state Potts and RSOS models. In particular, $\beta = 1$ corresponds to percolation and $\beta = \sqrt{2}$ to the Ising model.

Reducible but indecomposable modules are essential for a proper understanding of non-local observables in such models, *eg.* crossing probabilities and fractal dimensions.

Ex. 4: Virasoro algebras

The Virasoro algebra controls 2D conformal field theories.

It is the Lie algebra $\mathfrak{Vir} = \langle L_n, C \mid n \in \mathbb{Z} \rangle$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C, \quad [L_m, C] = 0.$$

Here, the eigenvalue of C on a module is its central charge c.

Irreducible highest-weight modules $\mathcal{L}_h$ of $\mathfrak{Vir}$ are labelled by a conformal dimension h, the L_0-eigenvalue of the highest-weight vector.

These modules are the building blocks of the rational CFTs called Virasoro minimal models. *eg.*, the local observables of the Ising model[2] are modelled by the $c = \frac{1}{2}$ minimal model $M(3,4)$ with state space

$$\mathcal{H} = (\mathcal{L}_0 \otimes \mathcal{L}_0) \oplus (\mathcal{L}_{1/16} \otimes \mathcal{L}_{1/16}) \oplus (\mathcal{L}_{1/2} \otimes \mathcal{L}_{1/2}).$$

[2] At the critical point and in an appropriate scaling limit.

Non-local observables require an as yet poorly understood CFT called a logarithmic minimal model. Such models are built from Kac modules for $\mathfrak{Vir}$. These are (usually) reducible but indecomposable.

There is a tensor product $\times$ for modules in a CFT called fusion. In logarithmic CFT, this is one reliable way to produce reducible but indecomposable modules from irreducibles, *eg.*

$$\mathcal{L}_{1/3} \times \mathcal{L}_{1/3} \cong \mathcal{L}_{1/3} \oplus \left(\begin{array}{ccccc} & & \mathcal{L}_2 & & \\ & \swarrow & & \searrow & \\ \mathcal{L}_0 & & & & \mathcal{L}_5 \\ & \searrow & & \swarrow & \\ & & \mathcal{L}_2 & & \end{array} \right). \qquad (c = 0)$$

This indecomposable arises in the scaling limit of critical percolation and is a part of the structure that underlies Cardy's celebrated formula for the horizontal crossing probability.

Ex. 5: Affine Kac–Moody algebras

Wess–Zumino–Witten models describe strings propagating on Lie groups.

The underlying CFTs have affine Kac–Moody algebras $\widehat{\mathfrak{g}}$ for symmetries:

$$[J^a_m, J^b_n] = [J^a, J^b]_{m+n} + m\kappa(J^a, J^b)\delta_{m+n,0}K, \quad [J^a_m, K] = 0.$$

For compact simple Lie groups, consistency forces the K-eigenvalue k (the level) to be a non-negative integer.

However, mathematical considerations (coset constructions, orbifolds, quantum hamiltonian reduction) and physical constructions (4d/2d duality) require the existence of similar CFTs with fractional levels.

Kac and Wakimoto famously asserted that there are fractional levels at which there is a finite set of irreducible highest-weight $\widehat{\mathfrak{g}}$-modules whose characters span a representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$!

But the CFT isn't rational: just as famously, the celebrated Verlinde formula for fusion coefficients does not work at these levels...

The problem here was that Kac and Wakimoto forgot to consider the convergence regions of the highest-weight characters.

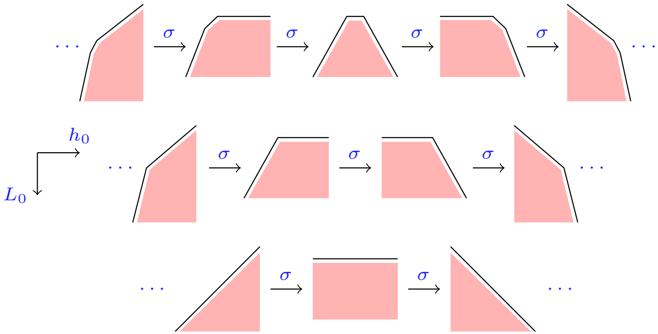
To get truly modular characters, one has to generalise to relaxed highest-weight modules and their spectral flow twists.

The resulting CFT is logarithmic and its spectrum is unbounded below. Nevertheless, it is mathematically consistent.

For $\widehat{\mathfrak{sl}}_2$, this is very well understood:

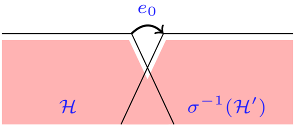
- The relaxed highest-weight modules are generically irreducible (the typical case) but are reducible but indecomposable for a finite set of parameters (the atypical case).
- The atypical relaxed modules consist of a highest-weight module glued to the spectral flow of another.
- Together, the characters of the typical and atypical relaxed modules carry a representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$.
- The Verlinde formula works!

We illustrate the action of spectral flow on highest-weight (top) and relaxed highest-weight (bottom) $\widehat{\mathfrak{sl}}_2$-modules:

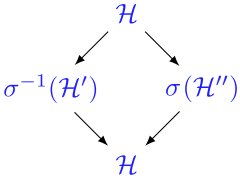


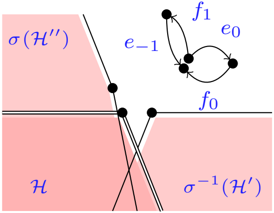
Notice that all but 1, 2 or 3 modules in each orbit have spectra that are unbounded below.

We picture (some of) the atypical relaxed $\widehat{\mathfrak{sl}}_2$-modules as follows:



There also exist more complicated reducible but indecomposable modules, *eg.*



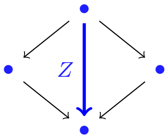


Of course, visualisations like this are very helpful for $\widehat{\mathfrak{sl}}_2$, but aren't of much use in higher ranks...

Diamonds in the rough

Many of the reducible but indecomposable modules that we've exhibited have a diamond "shape" (though sometimes truncated) reflecting the gluing of 4 irreducibles.

In each case, the algebra contains an element Z that is central and self-adjoint, even though its action is not diagonalisable:



Z is the quadratic Casimir for $\mathfrak{gl}(1|1)$ and $\mathsf{U}_q(\mathfrak{sl}_2)$, while it is the "braid transfer matrix" for $TL_n(\beta)$ and $\cos(2\pi L_0)$ for $\mathfrak{Vir}$ and $\widehat{\mathfrak{sl}}_2$.

In each case, Z has Jordan blocks of rank at most 2.

[This is evidently the simplest non-trivial possibility and therefore the nicest...]

In the examples discussed, these "diamond modules" have further nice mathematical properties. They are:

- Projective — they do not appear as a quotient of a strictly larger reducible but indecomposable module.
- Injective — they do not appear as an submodule of a strictly larger reducible but indecomposable module.

[This requires a (sometimes not obvious) choice for the physically relevant module category.]

These diamonds are thus "maximal" reducible but indecomposable modules. [However, one can glue more than four irreducibles together indecomposably.]

Moreover, this identification allows one to start applying the machinery of homological algebra to compute extension groups ($\sim$ allowed glueings). These have some physical relevance, *eg.* to logarithmic couplings and the algebra of functions on (4d $N = 2$ SUSY) Coulomb branches.

It also generalises the situation for completely reducible modules, for which every irreducible is both projective and injective.

These examples also have the property that the atypical irreducibles are in bijection with the projective and injective "diamond modules".

Define a matrix C, the Cartan matrix, whose (i, j) entry is the number of times the i-th irreducible appears in the j-th diamond.

Then, C factorises as $DD^{\intercal}$, where D is called the decomposition matrix.

D counts the number of times that the i-th irreducible appears in a j-th intermediate, called a standard module. In our examples:

- For $\mathfrak{gl}(1|1)$, the standard modules are the Kac (Verma) modules [Kac].
- For $\mathsf{U}_{\mathfrak{i}}(\mathfrak{sl}_2)$, they are the 2-dimensional highest-weight modules.
- For $TL_n(\beta)$, they are the "half-diagram" (cell) modules [Graham–Lehrer].
- For $\mathfrak{Vir}$, they are (probably) the Kac modules [Pearce–Rasmussen–Zuber].
- For $\widehat{\mathfrak{sl}}_2$, they are the spectral flows of the relaxed highest-weight modules [Creutzig–DR].

Along with another similar property (involving the duals of the standard modules), this makes the "diamond modules" into tilting modules.

The factorisation $C = DD^{\mathsf{T}}$ is then referred to as BGG reciprocity (*cf.* irreducibles, Vermas and projectives in the category $\mathscr{O}$ of modules of a semisimple Lie algebra [Bernšteĭn–Gel'fand–Gel'fand]).

The corresponding module categories are then highest-weight categories in the sense of [Cline–Parshall–Scott] and BGG categories in the sense of [Irving].

These features are of course extremely rare from a mathematical sense. It is therefore quite interesting to see them arising in physically relevant representation theories. Why is it so!?

Even if these are "relatively simple" cases, observations like these may provide a guide to the mathematical structures that will be uncovered in more complicated models.

Outlook

Reducible but indecomposable modules are notoriously difficult to classify in general (*cf.* tame vs wild). However, classifying the physically relevant indecomposables may be easier.

For example, for "bulk" CFTs, we only need the detailed structures of the projective modules to understand the quantum state space.

However, knowledge of many other types of reducible but indecomposable modules may be required to understand boundary sectors and defects.

The "diamond modules" that we've exhibited here are just the tip of the iceberg. More complicated physical models often lead to much more complicated indecomposables. [Just wait...]

Interest in this has been steadily growing, even though the language needed is not part of the traditional education of math physicists.

One last observation worth mentioning is that many of the examples discussed today have another common feature: modularity.

More precisely, when the modules are completely reducible, the corresponding categories for quantum groups, subfactors and CFTs all admit an action of $\mathsf{SL}(2;\mathbb{Z})$ that's consistent with the Verlinde formula.

This is the setting of modular tensor categories [Moore–Seiberg, Turaev].

There is currently a lot of interest in generalising this to categories containing reducible but indecomposable modules, largely because of proposed generalised Verlinde formulae for logarithmic CFTs. [Creutzig–DR]

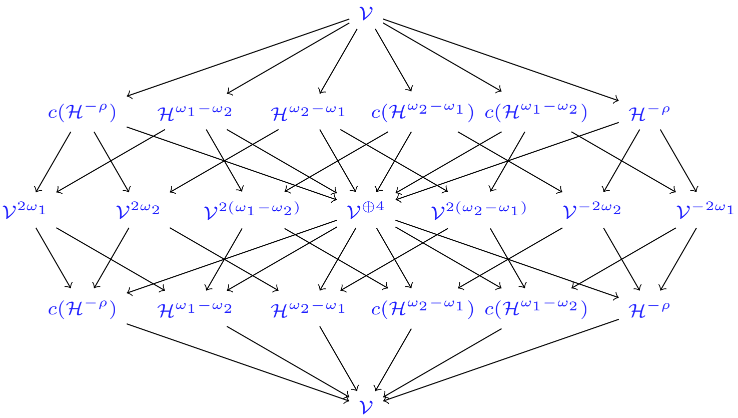
Mathematicians have dubbed (some of) these "relative modular categories", tailoring their definitions to 3-manifold invariants and TQFT applications. [Constantino–Geer–Patureau-Mirand, De Renzi]

Time will tell if these are good enough for the modularity of logCFTs...

"*Only one who attempts the absurd is capable of achieving the impossible.*"

— Miguel de Unamuno

Finally, as an example of going beyond diamonds, here is a recent conjecture for the structure of the projective cover of the vacuum module in the logCFT corresponding to $\widehat{\mathfrak{sl}}_3$ at level $k = -\frac{3}{2}$.



[arXiv:2112.13167 [math.RT]]