

# ELLIPTIC DEFORMATIONS

OF  $AdS_3 \times S^3$  S-MATRIX

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MATRIX ~~X~~ 2024

The success of integrability in  $AdS_5/CFT_4$  made people ask questions about lower dimensional AdS backgrounds

$AdS_4/CFT_3$

$AdS_3/CFT_2$

$AdS_2/CFT_1$

Less and less  
is known

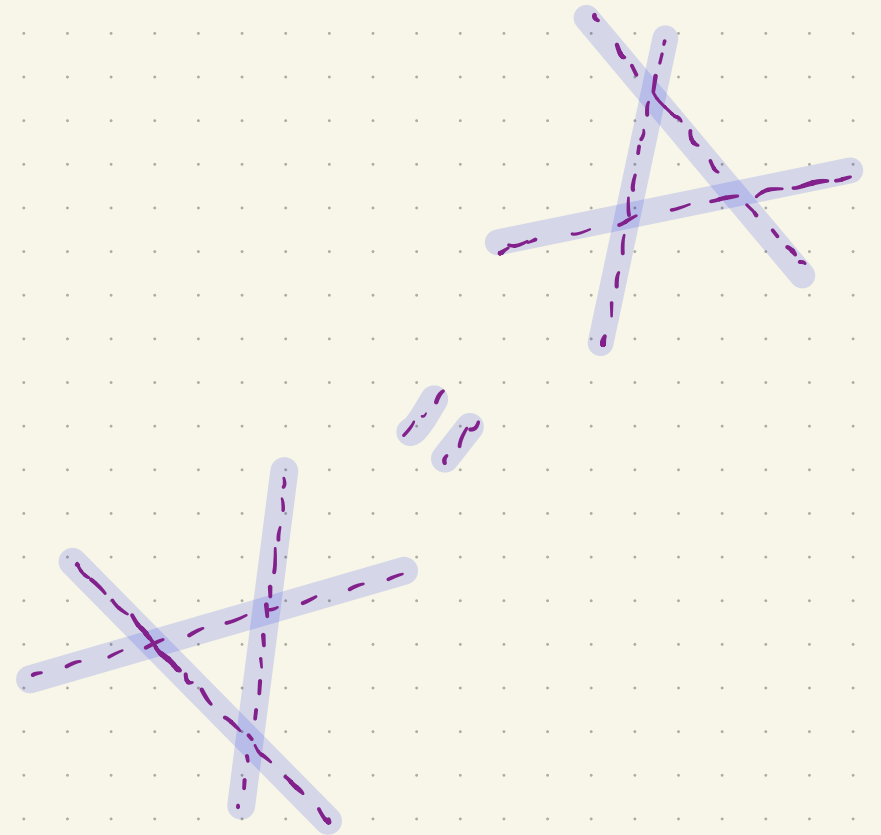
But integrability is present!

$$AdS_3 \times S^3 \times T^4$$

worldsheet S-matrix

is

INTEGRABLE



Cagnazzo, Zarembo, Babichenko, Stefanski, Lloyd, Sax, Sfondrini, Baggio, ...

The Green-Schwarz action is given by

$$S = -\frac{T}{2} \int d\tau d\sigma \left[ \delta^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu} \right] \partial_\alpha \gamma^\mu \partial_\beta \gamma^\nu$$

+ fermions

$$\alpha, \beta = 1, 2$$

$$\mu, \nu = 0, \dots, 9$$

$$\underbrace{\text{AdS}_3}_{0,1,2} \times \underbrace{S^3}_{3,4,5} \times \underbrace{T^4}_{6,7,8,9}$$

$G_{\mu\nu} \rightarrow$  metric

$B_{\mu\nu} \rightarrow$  anti-symmetric B-field

- We can only quantise this theory perturbatively
- Theory has a lot of redundancies... so the first step is fixing the gauge
- $\mathcal{L}^{\text{g.f.}} = \mathcal{L}_2 + \frac{1}{\sqrt{T}} \mathcal{L}_3 + \frac{1}{T} \mathcal{L}_4 + \dots$
- Compute the tree-level S-matrix
- Investigate hidden symmetries
- Assume that they hold at quantum level and use them to bootstrap the quantum S-matrix

$$\rightarrow S = 1 + \frac{i}{T} \mathcal{T} + \dots$$

↑  
tree-level S-matrix

It satisfies the classical YBE

$$[\mathcal{T}_{12}, \mathcal{T}_{13}] + [\mathcal{T}_{12}, \mathcal{T}_{23}] + [\mathcal{T}_{13}, \mathcal{T}_{23}] = 0$$

→ Bootstrap the quantum S-matrix using off-shell symmetries...

→ Check quantum integrability

Worksheet  
S-matrix  $S$

satisfies the

QUANTUM  
YANG-BAXTER  
EQUATION

but more than that

↙

$$\check{S} = \check{S} \otimes \check{S}$$

where

$$\check{S} = \left( \begin{array}{c|c} \check{S}^{LL} & \check{S}^{LR} \\ \hline \check{S}^{RL} & \check{S}^{RR} \end{array} \right)$$

FACTORIZATION

BLOCK STRUCTURE

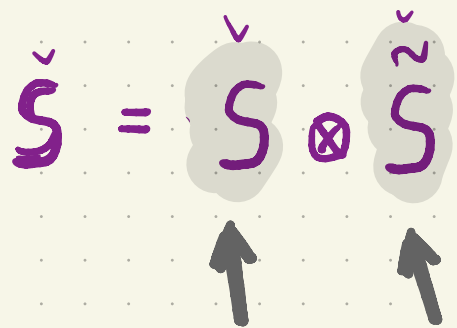
only schematically. In reality each of this blocks is  $8 \times 8$  and the complete (+) is  $16 \times 16$ .

Worldsheet  
S-matrix  $S$

satisfies the

QUANTUM  
YANG-BAXTER  
EQUATION

but more than that

$$\check{S} = \check{S} \otimes \check{S}$$


FACTORIZATION

where

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BLOCK STRUCTURE



Worldsheet  
S-matrix  $S$

satisfies the

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$$\check{S} = \check{S} \otimes \check{S}$$

where

$$\check{S} = \begin{pmatrix} \check{S}^{LL} & \check{S}^{LR} \\ \check{S}^{RL} & \check{S}^{RR} \end{pmatrix}$$

FACTORIZATION

BLOCK STRUCTURE

$$\check{S} = \left( \begin{array}{c|c} \check{S}^{LL} & \check{S}^{LR} \\ \hline \check{S}^{RL} & \check{S}^{RR} \end{array} \right)$$

$$V^L \otimes V^L \otimes V^R \mapsto V^L \otimes V^L \otimes V^R$$

$$\mathcal{S}_{12}^{LL}(u_1, u_2) \mathcal{S}_{13}^{LR}(u_1, u_3) \mathcal{S}_{23}^{LR}(u_2, u_3)$$

$$= \mathcal{S}_{23}^{LR}(u_2, u_3) \mathcal{S}_{13}^{LR}(u_1, u_3) \mathcal{S}_{12}^{LL}(u_1, u_2)$$

In particular,

$\mathcal{S}^{LL}$  and  $\mathcal{S}^{RR}$  are  
6-vertex models

$$\begin{pmatrix} \mathfrak{D}_{11} & 0 & 0 & 0 \\ 0 & \mathfrak{D}_{22} & \mathfrak{D}_{23} & 0 \\ 0 & \mathfrak{D}_{32} & \mathfrak{D}_{33} & 0 \\ 0 & 0 & 0 & \mathfrak{D}_{44} \end{pmatrix}$$

$$\mathfrak{D}_{ij} \equiv \mathfrak{D}_{ij}(p_i, p_j) \neq \mathfrak{D}_{ij}(p_i - p_j)$$

"Similar" to XXX R-matrix

But it is of Free-Fermion type:

$$\mathfrak{D}_{11} \mathfrak{D}_{44} + \mathfrak{D}_{22} \mathfrak{D}_{33} = \mathfrak{D}_{21} \mathfrak{D}_{32} + \mathfrak{D}_{14} \mathfrak{D}_{41}$$

This model has a lot of symmetry and as a consequence

$$\downarrow$$
$$[su(1|1)_L \oplus su(1|1)_R]_{ce}^2$$

→ FACTORISATION

→ BLOCK STRUCTURE

→ INTEGRABILITY

This model has a lot of symmetry and as a consequence

$$\downarrow \\ [su(1|1)_L \oplus su(1|1)_R]_{ce}^2$$

→ FACTORISATION

A lot is known

→ BLOCK STRUCTURE

about the TBA for

→ INTEGRABILITY

example (Dei, Sfondrini, Baggio,  
Frolov, Polvora, Torricelli,  
Borsato, Sax, Stefański)

For more see

## Exact approaches on the string worldsheet

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arXiv:2312.12930v2

There are also deformations ... (Hoare, Seibold, Tseytlin, Sfondrini, Tongeren, Vicedo, Delduc, Kameyama, Lacroix, Magro, ...)

For example, a  $q$ -deformation (trigonometric)

$\mathcal{S}_{q\text{-def.}}^{LL}$  is again of form

$$\begin{pmatrix} \mathfrak{D}_{11} & 0 & 0 & 0 \\ 0 & \mathfrak{D}_{22} & \mathfrak{D}_{23} & 0 \\ 0 & \mathfrak{D}_{32} & \mathfrak{D}_{33} & 0 \\ 0 & 0 & 0 & \mathfrak{D}_{44} \end{pmatrix}$$

For  $q \rightarrow 1$ ,  $\Rightarrow$  undeformed  $\mathcal{S}^{LL}$

"similar" to  $XXZ$

but free-fermion

WHAT ABOUT AN ELLIPTIC

DEFORMATION OF

$$AdS_3 \times S^3 \times T^4 ?$$



From the point of view of INTEGRABILITY,

can we find an 8-vertex models such that

$$\check{S} = \left( \begin{array}{c|c} \check{S}^{LL} & \check{S}^{LR} \\ \hline \check{S}^{RL} & \check{S}^{RR} \end{array} \right)$$

under some limit

$S_{\text{ELLIPTIC}}$



$S_{\text{TRIG.}}$  ?

It depends!

$S_{\text{UNDEF.}}$  ?

YES!

BUT DOES SUCH AN S-MATRIX

COME FROM A STRING

SIGMA MODEL ?

# PLAN

arXiv: 2003.04332

arXiv: 2312.14031

① Method to construct integrable models.

(de Leeuw, Palleta, Pribytok, ALR, Ryan, 2020)

② Integrable and quasi-integrable elliptic deformation.

③ Do they come from a string  $\sigma$ -model?

(Hoare, ALR, Seibold, 2023)

Solving the Yang-Baxter equation is quite difficult

$$R_{12}(u_1, u_2) R_{13}(u_1, u_3) R_{23}(u_2, u_3) = R_{23}(u_2, u_3) R_{13}(u_1, u_3) R_{12}(u_1, u_2)$$

Many people contributed to this problem (Baxter, Jimbo, Perk, Yang, Kuniba, Bazhanov, Batchelor, Martins, Pimenta, Vieira, Links, Doikou, Fateev, Serban, Beisert, Hoare, Torrielli, Zamolodchikov, Faddeev, Izergin, Korepin, Alcaraz, Kulish, Sklyanin, ...)

Most of them were found using strategies that make use of

SYMMETRIES

and/or are of difference form

$$R_{12}(u, v) = R_{12}(u - v)$$

But the few known non-difference form models are very interesting!

Can we find more?

Can we have a systematic way to find non-difference form models ( $R(u,v) \neq R(u-v)$ ) without knowing in advance its algebraic structure?

$$H = \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & a_{24} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & 0 & 0 & a_{37} & 0 & 0 \\ 0 & a_{42} & 0 & a_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & 0 & a_{68} & 0 \\ 0 & 0 & a_{73} & 0 & 0 & 0 & a_{77} & 0 & a_{79} \\ 0 & 0 & 0 & 0 & 0 & a_{86} & 0 & a_{88} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{97} & 0 & a_{99} \end{bmatrix}$$

For which values of  $a_{ij}$  is this integrable?

And what is the corresponding  $R$ ?

de Leeuw, Pallata, Pribytok, ALR, Ryan, 2020

We can do that for

$$R(u, u) \sim P$$
$$\left. \frac{dR(u, v)}{du} \right|_{v=u} \sim PH(u)$$

The key ingredient is the boost operator (Links, Zhou, McKenzie, Gold, 2001)

$$B[\mathcal{Q}_2] = - \sum_n n H_{n, n+1}(u) + \frac{\partial}{\partial u}$$

It is very useful because

$$Q_{r+1} = [B[\theta_2], Q_r] \quad r > 1$$

↓  
Boost operator

$$Q_3 = [B[\theta_2], Q_2]$$

$$= \sum_{i=1}^L [H_{i-1,i}(u), H_{i,i+1}(u)] + \frac{d}{du} H(u)$$



It is very useful because

$$Q_{r+1} = [B[\theta_2], Q_r]$$

↓  
Boost operator

For  $R(u, r) = R(u-r)$

$$R'(0) = PH$$

$$Q_3 = [B[\theta_2], Q_2]$$

$$= \sum_{i=1}^L [H_{i-1, i}(u), H_{i, i+1}(u)] + \cancel{\frac{d}{du} H(u)}$$

(Tetel'man, 1982)

STEP 1: Propose an ansatz for  $H$ :

Example: 
$$H(u) = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}, \quad a_{ij} \equiv a_{ij}(u)$$

STEP 2: Construct  $Q_3$  using the boost operator

$$Q_3 = - \sum_{j=1}^L [H_{j-1,j}, H_{j,j+1}] + \frac{dH}{du}$$

STEP 3: Require  $[Q_2, Q_3] = 0$

and carefully solve all ODEs

STEP 4: Plug each ansatz on the Sutherland equation

$$\frac{\partial}{\partial u} YBE(u, v, w) \Big|_{v \rightarrow u} = 0$$

$$[R_{13} R_{23}, H_{12}(u)] = \dot{R}_{13} R_{23} - R_{13} \dot{R}_{23}$$

and solve for R using the boundary conditions

$$R(u, u) \sim P \quad \dot{R}(u, v) \Big|_{v=u} \propto PH(u)$$

For each you will find an R-matrix

It seems that  $[Q_2, Q_3] = 0$  is enough to fix  $H$

But transformations like

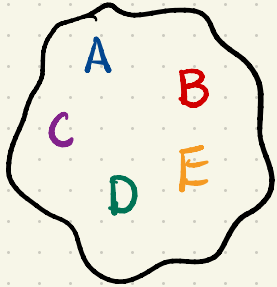
- $R(u, v) \rightarrow R(f(u), f(v))$  — reparametrisation
- $R(u, v) \rightarrow f(u, v) R(u, v)$  — normalisation
- $R(u, v) \rightarrow (V \otimes V) R(u, v) (V \otimes V)^{-1}$  — basis transformation
- $R(u, v) \rightarrow P R(u, v) P$
- $R(u, v) \rightarrow R(u, v)^T$  — transposition
- $R(u, v) \rightarrow (U(u) \otimes 1) R(u, v) (1 \otimes U(v)^{-1})$  — Twist

or

Preserve YBE

# Disadvantages:

Expectation:



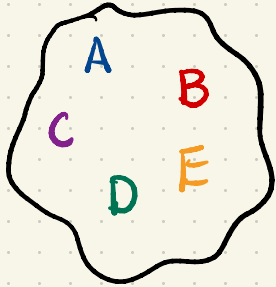
Some interesting  
solutions

## Disadvantages:

Normalisation  
Reparametrisation

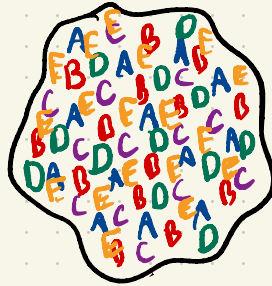
Basis transformations  
Discrete transformations  
Twists

Expectation:



Some interesting  
solutions

Reality:



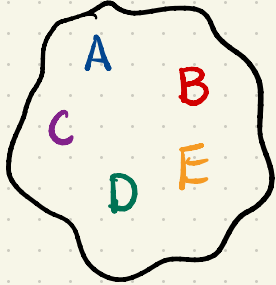
Lots of dependent  
solutions

Disadvantages:

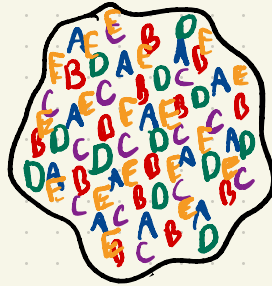
Normalisation  
Reparametrisation

Basis transformations  
Discrete transformations  
Twists

Expectation:



Reality:

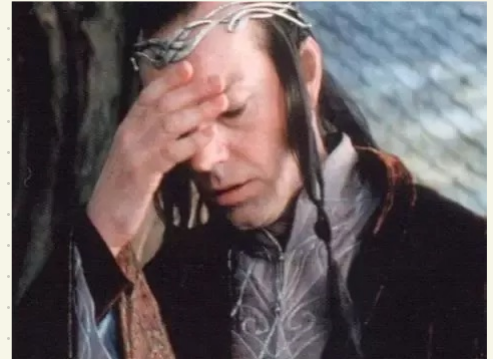


reasonable  
amount of  
work  
⇒



Some interesting  
solutions

Lots of dependent  
solutions



This disadvantage can be turned into an advantage

We can use this freedom to

- Put one diagonal element of the Hamiltonian to zero ;
- And one element to one ;

This makes the system much easier to solve



ANSATZ  
FOR  $H$



USE BOOST TO  
CONSTRUCT  $Q_3$



$[Q_2, Q_3] = 0$   
Solve system of  
ODEs



Main advantage:

quite efficient  
for non-difference  
form models

Check  
YBE



Solve Sutherland  
for each  $H$

# We applied the method to

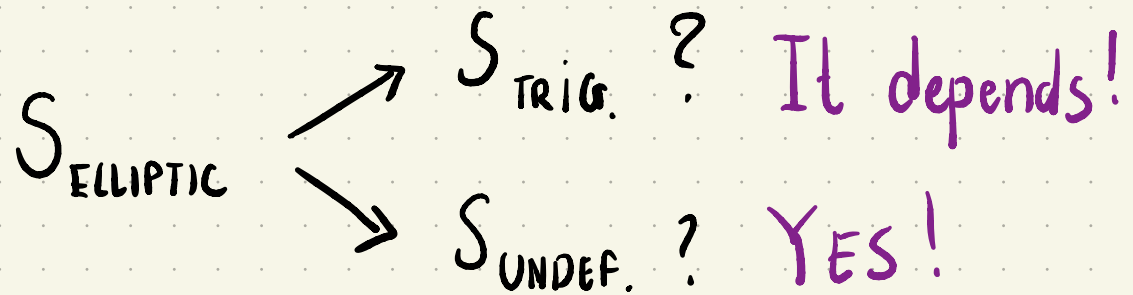
- $su(2) \oplus su(2)$  diff. and non-diff
  - 15-vertex model non-diff.
  - Full  $4 \times 4$  non-difference form
  - Flag models
  - Systematic construction of integrable Lindblad systems
  - $AdS_3$  and  $AdS_2$  integrable deformations
- de Leeuw, Paletta  
Pribytok, ALR, Ryan, 2020, 2021
- Corcoran, de Leeuw, 2023
- de Leeuw, Nepomechie, ALR, 2022
- de Leeuw, Paletta, Posguy, 2021
- de Leeuw, Paletta  
Pribytok, ALR, Ryan, 2020

②

From the point of view of **INTEGRABILITY**,  
can we find an 8-vertex models such that

$$\check{S} = \left( \begin{array}{c|c} \check{S}^{LL} & \check{S}^{LR} \\ \hline \check{S}^{RL} & \check{S}^{RR} \end{array} \right)$$

under some limit



Let us go back to the elliptic models

$$\int^{\text{LL}}(u, v) \sim P$$

$$r_1^{\text{LL}} = \frac{1}{\sqrt{\sin(\eta(u)) \sin(\eta(v))}} \left( -\cos \eta_+ \text{sn}_-^{\text{LL}} + \frac{\text{cn}_-^{\text{LL}}}{\text{dn}_-^{\text{LL}}} \sin \eta_+ \right),$$

$$r_2^{\text{LL}} = -\frac{1}{\sqrt{\sin(\eta(u)) \sin(\eta(v))}} \left( \cos \eta_- \text{sn}_-^{\text{LL}} - \frac{\text{cn}_-^{\text{LL}}}{\text{dn}_-^{\text{LL}}} \sin \eta_- \right),$$

$$r_3^{\text{LL}} = -\frac{1}{\sqrt{\sin(\eta(u)) \sin(\eta(v))}} \left( \cos \eta_- \text{sn}_-^{\text{LL}} - \frac{\text{cn}_-^{\text{LL}}}{\text{dn}_-^{\text{LL}}} \sin \eta_- \right),$$

$$r_4^{\text{LL}} = \frac{1}{\sqrt{\sin(\eta(u)) \sin(\eta(v))}} \left( \cos \eta_+ \text{sn}_-^{\text{LL}} + \frac{\text{cn}_-^{\text{LL}}}{\text{dn}_-^{\text{LL}}} \sin \eta_+ \right),$$

$$r_5^{\text{LL}} = \sqrt{\frac{g_L(v)}{g_L(u)}}, \quad r_6^{\text{LL}} = \sqrt{\frac{g_L(u)}{g_L(v)}},$$

$$r_7^{\text{LL}} = \frac{k\alpha}{\sqrt{g_L(u)g_L(v)}} \frac{\text{cn}_-^{\text{LL}} \text{sn}_-^{\text{LL}}}{\text{dn}_-^{\text{LL}}}, \quad r_8^{\text{LL}} = \frac{k\sqrt{g_L(u)g_L(v)}}{\alpha} \frac{\text{cn}_-^{\text{LL}} \text{sn}_-^{\text{LL}}}{\text{dn}_-^{\text{LL}}}.$$

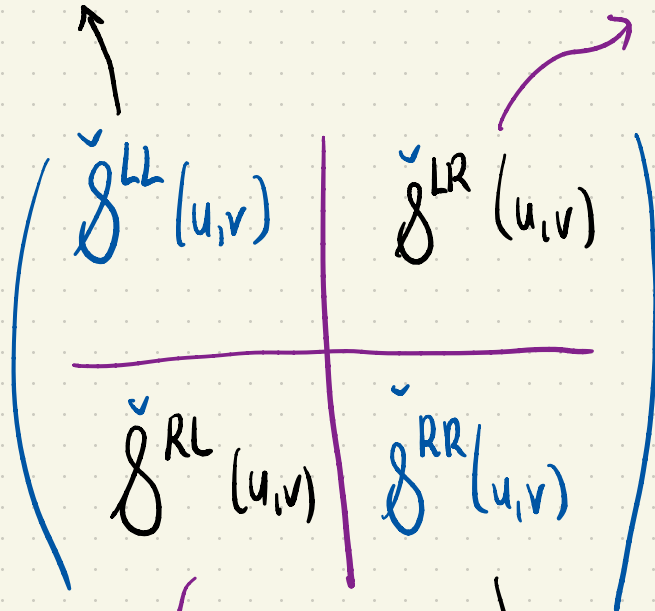
$$S^{\text{LL}} = \begin{pmatrix} r_1 & & & r_8 \\ & r_2 & r_6 & \\ & r_5 & r_3 & \\ r_7 & & & r_4 \end{pmatrix}$$

$$\eta_{\pm} = \frac{\eta(u) \pm \eta(v)}{2}$$

$$\text{cn}_-^{\text{LL}} = \text{JacobiCN}(F^{\text{L}}(u) - F^{\text{L}}(v), k_L^2)$$

$$\checkmark \checkmark \delta^{LL}(u,u) \sim P$$

$$\delta^{LR}(u,u) \neq P$$



$$\delta^{RL}(u,u) \neq P$$

$$\delta^{RR}(u,u) \sim P$$

$$k_R = k_L \equiv k$$

$$r_1^{\text{LR}} = \frac{1}{\sqrt{\sin(\eta(u)) \sin(\eta(v))}} \sqrt{\frac{g_R(v)}{g_L(u)}} \left( \cos \eta_- \text{sn}_+^{\text{LR}} + \frac{\text{cn}_+^{\text{LR}}}{\text{dn}_+^{\text{LR}}} \sin \eta_- \right),$$

$$r_2^{\text{LR}} = \frac{1}{\sqrt{\sin(\eta(u)) \sin(\eta(v))}} \sqrt{\frac{g_R(v)}{g_L(u)}} \left( \cos \eta_+ \text{sn}_+^{\text{LR}} - \frac{\text{cn}_+^{\text{LR}}}{\text{dn}_+^{\text{LR}}} \sin \eta_+ \right),$$

$$r_3^{\text{LR}} = \frac{1}{\sqrt{\sin(\eta(u)) \sin(\eta(v))}} \sqrt{\frac{g_R(v)}{g_L(u)}} \left( \cos \eta_+ \text{sn}_+^{\text{LR}} + \frac{\text{cn}_+^{\text{LR}}}{\text{dn}_+^{\text{LR}}} \sin \eta_+ \right),$$

$$r_4^{\text{LR}} = \frac{1}{\sqrt{\sin(\eta(u)) \sin(\eta(v))}} \sqrt{\frac{g_R(v)}{g_L(u)}} \left( -\cos \eta_- \text{sn}_+^{\text{LR}} + \frac{\text{cn}_+^{\text{LR}}}{\text{dn}_+^{\text{LR}}} \sin \eta_- \right)$$

$$r_5^{\text{LR}} = -\frac{k\alpha}{g_L(u)} \frac{\text{cn}_+^{\text{LR}} \text{sn}_+^{\text{LR}}}{\text{dn}_+^{\text{LR}}}, \quad r_6^{\text{LR}} = \frac{k g_R(v)}{\alpha} \frac{\text{cn}_+^{\text{LR}} \text{sn}_+^{\text{LR}}}{\text{dn}_+^{\text{LR}}},$$

$$r_7^{\text{LR}} = -\frac{g_R(u)}{g_L(v)}, \quad r_8^{\text{LR}} = 1.$$

$$\text{cn}_+^{\text{LR}} = \text{Jacobi CN} \left[ F^L(u) \pm \underline{F}^R(v), k^2 \right]$$

$$\eta_{\pm} = \frac{\eta(u) \pm \eta(v)}{2}$$

$$S_{\text{ELLIPTIC}}(u, v) : \begin{pmatrix} \check{S}^{LL} & \check{S}^{LR} \\ \check{S}^{RL} & \check{S}^{RR} \end{pmatrix}$$

BLOCK STRUCTURE!

YBE!

CROSSING SYMMETRY!

→  $\hbar \rightarrow 0$ , We can define  $F, \eta, g$  such that

$S_{\text{ELLIPTIC}}$  reduces to  $\text{AdS}_3 \times S^3 \times T^4$  S-matrix

→ But does NOT reduce to the  $q$ -deformed

$\text{AdS}_3 \times S^3 \times T^4$  S-matrix unless we use a twist

that breaks integrability.

BUT DOES THIS COMES FROM

A STRING SIGMA MODEL ?

( Ben Moore, ALR, Fiona Scibdd )



# Principal Chiral Model (basic ideas)

$g(z, \sigma)$  valued in a Lie Group  $G$  (Plays the role of the target space)

$$j = g^{-1} dg \in \mathfrak{g} = \text{Lie}(G)$$

$$k = -dg g^{-1} \in \mathfrak{g} = \text{Lie}(G)$$

$$S_{\text{PCM}} = \frac{T}{2} \int dt d\sigma \text{tr} (g^{-1} \partial_\alpha g g^{-1} \partial^\alpha g)$$

$j, k$  satisfy the zero curvature equation

$$\partial_\alpha j_\beta - \partial_\beta j_\alpha + [j_\alpha, j_\beta] = 0$$

Global symmetry :  $G_L \times G_R$

$$g \rightarrow g_L g g_R \quad g_L, g_R \in G$$

$$x^\pm = \frac{z \pm \sigma}{2} \quad ,$$

$$\mathcal{L}_\pm(u) = \frac{j^\pm}{1 \mp u} \rightarrow \text{Lax}$$

$$S_{\text{PCM}} = \frac{T}{2} \int dz d\sigma \text{tr} (g^{-1} \partial_\alpha g g^{-1} \partial^\alpha g) = \frac{T}{2} \int dz d\sigma \text{tr} j_\mu j^\mu$$

$$= \frac{T}{2} \int dz d\sigma (\gamma^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}) \partial_\alpha \Psi^\mu \partial_\beta \Psi^\nu$$

Work with Fiona Seibold and Ben Hoare  
(arxiv 2312.14031)

Focus on the bosonic strings on  $AdS_3 \times S^3$

$$AdS_3 = \frac{SO(2,2)}{SO(1,2)} \cong \frac{SL(2;\mathbb{R}) \times SL(2;\mathbb{R})}{SL(2;\mathbb{R})}$$

$$S^3 = \frac{SO(4)}{SO(3)} \cong \frac{SU(2) \times SU(2)}{SU(2)}$$

One way to think about it is to describe this as a  
Principal Chiral Model with

$$G = SL(2;\mathbb{R}) \times SU(2)$$

from  
AdS<sub>3</sub>

from S<sup>3</sup>

$$G = SL(2; \mathbb{R}) \times SU(2)$$

SL(2; R) algebra

SU(2) algebra

$$L_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$J_j = \begin{pmatrix} 0 & 0 \\ 0 & i\sigma_j \end{pmatrix}$$

$$g = e^{TL_3} e^{UL_3} e^{VL_1} e^{\Phi J_2} e^{XJ_3} e^{YJ_1}$$

{ T, U, V } coordinates in AdS<sub>3</sub>

{ Φ, X, Y } coordinates in S<sup>3</sup>

For now focus on  $AdS_3 \times S^3$

STEP 1:

Deform the PCM for  $G = SL(2; \mathbb{R}) \times SU(2)$

$$\mathcal{S} = \frac{1}{4} \int dt d\sigma (\delta^{\alpha\beta} + \epsilon^{\alpha\beta}) \text{Tr} [g^{-1} \partial_\alpha g \Theta g^{-1} \partial_\beta g]$$

$$\Theta(L_j) = -\alpha_j L_j \quad j=1,2,3$$

$$\Theta(J_j) = \beta_j J_j$$

$\alpha_1 \neq \alpha_2 \neq \alpha_3$  elliptic

$\alpha_1 = \alpha_3 \neq \alpha_2$  trigonometric

$\alpha_1 = \alpha_2 = \alpha_3$  rational / undeformed

STEP 2:

Compare it with Green-Schwarz action

$$B_{\mu\nu} = 0$$

and the metric  $ds^2 = ds_a^2 + ds_b^2$

$$\begin{aligned} ds_a^2 &= (\alpha_1 \sinh^2 2U + \cosh^2 2U (-\alpha_2 \cosh^2 2V + \alpha_3 \sinh^2 2V)) dT^2 + \alpha_1 dV^2 + \\ &\quad + (\alpha_3 \cosh^2 2V - \alpha_2 \sinh^2 2V) dU^2 - 2\alpha_1 \sinh 2U dT dV + (\alpha_3 - \alpha_2) \cosh 2U \sinh 4V dT dU \\ ds_b^2 &= (\beta_1 \sin^2 2X + \cos^2 2X (\beta_2 \cos^2 2Y + \beta_3 \sin^2 2Y)) d\Phi^2 + \beta_1 dY^2 + \\ &\quad + (\beta_3 \cos^2 2Y + \beta_2 \sin^2 2Y) dX^2 - 2\beta_1 \sin 2X d\Phi dY + (\beta_3 - \beta_2) \cos 2X \sin 4Y d\Phi dX . \end{aligned}$$



$$\begin{aligned}
ds_a^2 &= (\alpha_1 \sinh^2 2U + \cosh^2 2U (-\alpha_2 \cosh^2 2V + \alpha_3 \sinh^2 2V)) dT^2 + \alpha_1 dV^2 + \\
&\quad + (\alpha_3 \cosh^2 2V - \alpha_2 \sinh^2 2V) dU^2 - 2\alpha_1 \sinh 2U dT dV + (\alpha_3 - \alpha_2) \cosh 2U \sinh 4V dT dU \\
ds_b^2 &= (\beta_1 \sin^2 2X + \cos^2 2X (\beta_2 \cos^2 2Y + \beta_3 \sin^2 2Y)) d\Phi^2 + \beta_1 dY^2 + \\
&\quad + (\beta_3 \cos^2 2Y + \beta_2 \sin^2 2Y) dX^2 - 2\beta_1 \sin 2X d\Phi dY + (\beta_3 - \beta_2) \cos 2X \sin 4Y d\Phi dX .
\end{aligned}$$

STEP 3:

FIX LIGHT-CONE GAUGE



STEP 4:

DECOMPACTIFY AND COMPUTE  
THE SCATTERING MATRIX

Perturbatively!

$$ds_a^2 = (\alpha_1 \sinh^2 2U + \cosh^2 2U (-\alpha_2 \cosh^2 2V + \alpha_3 \sinh^2 2V)) dT^2 + \alpha_1 dV^2 +$$

$$+ (\alpha_3 \cosh^2 2V - \alpha_2 \sinh^2 2V) dU^2 - 2\alpha_1 \sinh 2U dT dV + (\alpha_3 - \alpha_2) \cosh 2U \sinh 4V dT dU$$

$$ds_b^2 = (\beta_1 \sin^2 2X + \cos^2 2X (\beta_2 \cos^2 2Y + \beta_3 \sin^2 2Y)) d\Phi^2 + \beta_1 dY^2 +$$

$$+ (\beta_3 \cos^2 2Y + \beta_2 \sin^2 2Y) dX^2 - 2\beta_1 \sin 2X d\Phi dY + (\beta_3 - \beta_2) \cos 2X \sin 4Y d\Phi dX .$$

$$\sqrt{\omega_{\pm}^a (p)^2 + \tilde{\gamma}_2^2} = \sqrt{p^2 + \tilde{\gamma}_1^2} \pm \tilde{\gamma}_3 \quad \leftarrow \text{AdS}_3$$

$$\sqrt{\omega_{\pm}^b (p)^2 + \gamma_2^2} = \sqrt{p^2 + \gamma_1^2} \pm \gamma_3 \quad \leftarrow S^3$$

## DISPERSION RELATIONS





$$ds_a^2 = (\alpha_1 \sinh^2 2U + \cosh^2 2U (-\alpha_2 \cosh^2 2V + \alpha_3 \sinh^2 2V)) dT^2 + \alpha_1 dV^2 + \\ + (\alpha_3 \cosh^2 2V - \alpha_2 \sinh^2 2V) dU^2 - 2\alpha_1 \sinh 2U dT dV + (\alpha_3 - \alpha_2) \cosh 2U \sinh 4V dT dU$$

$$ds_b^2 = (\beta_1 \sin^2 2X + \cos^2 2X (\beta_2 \cos^2 2Y + \beta_3 \sin^2 2Y)) d\Phi^2 + \beta_1 dY^2 + \\ + (\beta_3 \cos^2 2Y + \beta_2 \sin^2 2Y) dX^2 - 2\beta_1 \sin 2X d\Phi dY + (\beta_3 - \beta_2) \cos 2X \sin 4Y d\Phi dX .$$

BUT the tree-level S-MATRIX is DIAGONAL  
(bosonic) ↓

IT HAS AN UNEXPECTED

U(1) SYMMETRY

It automatically satisfies the classical YBE!

$$J = J^{\text{LL}} = \begin{pmatrix} r_1 & 0 & 0 & s_4 \\ 0 & r_2 & s_3 & 0 \\ 0 & s_2 & r_3 & 0 \\ s_1 & 0 & 0 & r_4 \end{pmatrix}$$

Tried to find  $(s_i, i=1, \dots, 4)$

such that

$$[T_{12}, T_{13}] + [T_{12}, T_{23}] + [T_{13}, T_{23}] = 0$$

IT IS IMPOSSIBLE!

When we include the fermions, one of the following will happen

1) It breaks

$$\check{S} = \check{S} \otimes \check{S}$$

FACTORIZATION

2) It breaks

$$\left( \begin{array}{c|c} \check{S}^{LL} & \check{S}^{LR} \\ \hline \check{S}^{RL} & \check{S}^{RR} \end{array} \right)$$

BLOCK STRUCTURE

3) IT BREAKS INTEGRABILITY!

4) IT "WEAKLY" BREAKS INTEGRABILITY!

# CONCLUSIONS

I presented a method to construct R-matrices

Showed one elliptic integrable deformation  
coming from integrability and another from the  
sigma model.

## NEXT STEPS :

(in progress with B. Hoare and F. Seibold)

- Compute the tree-level  $S$ -matrix including fermions

- Investigate the existence of hidden symmetries.

- Bootstrap the exact  $S$ -matrix

- Connection with  $AdS_2$  ?



THANK YOU!