Hagedorn transitions in exact $U_q(sl_2)$ S-matrix theories with arbitrary spins

Changrim Ahn, Tommaso Franzini and Francesco Ravanini

EWHA WOMANS UNIVERSITY

University of Hertfordshire UH

ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

INFN Istituto Nazionale di Fisica Nucleare

Scattering theory in 2D

Integrability → Factorised S-matrices that satisfy:

- YBE
- Unitarity
- Crossing symmetry
- Bootstrap

	Sine-Gordon → s = ½	Sausage → s = 1
Factorised scattering	Doublet soliton-antisoliton	Triplet of solitons
S-matrix prop. to	R-matrix of spin ½ x ½	R-matrix of spin 1 x 1

Why not try higher spin s ?

Sine-Gordon S-matrix

- Theory with a bosonic lagrangian
- However in its repulsive regime it is described by the scattering of solitons (isospin ½) and antisolitons (isospin -½) forming a doublet in irrep s = ½
- S-matrix well known since Zam-Zam work (1979) to be factorized in two body S-matrices proportional to the R-matrix of $U_q(sl_2)$ in repr ½ x ½
- So by construction it satisfies YBE
- The deformation parameter *q* is related to the sG coupling
- Overall scalar function $S_0(\theta)$ left free by YBE can be determined by unitarity, crossing and bootstrap
- Could in principle have CDD factors not adding new poles.
- We assume **minimality**: no additional CDD factors.

Inverse scattering program

Although S-matrices are on-shell, they can provide also off-shell information on the underlying QFT

- **Correlation functions** through the form factor program
- **Thermodynamics** of the system and **finite size effects** through TBA and/or NLIE
- Even describe **non-equilibrium physics**
- One could think that the S-matrix is a way to define a QFT alternative to the Lagrangian formulation
- It is a longstanding question, since the 1960’s, if any S-matrix theory has an underlying QFT description.

Hagedorn singularities & phase transitions

Is there a scale where the S-matrix is not valid anymore?

- For example, quarks replace hadrons at high temperature in QCD
- Also in string theory similar phenomena: **Hagedorn phase transitions**
- If S-matrix fails to describe the theory, TBA also fails (diverges) and this signals a singularity that we call **Hagedorn singularity**
- This behaviour has been shown in TTbar deformations of QFTs, but other irrelevant operators may play the same role
- TTbar is usually related to adding CDD factors to the scalar prefactor of S-matrices
- We explore this situation in minimal S-matrices, where no additional CDD factor appears

Factorised scattering of (iso-)spin s particles

$\mathcal{U}_{\mathsf{q}}(\mathfrak{su}_2)$ symmetry

$$[\lambda]_{\mathsf{q}} \equiv \frac{\mathsf{q}^{\lambda/2} - \mathsf{q}^{-\lambda/2}}{\mathsf{q}^{1/2} - \mathsf{q}^{-1/2}}$$

$$[\mathbb{J}_{\pm}, , \mathbb{J}_3] = \pm \mathbb{J}_{\pm} \quad , \quad [\mathbb{J}_{+}, , \mathbb{J}_{-}] = [2\mathbb{J}_3]_{\mathsf{q}}$$

2 particle S-matrix:

$$\mathsf{S}(\theta) = \sigma \left(\mathsf{P} \sum_{J=0}^{2s} f_{\mathsf{q}}^{[J]}(\theta)\, \mathbb{P}_{\mathsf{q}}^{[J]} \right) \sigma^{-1}$$

gauge permutator q-Projectors

Case q = 1 introduced in Aladim, Martins 1994

q-Projectors

$$\mathbb{P}_{\mathsf{q}}^{[J]}{}^{m_1' m_2'}_{\ m_1 m_2} = \sum_{M=-J}^{J} \langle s, m_1'; s, m_2' | J, M \rangle_{\mathsf{q}} \langle J, M | s, m_1; s, m_2 \rangle_{\mathsf{q}}$$

- q-Clebsch-Gordan

$$\langle s, m_1; s, m_2 | J, M \rangle_{\mathsf{q}} = f(J) \cdot \mathsf{q}^{(2s-J)(2s+J+1)/4 + s(m_2 - m_1)/2}$$

$$\times \quad \{[s+m_1]![s-m_1]![s+m_2]![s-m_2]![J+M]![J-M]!\}^{1/2} \sum_{\nu \geq 0} (-1)^{\nu} \frac{\mathsf{q}^{-\nu(2s+J+1)/2}}{\mathcal{D}_{\nu}}$$

$$\mathcal{D}_{\nu} \quad = \quad [\nu]![2s-J-\nu]![s-m_1-\nu]![s+m_2-\nu]![J-s+m_1+\nu]![J-s-m_2+\nu]!$$

$$f(J) \quad = \quad \left\{ \frac{[2J+1]_{\mathsf{q}}([J]!)^2 [2s-J]!}{[2s+J+1]!} \right\}^{1/2}$$

Rapidity functions & prefactor

$$\mathsf{q} = e^{2\pi i\gamma}$$

$$f_{\mathsf{q}}^{[J]}(\theta) = S_0(\theta)\prod_{k=1}^{J}\frac{\sinh\left[\gamma(ik\pi-\theta)\right]}{\sinh\left[\gamma(ik\pi+\theta)\right]}, \qquad J = 0, 1, \cdots, 2s.$$

Unitarity: $S_0(\theta)S_0(-\theta) = 1$

Crossing: $S_0(i\pi-\theta) = \prod_{k=1}^{2s}\frac{\sinh\left[\gamma(i(k+1)\pi-\theta)\right]}{\sinh\left[\gamma(ik\pi+\theta)\right]}S_0(\theta)$

$$S_0(\theta) = \prod_{k=1}^{2s}\left[\frac{\sinh\left[\gamma(i\pi k+\theta)\right]}{\sinh\left[\gamma(i\pi k-\theta)\right]}\left(\prod_{\ell=1}^{\infty}\frac{\sinh\left[\gamma(i\pi(k+\ell)-\theta)\right]\sinh\left[\gamma(i\pi(k-\ell)-\theta)\right]}{\sinh\left[\gamma(i\pi(k+\ell)+\theta)\right]\sinh\left[\gamma(i\pi(k-\ell)k+\theta)\right]}\right)\right]$$

More on prefactor

When s is integer the prefactor greatly symplifies

$$S_0(\theta) = \prod_{m=1}^{s} \frac{\sinh\left[\gamma(\theta + i2m\pi)\right]}{\sinh\left[\gamma(\theta - i2m\pi)\right]} \quad \Longrightarrow \quad \mathsf{S}_{ss}^{ss}(\theta) = \prod_{m=1}^{s} \frac{\sinh\left[\gamma(\theta - i(2m-1)\pi)\right]}{\sinh\left[\gamma(\theta + i(2m-1)\pi)\right]}$$

When s is half-integer it gives rise to the infinite Γ product

$$\begin{aligned} S_0(\theta) \;&=\; \prod_{m=1}^{2s} \Bigg\{ \frac{1}{i\pi} \sinh\left[\gamma(\theta + im\pi)\right] \Gamma\left[1 - \gamma(m-1) + \frac{i\gamma\theta}{\pi}\right] \Gamma\left[1 - \gamma m - \frac{i\gamma\theta}{\pi}\right] \\ &\quad \times \prod_{n=1}^{\infty} \left[\frac{R_n^{[s,m]}(\theta) R_n^{[s,m]}(i\pi - \theta)}{R_n^{[s,m]}(0) R_n^{[s,m]}(i\pi)} \right] \Bigg\} \end{aligned}$$

In both cases the integral representation holds

$$\mathsf{S}_{ss}^{ss}(\theta) = \exp \int_{-\infty}^{\infty} \frac{dk}{k} \frac{\sinh(\pi k s) \sinh \pi k (s - \frac{1}{2\gamma})}{\sinh \frac{\pi k}{2\gamma} \sinh \pi k} e^{ik\theta}$$

Examples

- s = ½ : **Sine-Gordon**
- s = 1 : **Sausage**
- s = 3/2 :

$$\mathsf{S}_{11}^{11} = 1,\ \mathsf{S}_{12}^{12} = \frac{(0)}{(3)},\ \mathsf{S}_{12}^{21} = \frac{s_3}{(3)},\ \mathsf{S}_{13}^{13} = \frac{(0)(-1)}{(2)(3)},\ \mathsf{S}_{13}^{22} = \frac{s_2\sqrt{s_3/s_1}(0)}{(2)(3)},$$

$$\mathsf{S}_{13}^{31} = \frac{(s_1 s_4 + 2s_2)(0)}{(2)(3)},\ \mathsf{S}_{22}^{22} = \frac{f_1}{(2)(3)},\ \mathsf{S}_{14}^{14} = \frac{(0)(-1)(-2)}{(1)(2)(3)},\ \mathsf{S}_{14}^{23} = \frac{s_3(0)(-1)}{(1)(2)(3)}$$

$$\mathsf{S}_{14}^{32} = \frac{s_2 s_3(0)}{(1)(2)(3)},\ \mathsf{S}_{14}^{41} = \frac{s_1 s_2 s_3}{(1)(2)(3)},\ \mathsf{S}_{23}^{23} = \frac{(0)f_1}{(1)(2)(3)},\ \mathsf{S}_{23}^{32} = \frac{s_2 f_2}{(1)(2)(3)},$$

$$(n) \equiv 2\sinh\left[\gamma(\theta - i\pi n)\right], \quad s_n \equiv 2\sinh(in\pi\gamma),$$

$$f_1 = 2\cosh\left[\gamma(2\theta - i\pi)\right] + \frac{s_{10}}{s_5} - 2\frac{s_2}{s_1},\ f_2 = 2\frac{s_2}{s_1}\cosh\left[\gamma(2\theta - i\pi)\right] + s_2^2 - 2s_1^2 - 4$$

Thermodynamic Bethe Ansatz

- Bethe Yang equation

$$e^{iR\mathsf{m}\sinh\theta_j}\mathbb{T}(\theta_j|\{\theta_i\}) = 1,$$

$$\mathbb{T}(\theta_j|\{\theta_i\})^{m'_1,\cdots,m'_{\mathcal{N}}}_{m_1,\cdots,m_{\mathcal{N}}} = \sum_{n_1,\cdots,n_{\mathcal{N}}} \mathsf{S}^{n_2 m'_1}_{n_1 m_1}(\theta_1-\theta_j)\mathsf{S}^{n_3 m'_2}_{n_2 m_2}(\theta_2-\theta_j)\cdots\mathsf{S}^{n_1 m'_{\mathcal{N}}}_{n_{\mathcal{N}} m_{\mathcal{N}}}(\theta_{\mathcal{N}}-\theta_j)$$

- String hypothesis

$$\lambda^{(n)}_{j,\alpha} = \lambda^{(n)}_j + \frac{i\pi}{2}(n+1-2\alpha), \quad \alpha = 1,2,\cdots,n$$

we restrict to the simpler case $\gamma = \frac{1}{N}$

- Diagonalisation in terms of Bethe Ansatz of XXZ higher spin chains [Kulish Reshetikhin]
- Thermodynamic limit and intergal eqs. for densities of centers of strings σ and $\tilde{\sigma}$
- Minimization of free energy and TBA

$$\sigma_n(\theta) + \tilde{\sigma}_n(\theta) = \delta_{n0}\mathsf{m}\cosh\theta - \nu_n\sum_{m=0}^{N} K_{nm}\star\sigma_n(\theta) \qquad K_{nm}(\theta) = \frac{1}{2\pi i}\frac{d}{d\theta}\ln S_{nm}(\theta)$$

TBA equations: free energy & scaling fct.

Pseudoenergies

$$\epsilon_0(\theta) = \log \frac{\tilde{\sigma}_0}{\sigma_0}, \qquad \epsilon_n(\theta) = \log \frac{\sigma_n}{\tilde{\sigma}_n}, \quad n = 1, \ldots, N-1, \qquad \epsilon_N(\theta) = \log \frac{\tilde{\sigma}_N}{\sigma_N}$$

Universal kernel $p(\theta) = \dfrac{1}{2\pi \cosh \theta}$

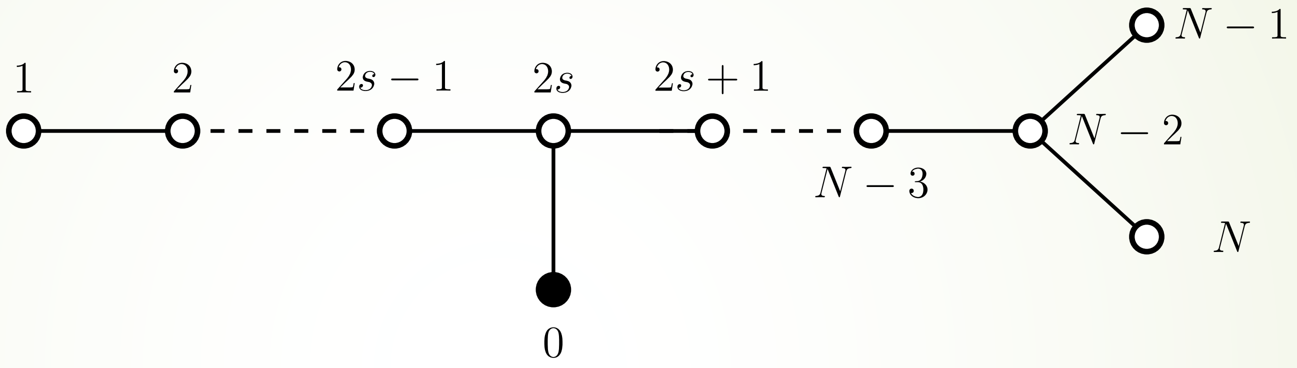
Incidence matrix

$$\epsilon_n(\theta) = \delta_{n,0} \mathrm{m} L \cosh \theta - \sum_{m=0}^{N} \mathbb{I}_{nm}\, p \star \log\left(1 + e^{-\epsilon_m}\right)(\theta)$$

Free energy

$$\frac{f(T)}{T} = -\int_{-\infty}^{\infty} \frac{\mathrm{m}}{2\pi} \cosh \theta \ln\left(1 + e^{-\epsilon_0(\theta)}\right) d\theta$$

TBA graph: not a Dynkin diagram



Finite size vacuum energy of the mirror theory $E_0(T) = \frac{f(T)}{T}$

Dimensionless parameter $r = \mathsf{m}/T$ measures the size

Scaling function

$$\widetilde{c}(r) = \frac{3}{\pi^2}\mathsf{m}\int_{-\infty}^{\infty} r\cosh(\theta) L_0(\theta) d\theta$$

$$\lim_{r\to 0}\widetilde{c}(r) = c - 24\Delta_{\min}$$

Plateaux or not plateaux?

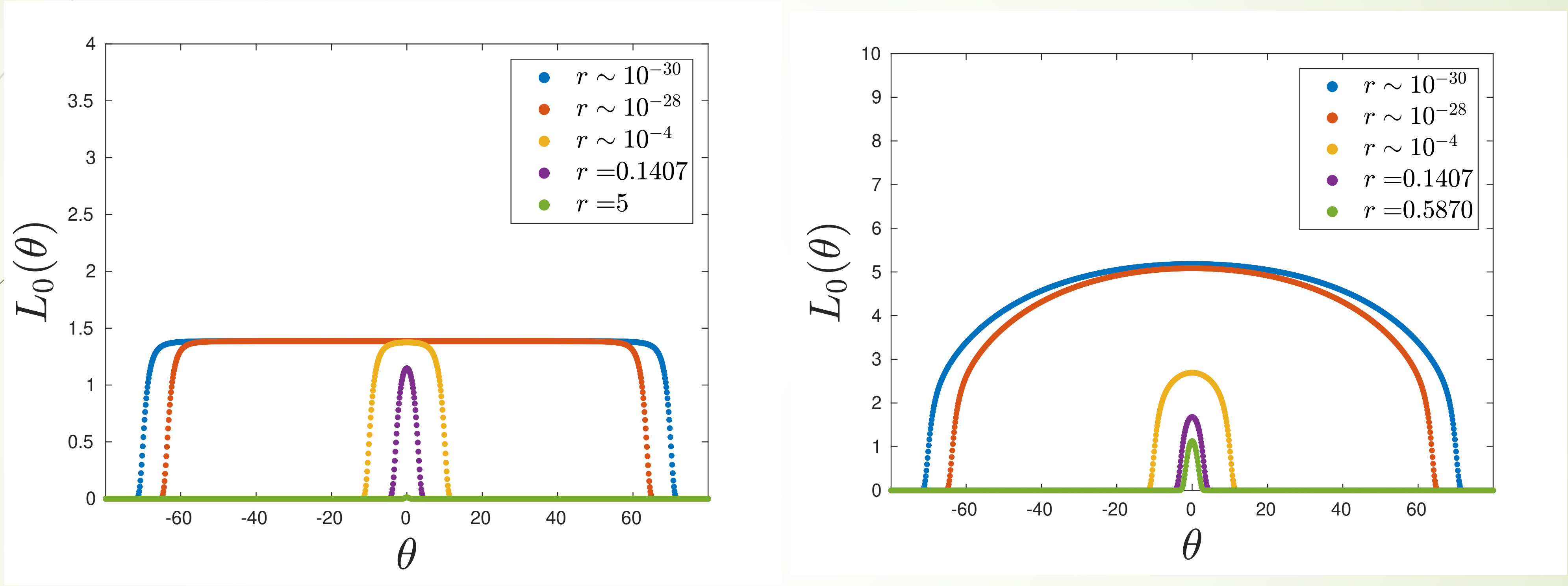


Figure 3: The functions $L_0(\theta)$ for spin $s = 1/2$ (left) and $s = 1$ (right) with $\gamma = 1/7$, for different values of r. One can see that for smaller values of r the plateau starts to form.

Behaviour for various s

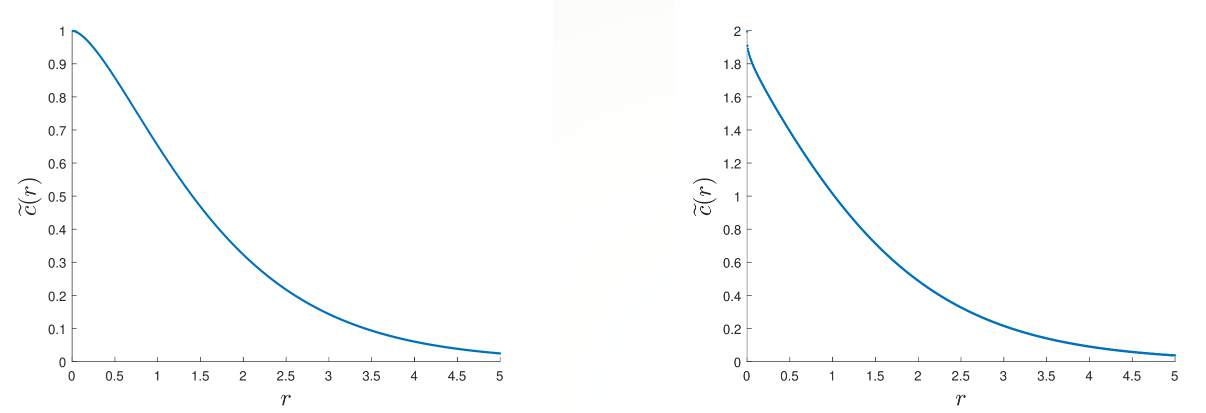


Figure 4: Scaling functions for spin $s = 1/2$ (left) and $s = 1$ (right).

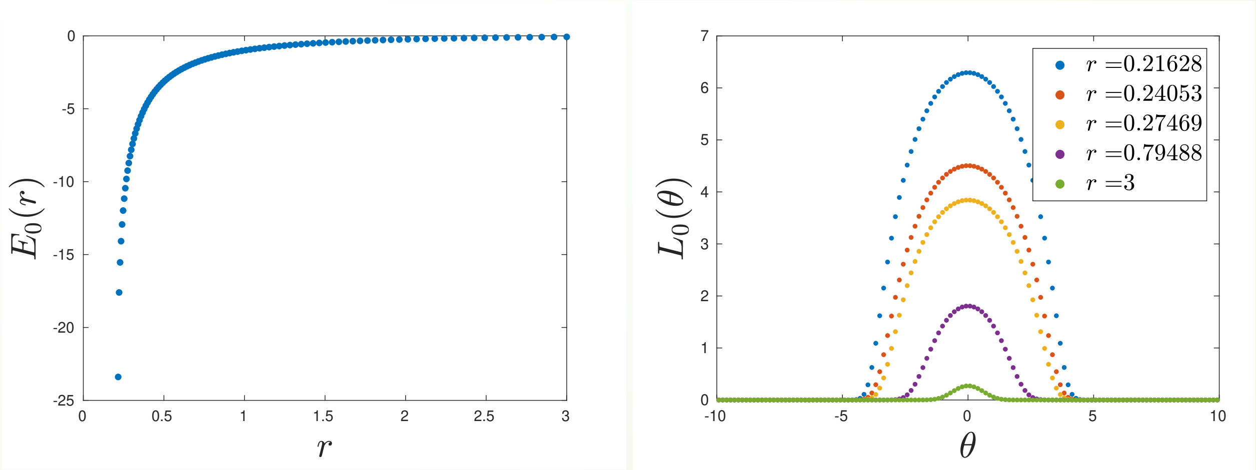


Figure 5: Left: the vacuum energy $E_0(r)$ as it approaches the singular point $r^* = 0.21628(2)$; right: the kernel $L_0(\theta)$ at different values of r. Both were obtained for $s = 5/2$ and $N = 12$.

Critical temperature

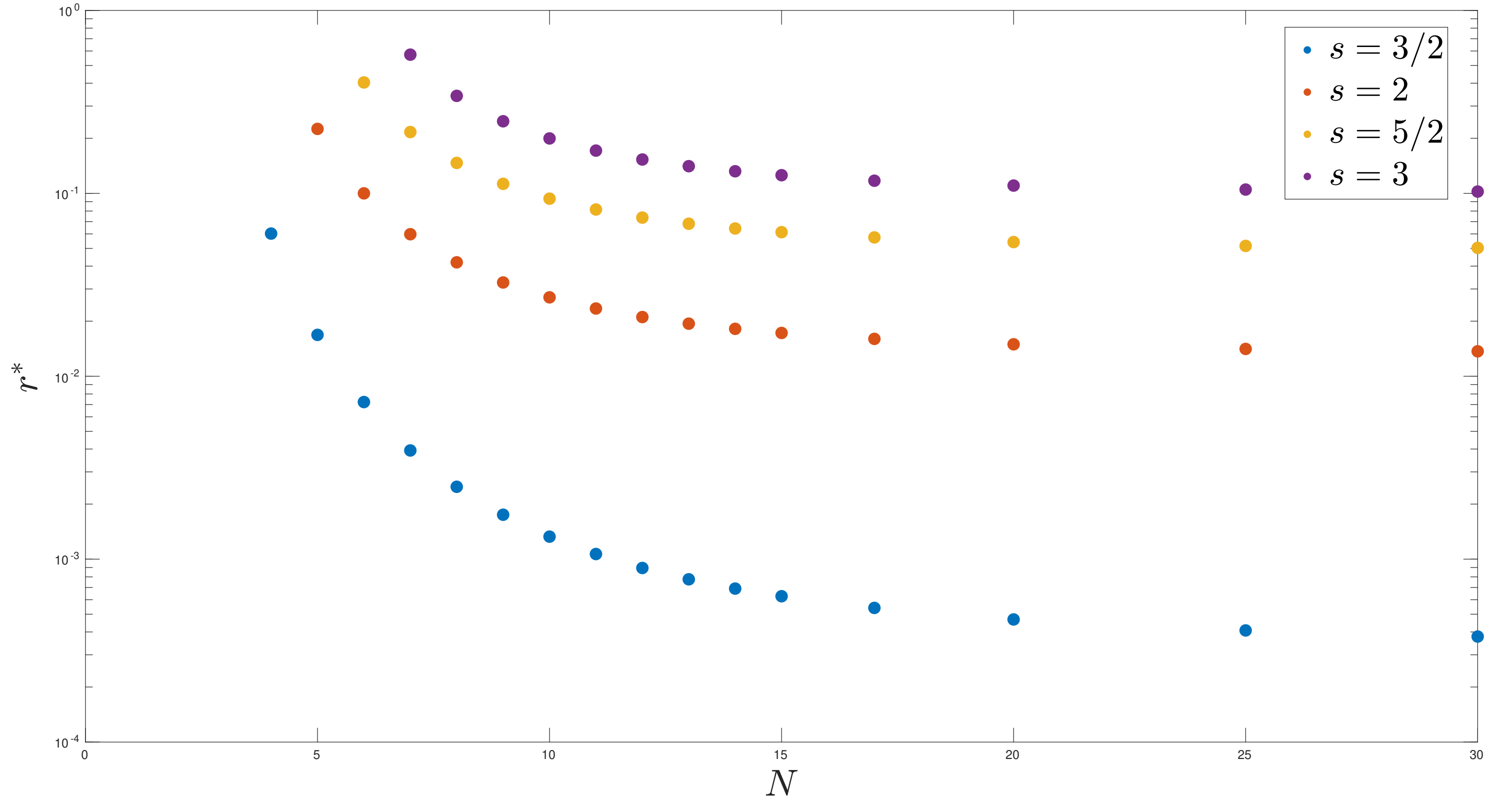


Figure 6: Value of the singular point r^*, for different values of spin and coupling constant $\gamma = 1/N$. The values are computed with precision to the 6th decimal digit. The r^*-axis is log-scaled.

Hagedorn transition ?

$$E_0(r) \sim_{r \to r^*} c_0 + c_{1/2}\sqrt{r - r^*}$$

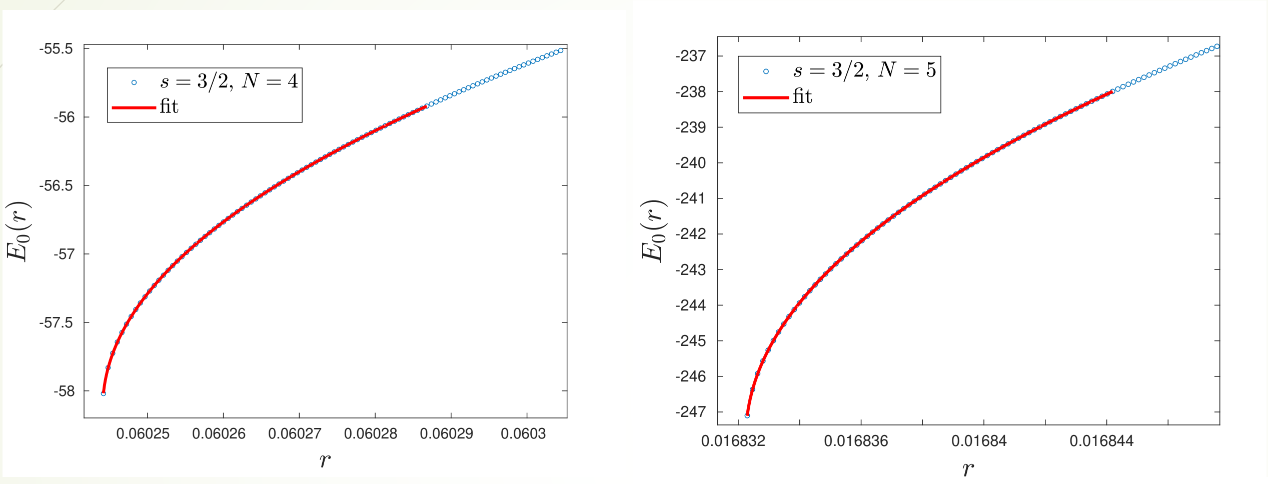
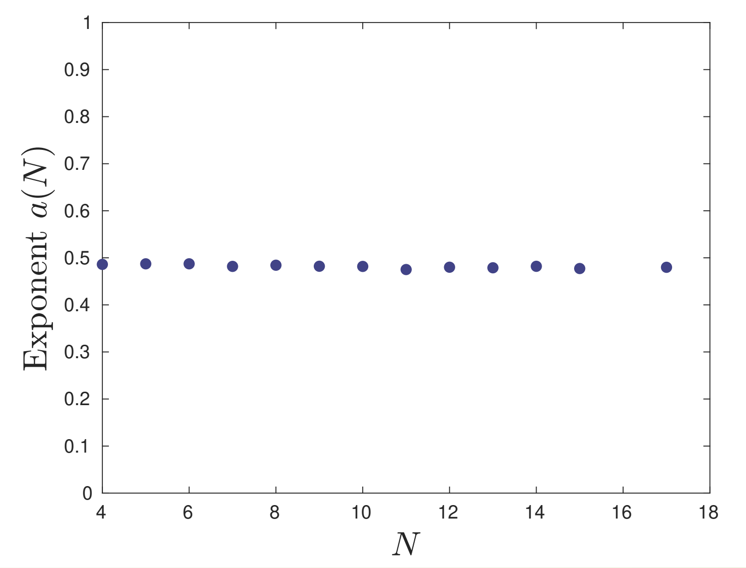


Figure 7: Examples of fitting for $s = 3/2$ and $N = 4$ (left) and $N = 5$ (right).

$$b(r - r^*)^a + c_0$$



Conclusions

- New S-matrices of isospin s, consistent with crossing symmetry, unitarity and bootstrap (repulsive regime) with minimality condition on the CDD factors.
- Describe scattering of solitonic multiplets
- In the case s = ½ coincide with sG, for s = 1 with Sausage models
- Focus on thermodynamics, for s > 1 a Hagedorn singularity appears in the free energy
- The TBA equations can be encoded on a diagram. This s a Dynkin diagram for s = ½ , 1 but not for higher s.
- We suspect that it is not a coincidence that Hagedorn singularity appears where the diagram is not Dynkin.
- Universality of the singular behaviour of free energy: exponent = ½ suggesting that the responsible field could be something related to TTbar

Thank you