Modular covariant torus partition functions of dense $A_1^{(1)}$ and dilute $A_2^{(2)}$ loop models

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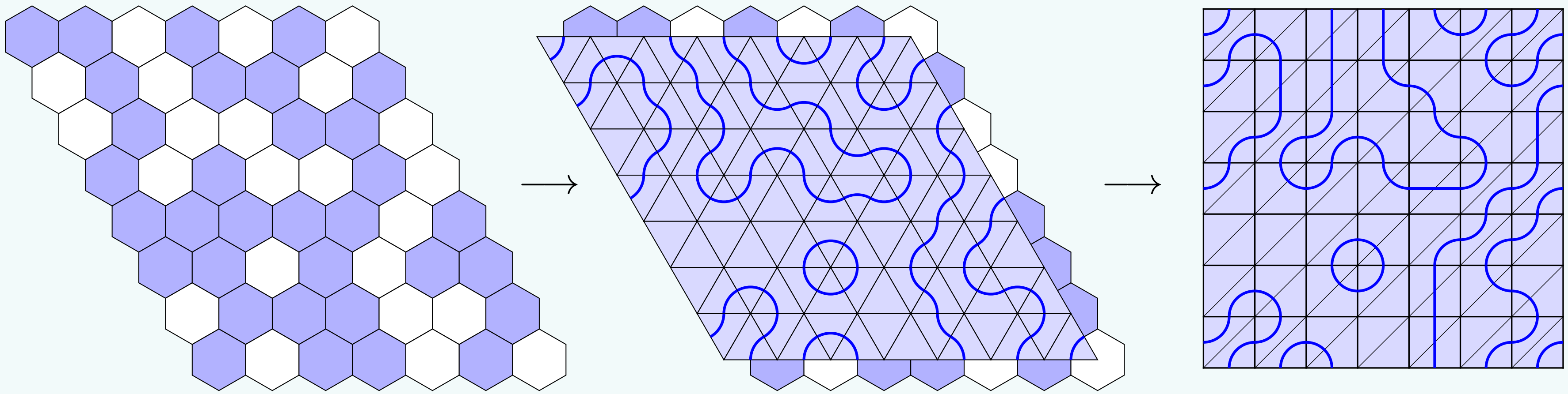
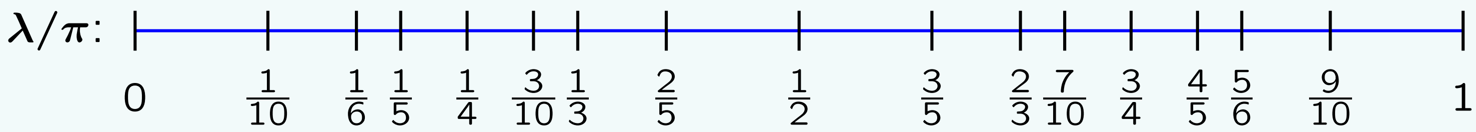


- A.Morin-Duchesne, A.Klümper, PAP, *Modular covariant torus partition functions of dense $A_1^{(1)}$ and dilute $A_2^{(2)}$ loop models*, to be submitted, ≈30p (2024).
- A.Morin-Duchesne, A.Klümper, PAP, **Critical site percolation on the triangular lattice:** *From integrability to conformal partition functions*, J.Stat.Mech. (2023) 043103, 105p: $sl(3)$
- A.Morin-Duchesne, A.Klümper, PAP, *Conformal partition functions of critical percolation from D_3 TBA equations*, J.Stat.Mech. (2017) 083101, 85p (**Critical bond percolation on the square lattice**): $sl(2)$
- A.Morin-Duchesne, PAP, *Fusion hierarchies, T-systems and Y-systems for the dilute $A_2^{(2)}$ loop models*, J.Stat.Mech. (2019) 094007, 35p.
- A.Morin-Duchesne, A.Klümper, PAP, *Groundstate finite-size corrections and dilogarithm identities for the twisted $A_1^{(1)}$, $A_2^{(1)}$ and $A_2^{(2)}$ models*, J.Stat.Mech. (2021) 033105, 95p.

“Coulomb Line”



- In the continuum scaling limit, the modular invariant conformal partition functions of many critical 2d lattice models (RSOS, loop, vertex) can be expressed [FSZ87] in terms of the partition functions of the 2d Coulomb gas with logarithmic interactions.

- In their simplest regimes, the critical manifolds of the 6-vertex and 19-Vertex Izergin-Korepin models (respectively the isospectral $A_1^{(1)}$ and $A_2^{(2)}$ loop models) are curves in the thermodynamic space parametrized by the crossing parameter $\frac{\lambda}{\pi} \in (0,1)$. The models are Yang-Baxter integrable along these lines.

- The transfer matrix eigenenergies cross at the rational points $\frac{\lambda}{\pi} \in \mathbb{Q}$. At each such point, $\frac{\lambda}{\pi} = \frac{p'-p}{p'} \in \mathbb{Q}$ in this dense set, there exists an extended loop algebra symmetry [DFM2001] reflected by degeneracies in the eigenenergies.

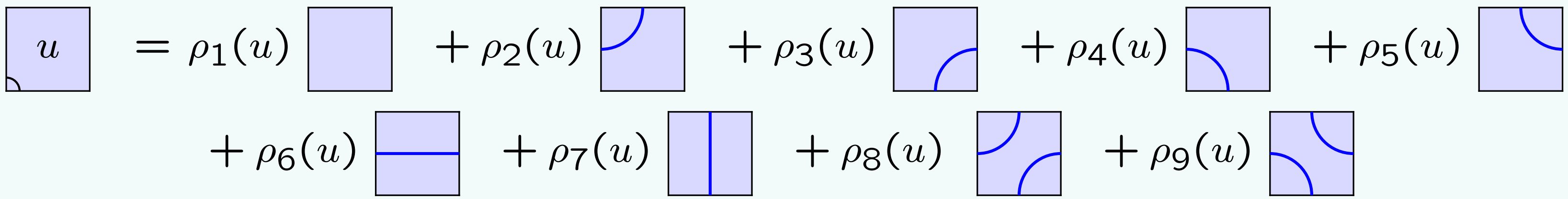
- For generic values of the crossing parameter $\frac{\lambda}{\pi} \in (0,1)$ and twist $\gamma \in \mathbb{R}$, the partition functions are given as (infinite) sesquilinear forms in generic Verma characters.

- We will see that for zero twist ($\gamma = 0$) and $\frac{\lambda}{\pi} \in \mathbb{Q}$, the modular covariant torus partition functions can be written in terms of a generalized Coulomb partition function and also as sesquilinear forms in affine $u(1)$ characters.

- In this talk, it is convenient to work in the language of loop models but the results apply equally to both loop and the associated (isospectral) 6- and 19-vertex (Izergin-Korepin) models. Here I focus on results and omit most calculations.

Dense and Dilute Loop Models

- The face operators of the dense $A_1^{(1)}$ and dilute $A_2^{(2)}$ loop models take the form



- Set $s(u) = \frac{\sin u}{\sin \lambda}$. In the simplest regimes $0 < u < \lambda < \pi$. In terms of the spectral parameter u and crossing parameter λ, the tile weights $\rho_i = \rho_i(u)$, corresponding to critical manifolds, are

dense: $\rho_1 = \rho_2 = \cdots = \rho_7 = 0, \qquad \rho_8 = s(\lambda - u), \qquad \rho_9 = s(u)$

dilute:
$$\begin{cases} \rho_1 = s(2\lambda)s(3\lambda) + s(u)s(3\lambda - u), & \rho_6 = \rho_7 = s(u)s(3\lambda - u) \\ \rho_2 = \rho_3 = s(2\lambda)s(3\lambda - u), & \rho_8 = s(2\lambda - u)s(3\lambda - u) \\ \rho_4 = \rho_5 = s(2\lambda)s(u), & \rho_9 = -s(u)s(\lambda - u) \end{cases}$$

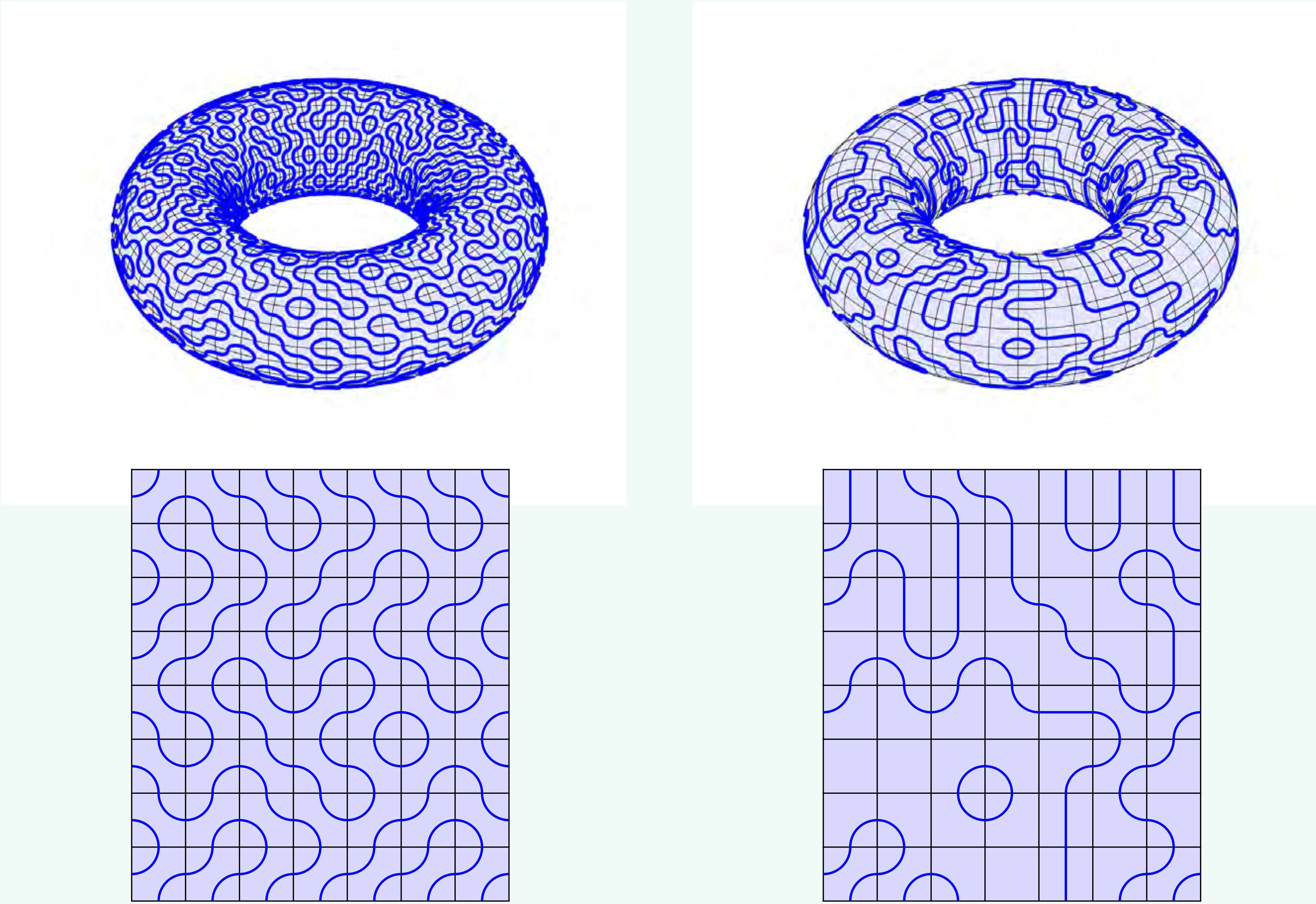
- The fugacities of non-contractible and contractible (closed) loops on the torus are

$$\alpha = \omega + \omega^{-1}, \quad \omega = e^{i\gamma}, \quad \gamma = \text{twist}; \qquad \beta = \begin{cases} 2\cos\lambda, & \text{dense} \\ -2\cos 4\lambda, & \text{dilute} \end{cases}$$

- Let $n_\beta(\sigma)$, $n_\alpha(\sigma)$, $n_i(\sigma)$ count contractible, non-contractible loops and the occurences of the i-th tile in σ. The lattice partition functions are then defined as a sum over configurations σ

$$Z = \sum_\sigma \alpha^{n_\alpha(\sigma)} \beta^{n_\beta(\sigma)} \prod_{i=1}^{9} \rho_i^{n_i(\sigma)}$$

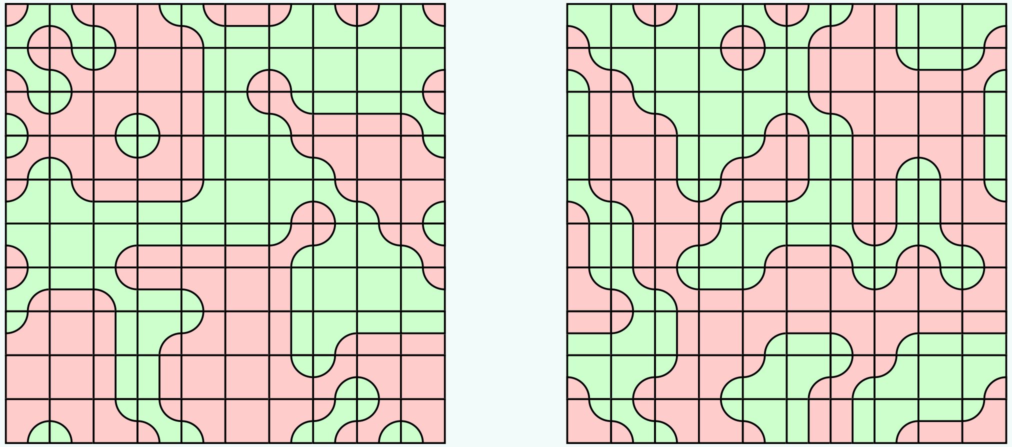
Typical Loop Configurations



- Typical configurations of dense (left) and dilute (right) loop models. Upper panels show typical configurations on the torus. Lower panels show projections onto a doubly periodic rectangle.

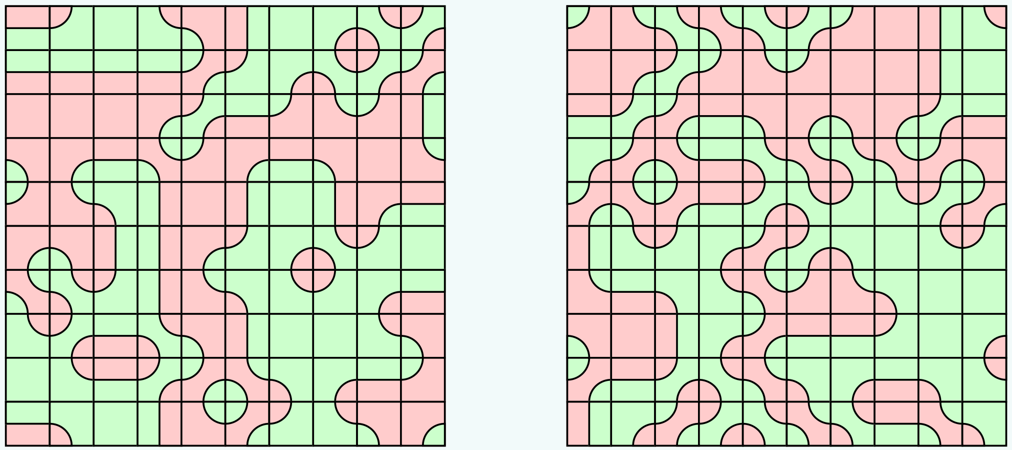
Periodic and Anti-Periodic Boundary Conditions

$(h, v) = (0, 0)$



$(h, v) = (0, 1)$

$(h, v) = (1, 0)$



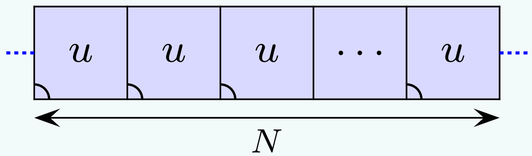
$(h, v) = (1, 1)$

- Sample loop configurations for the dilute $A_2^{(2)}$ loop model with combinations of periodic $(h, v = 0)$ and anti-periodic $(h, v = 1)$ boundary conditions horizontally and vertically on an $N \times M$ lattice. The left/right edges and top/bottom edges are identified to form a torus. Loop configurations for the dense $A_1^{(1)}$ loop model are similar but with each square face containing two loop segments. In this case h, v are simply the $\mathbb{Z}_2$ parities of N and M.

Transfer Matrices and TL Algebras

- The dense and dilute loop models are integrable since their face operators satisfy the Yang-Baxter equation. The commuting transfer matrices $\boldsymbol{T}(u)$ of the dense and dilute loop models

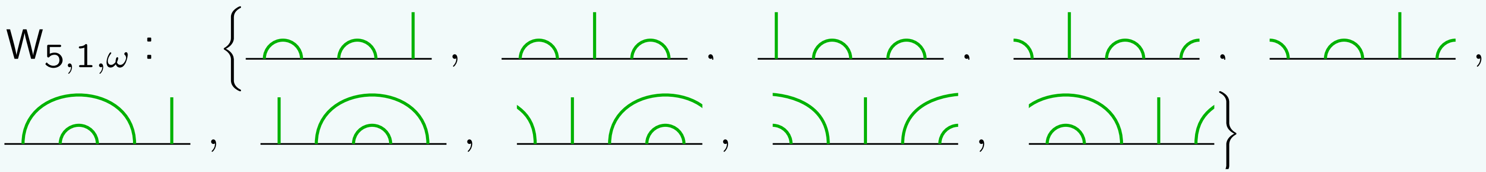
$$\boldsymbol{T}(u) =$$



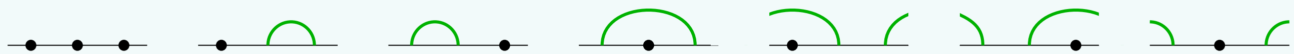
are elements of the periodic dense/dilute Temperley-Lieb algebras

- **Standard Modules:** The periodic loop transfer matrices act diagrammatically on the vector spaces $\mathsf{W}_{N,d,\omega}$ spanned by planar link states on N nodes with d defects ($0 \le d \le N$). Examples:

dense: $\mathsf{W}_{5,1,\omega}$:

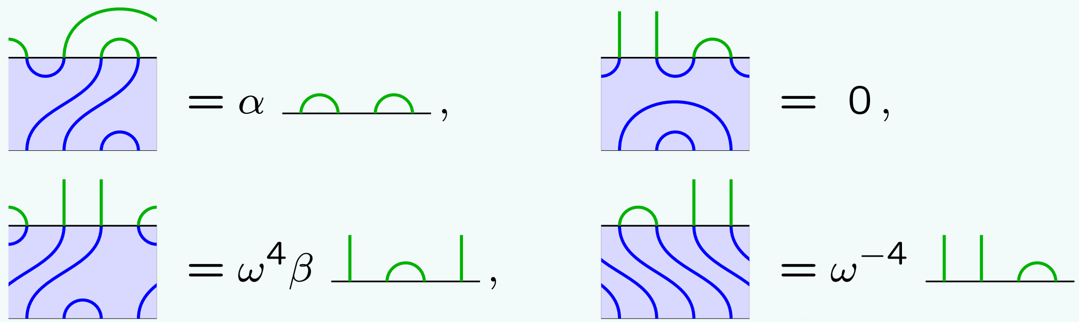


dilute: $\mathsf{W}_{3,0,\omega}$:



dilute: $\mathsf{W}_{3,1,\omega}$:

- Examples of the (defect preserving) action of the periodic TL algebra on the cylinder are:



Torus Lattice Partition Functions

- Set $\alpha = \omega + \omega^{-1}$ and let $\mathsf{W}_{N,d,\omega}$ denote the standard module over the periodic dense/dilute Temperley-Lieb algebras where d is the number of defects. In both dense/dilute cases, the (matrix) trace over standard modules of the M-th power of the transfer matrix decomposes as a Laurent series

$$\mathrm{tr}_{\mathsf{W}_{N,d,\omega}} \boldsymbol{T}(u)^M = \sum_{j=-M}^{M} \omega^{-j} C_{d,j}, \qquad C_{d,j} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma\, e^{i\gamma j}\, \mathrm{tr}_{\mathsf{W}_{N,d,e^{i\gamma}}} \widehat{\boldsymbol{T}}(u)^M$$

where $C_{d,j}$ are independent of $\omega = e^{i\gamma}$.

- The Markov trace glues the top and bottom edges of the cylinder together to form a torus.

- Using the Markov trace, it follows that the torus lattice partition functions are

$$\begin{aligned} Z^{(h,v)}_{\text{dense}} &= \sum_{\substack{-N\le d\le N\\ d\equiv h \bmod 2}} \sum_{\substack{-M\le j\le M\\ j\equiv v \bmod 2}} T_{d\wedge j}(\tfrac{\alpha}{2})\, C_{d,j} \\ Z^{(h,v)}_{\text{dilute}} &= \sum_{\substack{-N\le d\le N\\ d\equiv h \bmod 2}} \sum_{\substack{-M\le j\le M\\ j\equiv v \bmod 2}} \mathbf{2}\, T_{d\wedge j}(\tfrac{\alpha}{2})\, C_{d,j} \end{aligned}$$

where $d\wedge j = \gcd(d,j)$, $C_{-d,j} = C_{d,-j}$ and $T_n(x)$ is the n-th Chebyshev polynomial of the first kind $T_n(\cos\theta) = \cos n\theta$. Note that there is an extra factor of 2 for the dilute loop model.

Continuum Scaling Limit

- In the continuum scaling limit ($R' = aN, R = aM$; $a \to 0$; $M, N \to \infty$), the dense and dilute loop models are described by CFTs with effective central charges depending on the twist γ.

- Consider the untwisted models ($\gamma = 0$, $\alpha = 2$). If λ/π is irrational then the CFT is irrational in the sense that its central charge is irrational. In this case, the models exhibit generic Virasoro symmetry. Otherwise, if λ/π is rational, then the CFT has a rational central charge and is both nonunitary and logarithmic but not rational. In these cases, higher degeneracies occur in the spectra and an affine $u(1)$ symmetry emerges.

- For λ/π rational, we parametrize the crossing parameter by

$$\lambda = \begin{cases} \frac{\pi(p'-p)}{p'}, & \text{dense} \\ \frac{\pi(2p'-p)}{4p'}, & \text{dilute} \end{cases} \quad \Rightarrow \quad \beta = 2\cos\frac{\pi(p'-p)}{p'}, \qquad 0 < p < p' \quad (p, p' \text{ coprime})$$

- For $\gamma = 0$, the central charge c, conformal weights $\Delta_{r,s}$ and affine $u(1)$ characters are then

$$c = 1 - \frac{6(p-p')^2}{pp'}, \qquad \Delta_{r,s} = \Delta_{r,s}^{p,p'} = \frac{(p'r - ps)^2 - (p-p')^2}{4pp'}$$

$$\varkappa_j^n(q) = \varkappa_j^n(1,q), \quad \varkappa_j^n(z,q) = \frac{\Theta_{j,n}(q,z)}{q^{1/24}(q)_\infty} = \frac{q^{-1/24}}{(q)_\infty}\sum_{k\in\mathbb{Z}} z^k q^{(j+2kn)^2/4n}, \quad (q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$$

at level $n = pp'$ with associated conformal weights

$$\Delta_j^n = \min\left[\frac{j^2}{4n}, \frac{(2n-j)^2}{4n}\right], \qquad j = 0, 1, \ldots, 2n$$

- For $(p, p') = (2, 3)$ or $\lambda = \pi/3$, the dense case is critical bond percolation on the square lattice and the dilute case is critical site percolation on the triangular lattice. For $(p, p') = (1, 2)$ or $\lambda = \pi/2$, these are models of critical dense and dilute polymers on the square lattice.

Scaling Limit of Torus Partition Functions

- **Conjecture:** The scaling limit of the standard module traces are conjectured to be

$$\lim_{\substack{M,N\to\infty\\ M/N\to\delta\\ M\equiv\epsilon \text{ mod } 2}} e^{MNf_{\text{bulk}}(u)}\,\text{tr}_{W_{N,d,\omega}}\boldsymbol{T}(u)^M = \begin{cases} \dfrac{(q\bar q)^{-c/24}}{(q)_\infty(\bar q)_\infty}\displaystyle\sum_{\ell=-\infty}^{\infty}(-1)^{\epsilon\ell}q^{\Delta^{p,p'}_{\gamma/\pi-\ell,d/2}}\bar q^{\Delta^{p,p'}_{\gamma/\pi-\ell,-d/2}}, & \text{dense}\\ \dfrac{(q\bar q)^{-c/24}}{(q)_\infty(\bar q)_\infty}\displaystyle\sum_{\ell=-\infty}^{\infty}q^{\Delta^{p,p'}_{\gamma/\pi-2\ell,d/2}}\bar q^{\Delta^{p,p'}_{\gamma/\pi-2\ell,-d/2}}, & \text{dilute}\end{cases}$$

where $f_{\text{bulk}}(u)$ is a known bulk free energy, q is the modular nome, δ is the aspect ratio and

$$(q)_\infty=\prod_{n=1}^{\infty}(1-q^n),\qquad \omega=e^{i\gamma},\qquad q=e^{2\pi i\tau}=\begin{cases}\exp\left(-2\pi i\delta\, e^{-i\frac{\pi u}{\lambda}}\right), & \text{dense}\\ \exp\left(-2\pi i\delta\, e^{-i\frac{\pi u}{3\lambda}}\right), & \text{dilute}\end{cases}$$

- These (infinite) sesquilinear Verma forms are periodic functions in γ with period 2π.

Taking the scaling limit, we define scaled coefficients and partition functions by

$$\mathcal{C}_{d,j}=\lim_{\substack{M,N\to\infty\\ M/N\to\delta}} e^{MNf_{\text{bulk}(u)}}C_{d,j},\qquad \mathcal{Z}^{(h,v)}=\lim_{\substack{M,N\to\infty\\ M/N\to\delta}} e^{MNf_{\text{bulk}}(u)}Z^{(h,v)}$$

- For both dense and dilute cases, we find (treating $g=\frac{p}{4p'}$ as a continuous variable)

$$\mathcal{C}_{d,j}=\mathcal{Z}_{d,j}\Big(\tfrac{p}{4p'}\Big),\qquad \mathcal{Z}_{m,m'}(g)=\Big(\frac{g}{\tau_i}\Big)^{1/2}\frac{1}{\eta(q)\eta(\bar q)}\exp\Big[-\frac{\pi g}{\tau_i}\big|m\tau-m'\big|^2\Big]$$

$$\mathcal{Z}^{(h,v)}=\mathcal{Z}^{(h,v)}_{\text{dense}}=\mathcal{Z}^{(h,v)}_{\text{dilute}}=\sum_{d\in 2\mathbb{Z}+h}\ \sum_{j\in 2\mathbb{Z}+v} 2\,T_{d\wedge j}(\tfrac{\alpha}{2})\,\mathcal{Z}_{d,j}\Big(\tfrac{p}{4p'}\Big)$$

where $\eta(q)=q^{1/24}(q)_\infty$ is the Dedekind eta function, $\tau_i=\text{Im}\,\tau$ and the functions $\mathcal{Z}_{m,m'}(g)$ are well-known in the Coulomb gas formalism [FSZ87].

Modular Covariance

- Under the action of the modular group with generators $T:\tau\mapsto\tau+1$ and $S:\tau\mapsto-\frac{1}{\tau}$, we find

$$\mathcal{Z}_{d,j}(g,\tau+1)=\mathcal{Z}_{d,j-d}(g,\tau),\qquad \mathcal{Z}_{d,j}(g,-\tfrac{1}{\tau})=\mathcal{Z}_{j,-d}(g,\tau),\qquad q=e^{2\pi i\tau}$$

so the partition functions satisfy

$$\begin{array}{ll}
\mathcal{Z}^{(0,0)}(\tau+1)=\mathcal{Z}^{(0,0)}_{\text{tor}}(\tau), & \mathcal{Z}^{(0,0)}(-\frac{1}{\tau})=\mathcal{Z}^{(0,0)}(\tau)\\
\mathcal{Z}^{(0,1)}(\tau+1)=\mathcal{Z}^{(0,1)}_{\text{tor}}(\tau), & \mathcal{Z}^{(0,1)}(-\frac{1}{\tau})=\mathcal{Z}^{(1,0)}(\tau)\\
\mathcal{Z}^{(1,0)}(\tau+1)=\mathcal{Z}^{(1,1)}_{\text{tor}}(\tau), & \mathcal{Z}^{(1,0)}(-\frac{1}{\tau})=\mathcal{Z}^{(0,1)}(\tau)\\
\mathcal{Z}^{(1,1)}(\tau+1)=\mathcal{Z}^{(1,0)}_{\text{tor}}(\tau), & \mathcal{Z}^{(1,1)}(-\frac{1}{\tau})=\mathcal{Z}^{(1,1)}(\tau)
\end{array}$$

- The fully periodic partition function $\mathcal{Z}^{(0,0)}$ is modular invariant whereas $\mathcal{Z}^{(0,1)}$, $\mathcal{Z}^{(1,0)}$ and $\mathcal{Z}^{(1,1)}$ are covariant under the modular group. Specifically, the action of the generators S and T on the ordered basis $\{\mathcal{Z}^{(0,0)},\mathcal{Z}^{(0,1)},\mathcal{Z}^{(1,0)},\mathcal{Z}^{(1,1)}\}$ yields a four-dimensional representation of the modular group

$$\mathsf{S}=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\quad \mathsf{T}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\qquad \mathsf{S}^2=(\mathsf{ST})^3=I$$

Under this action, T satisfies $\mathsf{T}^2=I$ so here T is also an involution.

Sesquilinear Verma Forms

- The covariant partition functions should be expressible as sesquilinear forms in CFT characters.

For $\frac{\lambda}{\pi} = \frac{p'-p}{p'} \in (0,1)$, we find the sesquilinear form in generic Verma characters $\frac{q^{-c/24}}{(q)_\infty} q^{\Delta^{p,p'}_{r,s}}$

$$\mathcal{Z}^{(h,v)} = \frac{(q\bar{q})^{-c/24}}{(q)_\infty(\bar{q})_\infty}\Big[\delta_{h=0 \bmod 2}\sum_{\ell=-\infty}^{\infty}(q\bar{q})^{\Delta^{p,p'}_{\frac{\gamma}{\pi}-2\ell,0}} + \sum_{d\in\mathbb{Z}\setminus\{0\}}\delta_{d=h \bmod 2}\sum_{m=0}^{2d-1}\Gamma^{(v)}_{m,d}\sum_{\ell=-\infty}^{\infty} q^{\Delta^{p,p'}_{\frac{m}{d}-2\ell,\frac{d}{2}}}\bar{q}^{\Delta^{p,p'}_{\frac{m}{d}-2\ell,-\frac{d}{2}}}\Big]$$

$$\Gamma^{(v)}_{m,d} = \frac{1}{|2d|}\sum_{j=0}^{2d-1} e^{i\pi jm/d}\big[1+(-1)^{j+v}\big]T_{d\wedge j}(\tfrac{\alpha}{2})$$

This holds for general α. Summing over $h,v=0,1$ for the full partition function, this agrees with an equivalent but different expression in [FSZ87]. Proving this requires some number theory (Möbius/Euler totient functions, Möbius inversion formula)!

- For $\alpha = 2$ (zero twist) and λ/π rational, these conformal partition functions simplify to

$$\begin{aligned}
\mathcal{Z}^{(h,v)}\Big|_{\alpha=2} &= \frac{(q\bar{q})^{-c/24}}{(q)_\infty(\bar{q})_\infty}\sum_{\ell\in\mathbb{Z}}\sum_{d\in 2\mathbb{Z}+h}(-1)^{v\ell} q^{\Delta^{p,p'}_{-\ell,d/2}}\bar{q}^{\Delta^{p,p'}_{-\ell,-d/2}} \\
&= \frac{(q\bar{q})^{-c/24}}{(q)_\infty(\bar{q})_\infty}\sum_{r,s\in\mathbb{Z}}(-1)^{vr} q^{\Delta^{p,p'}_{r,s+h/2}}\bar{q}^{\Delta^{p,p'}_{r,-s-h/2}} \\
&= \frac{1}{\eta(q)\eta(\bar{q})}\sum_{r,s-h/2\in\mathbb{Z}}(-1)^{vr} q^{\frac{(p'r-ps)^2}{4pp'}}\bar{q}^{\frac{(p'r+ps)^2}{4pp'}} = \mathcal{Z}^{(h,v)}(\tfrac{p}{p'})
\end{aligned}$$

$$\mathcal{Z}^{(h,v)}(g) = \frac{1}{\eta(q)\eta(\bar{q})}\sum_{r,s-h/2\in\mathbb{Z}}(-1)^{vr} q^{(r/\sqrt{g}-s\sqrt{g})^2/4}\bar{q}^{(r/\sqrt{g}+s\sqrt{g})^2/4}, \qquad h,v=0,1$$

and so involve a generalization of the usual Coulomb partition function $\mathcal{Z}(g) = \mathcal{Z}^{(0,0)}(g)$.

Affine $u(1)$ Sesquilinear Forms

- Let's now specialize to $\alpha = 2$ (zero twist). By shifting r by p and using symmetries, we can write

$$\mathcal{Z}^{(h,v)}(\tfrac{p}{p'}) = \frac{1}{2\eta(q)\eta(\bar{q})} \sum_{r,s-\frac{h}{2}\in\mathbb{Z}} (-1)^{vr} q^{\frac{(p'r-ps)^2}{4pp'}} \Big[\bar{q}^{\frac{(p'r+ps)^2}{4pp'}} + (-1)^{pv} \bar{q}^{\frac{(p'r+ps+2n)^2}{4pp'}} \Big] = \sum_{r=0}^{p-1} \sum_{s=0}^{2p'-1} (-1)^{vr} \mathcal{Z}_{r,s}^{(h,v)}$$

where we prove the large set of identities (valid for all h, v, r, s, p, p')

$$\begin{aligned} \mathcal{Z}_{r,s}^{(h,v)} &:= \frac{1}{\eta(q)\eta(\bar{q})} \sum_{r'\in 2p\mathbb{Z}+r} \; \sum_{s'\in 2p'\mathbb{Z}+s+h/2} q^{\frac{(p'r'-ps')^2}{4pp'}} \Big[\bar{q}^{\frac{(p'r'+ps')^2}{4pp'}} + (-1)^{pv} \bar{q}^{\frac{(p'r'+ps'+2n)^2}{4pp'}} \Big] \\ &= \varkappa^n_{p'r-p(s+\frac{h}{2})}\big((-1)^{pv}, q\big)\, \varkappa^n_{p'r+p(s+\frac{h}{2})}\big((-1)^{pv}, \bar{q}\big), \qquad n = pp' \end{aligned}$$

- It follows that the torus partition functions are sesquilinear forms in affine $u(1)$ characters

$$\mathcal{Z}^{(h,v)}(\tfrac{p}{p'}) = \sum_{r=0}^{p-1} \sum_{s=0}^{2p'-1} (-1)^{vr} \varkappa^n_{p'r-p(s+\frac{h}{2})}\big((-1)^{pv}, q\big)\, \varkappa^n_{p'r+p(s+\frac{h}{2})}\big((-1)^{pv}, \bar{q}\big)$$

where we note that the decomposition into affine $u(1)$ characters at level $4n = 4pp'$

$$\varkappa^n_j(\pm 1, q) = \varkappa^{4n}_{2j}(q) \pm \varkappa^{4n}_{4n-2j}(q)$$

Affine $u(1)$ Sesquilinear Forms (Zero Twist)

- In the examples below, we fix the notation

$$\varkappa_j^{n,\pm}(q) = \varkappa_j^n(q) \pm \varkappa_{n-j}^n(q)$$

- $(p,p') = (1,2)$, $c = -2$, critical dense/dilute polymers [MDKP17]:

$$\begin{aligned}
\mathcal{Z}^{(0,0)}(\tfrac{1}{2}) &= |\varkappa_0^2(q)|^2 + 2|\varkappa_1^2(q)|^2 + |\varkappa_2^2(q)|^2 = |\varkappa_0^{8,+}(q)|^2 + 2|\varkappa_2^{8,+}(q)|^2 + |\varkappa_4^{8,+}(q)|^2,\\
\mathcal{Z}^{(0,1)}(\tfrac{1}{2}) &= |\varkappa_0^2(-1,q)|^2 + 2|\varkappa_1^2(-1,q)|^2 + |\varkappa_2^2(-1,q)|^2 = |\varkappa_0^{8,-}(q)|^2 + 2|\varkappa_2^{8,-}(q)|^2 + |\varkappa_4^{8,-}(q)|^2,\\
\mathcal{Z}^{(1,0)}(\tfrac{1}{2}) &= 2|\varkappa_{1/2}^2(q)|^2 + 2|\varkappa_{3/2}^2(q)|^2 = 2|\varkappa_1^{8,+}(q)|^2 + 2|\varkappa_3^{8,+}(q)|^2,\\
\mathcal{Z}^{(1,1)}(\tfrac{1}{2}) &= 2|\varkappa_{1/2}^2(-1,q)|^2 + 2|\varkappa_{3/2}^2(-1,q)|^2 = 2|\varkappa_1^{8,-}(q)|^2 + 2|\varkappa_3^{8,-}(q)|^2.
\end{aligned}$$

- $(p,p') = (1,3)$ with $c = -7$

$$\begin{aligned}
\mathcal{Z}^{(0,0)}(\tfrac{1}{3}) &= |\varkappa_0^3(q)|^2 + 2|\varkappa_1^3(q)|^2 + 2|\varkappa_2^3(q)|^2 + |\varkappa_3^3(q)|^2\\
&= |\varkappa_0^{12,+}(q)|^2 + 2|\varkappa_2^{12,+}(q)|^2 + 2|\varkappa_4^{12,+}(q))|^2 + |\varkappa_6^{12,+}(q)|^2,\\
\mathcal{Z}^{(0,1)}(\tfrac{1}{3}) &= |\varkappa_0^3(-1,q)|^2 + 2|\varkappa_1^3(-1,q)|^2 + 2|\varkappa_2^3(-1,q)|^2 + |\varkappa_3^3(-1,q)|^2\\
&= |\varkappa_0^{12,-}(q)|^2 + 2|\varkappa_2^{12,-}(q)|^2 + 2|\varkappa_4^{12,-}(q))|^2 + |\varkappa_6^{12,-}(q)|^2,\\
\mathcal{Z}^{(1,0)}(\tfrac{1}{3}) &= 2|\varkappa_{1/2}^3(q)|^2 + 2|\varkappa_{3/2}^3(q)|^2 + 2|\varkappa_{5/2}^3(q)|^2\\
&= 2|\varkappa_1^{12,+}(q)|^2 + 2|\varkappa_3^{12,+}(q)^2 + 2|\varkappa_5^{12,+}(q)|^2,\\
\mathcal{Z}^{(1,1)}(\tfrac{1}{3}) &= 2|\varkappa_{1/2}^3(-1,q)|^2 + 2|\varkappa_{3/2}^3(-1,q)|^2 + 2|\varkappa_{5/2}^3(-1,q)|^2\\
&= 2|\varkappa_1^{12,-}(q)|^2 + 2|\varkappa_3^{12,-}(q)^2 + 2|\varkappa_5^{12,-}(q)|^2.
\end{aligned}$$

- $(p,p')=(2,3)$, $c=0$, critical bond [MDKP17] and critical site percolation [MDKP23]:

$$\begin{aligned}
\mathcal{Z}^{(0,0)}(\tfrac{2}{3}) &= |\varkappa_0^6(q)|^2 + 2\varkappa_1^6(q)\varkappa_5^6(\bar q) + 2|\varkappa_2^6(q)|^2 + 2|\varkappa_3^6(q)|^2 + 2|\varkappa_4^6(q)|^2 + 2\varkappa_5^6(q)\varkappa_1^6(\bar q) + |\varkappa_6^6(q)|^2 \\
&= |\varkappa_0^{24,+}(q)|^2 + 2\varkappa_2^{24,+}(q)\varkappa_{10}^{24,+}(\bar q) + 2|\varkappa_4^{24,+}(q)|^2 + 2|\varkappa_6^{24,+}(q)|^2 + 2|\varkappa_8^{24,+}(q)|^2 \\
&\qquad + 2\varkappa_{10}^{24,+}(q)\varkappa_2^{24,+}(\bar q) + |\varkappa_{12}^{24,+}(q)|^2, \\
\mathcal{Z}^{(0,1)}(\tfrac{2}{3}) &= |\varkappa_0^6(q)|^2 - 2\varkappa_1^6(q)\varkappa_5^6(\bar q) + 2|\varkappa_2^6(q)|^2 - 2|\varkappa_3^6(q)|^2 + 2|\varkappa_4^6(q)|^2 - 2\varkappa_5^6(q)\varkappa_1^6(\bar q) + |\varkappa_6^6(q)|^2 \\
&= |\varkappa_0^{24,+}(q)|^2 - 2\varkappa_2^{24,+}(q)\varkappa_{10}^{24,+}(\bar q) + 2|\varkappa_4^{24,+}(q)|^2 - 2|\varkappa_6^{24,+}(q)|^2 + 2|\varkappa_8^{24,+}(q)|^2 \\
&\qquad - 2\varkappa_{10}^{24,+}(q)\varkappa_2^{24,+}(\bar q) + |\varkappa_{12}^{24,+}(q)|^2, \\
\mathcal{Z}^{(1,0)}(\tfrac{2}{3}) &= \varkappa_0^6(q)\varkappa_6^6(\bar q) + 2|\varkappa_1^6(q)|^2 + 2\varkappa_2^6(q)\varkappa_4^6(\bar q) + 2|\varkappa_3^6(q)|^2 + 2\varkappa_4^6(q)\varkappa_2^6(\bar q) + 2|\varkappa_5^6(q)|^2 \\
&\qquad + \varkappa_6^6(q)\varkappa_0^6(\bar q) \\
&= \varkappa_0^{24,+}(q)\varkappa_{12}^{24,+}(\bar q) + 2|\varkappa_2^{24,+}(q)|^2 + 2\varkappa_4^{24,+}(q)\varkappa_8^{24,+}(\bar q) + 2|\varkappa_6^{24,+}(q)|^2 \\
&\qquad + 2\varkappa_8^{24,+}(q)\varkappa_4^{24,+}(\bar q) + 2|\varkappa_{10}^{24,+}(q)|^2 + \varkappa_{12}^{24,+}(q)\varkappa_0^{24,+}(\bar q), \\
\mathcal{Z}^{(1,1)}(\tfrac{2}{3}) &= -\varkappa_0^6(q)\varkappa_6^6(\bar q) + 2|\varkappa_1^6(q)|^2 - 2\varkappa_2^6(q)\varkappa_4^6(\bar q) + 2|\varkappa_3^6(q)|^2 - 2\varkappa_4^6(q)\varkappa_2^6(\bar q) + 2|\varkappa_5^6(q)|^2 \\
&\qquad - \varkappa_6^6(q)\varkappa_0^6(\bar q) \\
&= -\varkappa_0^{24,+}(q)\varkappa_{12}^{24,+}(\bar q) + 2|\varkappa_2^{24,+}(q)|^2 - 2\varkappa_4^{24,+}(q)\varkappa_8^{24,+}(\bar q) + 2|\varkappa_6^{24,+}(q)|^2 \\
&\qquad - 2\varkappa_8^{24,+}(q)\varkappa_4^{24,+}(\bar q) + 2|\varkappa_{10}^{24,+}(q)|^2 - \varkappa_{12}^{24,+}(q)\varkappa_0^{24,+}(\bar q).
\end{aligned}$$

- The modular invariants $\mathcal{Z}^{(0,0)}(\frac{p}{p'})$ agree with the conjectured sesquilinear forms of Pearce-Rasmussen [PR2011] by fixing the undetermined integers therein to $n_{p,p'}=-1$ for all p,p'.

- For the triplet model (cf. David's talk), $n_{p,p'}=2$ for all p,p'.

Bezout Conjugates

- The affine $u(1)$ characters have the periodicity

$$\varkappa^n_{j+P}\big((-1)^{pv},q\big)=\varkappa^n_j\big((-1)^{pv},q\big),\qquad P=\begin{cases}2n, & pv \text{ even}\\ 4n, & pv \text{ odd}\end{cases}\qquad n=pp'$$

- The Bezout conjugates $j+h'/2$ and $\overline{j+h'/2}$ are independent of v and defined by

$$j+\tfrac{h'}{2}=p'r-p(s+\tfrac{h}{2}) \bmod P,\qquad \overline{j+\tfrac{h'}{2}}=p'r+p(s+\tfrac{h}{2}) \bmod P$$

$$h'=\chi_p h=\begin{cases}1, & h=1,\ p \text{ odd}\\ 0, & \text{otherwise}\end{cases}\qquad \chi_p=\begin{cases}1, & p \text{ odd}\\ 0, & p \text{ even}\end{cases}$$

These are integers for $h'=0$, half-integers for $h'=1$. The Bezout construction gives a bijection (for $h'=0$ or 1) between the set of Kac labels $\mathbb{K}$ and the cyclic set of $u(1)$ indices $\mathbb{U}_{h'}$

$$\mathbb{K}=\{(r,s)\in\mathbb{Z}^2: 0\le r\le p-1,\ 0\le s\le \tfrac{P}{p}-1\}\ \leftrightarrow\ \mathbb{U}_{h'}=\{j+\tfrac{h'}{2}\in\mathbb{Z}+\tfrac{h'}{2}: 0\le j+\tfrac{h'}{2}<P\}$$

- The zero-twist modular covariant partition functions can then be written in the $u(1)$ form

$$\mathcal{Z}^{(h,v)}(\tfrac{p}{p'})=\begin{cases}\displaystyle\sum_{j=0}^{2n-1}(-1)^{vr(j)}\,\varkappa^n_{j+h'/2}(q)\varkappa^n_{\overline{j+h'/2}}(\bar q), & pv \text{ even}\\ \displaystyle\frac{1}{2}\sum_{j=0}^{4n-1}(-1)^{vr(j)}\,\varkappa^n_{j+h/2}(-1,q)\varkappa^n_{\overline{j+h/2}}(-1,\bar q), & pv \text{ odd}\end{cases}$$

with parity $r(j)=\big(j+h/2+\overline{j+h/2}\big)/(2p') \bmod 2$.

Kac Table of Bezout Conjugates

s \ r	0	1	2	3	4	5	6
8	0,0	4,4	8,8	12,12	16,16	20,20	0,0
7	3,21	7,1	11,5	15,9	19,13	23,7	3,21
6	6,18	10,22	14,2	18,6	22,10	2,14	6,18
5	9,15	13,19	17,23	21,3	1,7	5,11	9,15
4	12,12	16,16	20,20	0,0	4,4	8,8	12,12
3	15,9	19,13	23,17	3,21	7,1	11,5	15,9
2	18,6	22,10	2,14	6,18	10,22	14,2	18,6
1	21,3	1,7	5,11	9,15	13,19	17,23	21,3
0	0,0	4,4	8,8	12,12	16,16	20,20	0,0

$s+\frac{1}{2}$ \ r	0	1	2	3	4	5	6
$\frac{17}{2}$	$\frac{45}{2},\frac{51}{2}$	$\frac{53}{2},\frac{59}{2}$	$\frac{61}{2},\frac{67}{2}$	$\frac{69}{2},\frac{75}{2}$	$\frac{77}{2},\frac{83}{2}$	$\frac{85}{2},\frac{91}{2}$	$\frac{93}{2},\frac{3}{2}$
$\frac{15}{2}$	$\frac{51}{2},\frac{45}{2}$	$\frac{59}{2},\frac{53}{2}$	$\frac{67}{2},\frac{61}{2}$	$\frac{75}{2},\frac{69}{2}$	$\frac{83}{2},\frac{77}{2}$	$\frac{91}{2},\frac{85}{2}$	$\frac{3}{2},\frac{93}{2}$
$\frac{13}{2}$	$\frac{57}{2},\frac{39}{2}$	$\frac{65}{2},\frac{47}{2}$	$\frac{73}{2},\frac{55}{2}$	$\frac{81}{2},\frac{63}{2}$	$\frac{89}{2},\frac{71}{2}$	$\frac{1}{2},\frac{79}{2}$	$\frac{9}{2},\frac{87}{2}$
$\frac{11}{2}$	$\frac{63}{2},\frac{33}{2}$	$\frac{71}{2},\frac{41}{2}$	$\frac{79}{2},\frac{49}{2}$	$\frac{87}{2},\frac{57}{2}$	$\frac{95}{2},\frac{65}{2}$	$\frac{7}{2},\frac{73}{2}$	$\frac{15}{2},\frac{81}{2}$
$\frac{9}{2}$	$\frac{69}{2},\frac{27}{2}$	$\frac{77}{2},\frac{35}{2}$	$\frac{85}{2},\frac{43}{2}$	$\frac{93}{2},\frac{51}{2}$	$\frac{5}{2},\frac{59}{2}$	$\frac{13}{2},\frac{67}{2}$	$\frac{21}{2},\frac{75}{2}$
$\frac{7}{2}$	$\frac{75}{2},\frac{21}{2}$	$\frac{83}{2},\frac{29}{2}$	$\frac{91}{2},\frac{37}{2}$	$\frac{3}{2},\frac{45}{2}$	$\frac{11}{2},\frac{53}{2}$	$\frac{19}{2},\frac{61}{2}$	$\frac{27}{2},\frac{69}{2}$
$\frac{5}{2}$	$\frac{81}{2},\frac{15}{2}$	$\frac{89}{2},\frac{23}{2}$	$\frac{1}{2},\frac{31}{2}$	$\frac{9}{2},\frac{39}{2}$	$\frac{17}{2},\frac{47}{2}$	$\frac{25}{2},\frac{55}{2}$	$\frac{33}{2},\frac{63}{2}$
$\frac{3}{2}$	$\frac{87}{2},\frac{9}{2}$	$\frac{95}{2},\frac{17}{2}$	$\frac{7}{2},\frac{25}{2}$	$\frac{15}{2},\frac{33}{2}$	$\frac{23}{2},\frac{41}{2}$	$\frac{31}{2},\frac{49}{2}$	$\frac{39}{2},\frac{57}{2}$
$\frac{1}{2}$	$\frac{93}{2},\frac{3}{2}$	$\frac{5}{2},\frac{11}{2}$	$\frac{13}{2},\frac{19}{2}$	$\frac{21}{2},\frac{27}{2}$	$\frac{29}{2},\frac{35}{2}$	$\frac{37}{2},\frac{43}{2}$	$\frac{45}{2},\frac{51}{2}$

- Kac tables of Bezout conjugates $\{j,\bar{j}\}\big|_{h=0}$ and $\{j+\frac{1}{2},\overline{j+\frac{1}{2}}\}\big|_{h=1}$ for $(p,p')=(3,4)$. The periodicity is $P=2n$ in the left panel and $P=4n$ in the right panel. Only Bezout conjugates within the framed box in the lower left contribute to the modular covariant partition functions.

Checks on Conjectured Standard Module Traces

- For critical bond percolation (dense case with $(p,p')=(2,3)$ and $\gamma=0$), the conjecture can be confirmed because all of the eigenvalues are known analytically [MDKP2017].

- For critical site percolation (dilute case with $(p,p')=(2,3)$ and $\gamma=0,\pi$), the conjecture can be checked against the 162 leading eigenvalues obtained numerically [MDKP2023] by solving the logarithmic form of the Bethe ansatz equations.

- Indeed, for $(p,p')=(2,3)$ our conjecture agrees, in both dense and dilute cases, with our previous results for the four twisted conformal partition functions.

- Our conjecture, results from simply replacing $\Delta_{r,s}^{2,3}\mapsto\Delta_{r,s}^{p,p'}$. In the dense cases, for general (p,p'), our results agree with those of [PasquierSaleur90] based on mappings.

- For the "full" partition function (obtained by summing over $h,v=0,1$), with general γ, our conjecture leads to an infinite sesquilinear Verma form that (after some nontrivial number theory!) agrees with the result of [FSZ87].

- In all cases, our conjecture leads to modular covariant partition functions.

- For $\gamma=0$, the modular invariant partition functions $\mathcal{Z}^{(0,0)}(\frac{p}{p'})$ agree with the conjectured affine $u(1)$ sesquilinear forms of Pearce-Rasmussen [PR2011] with $n_{p,p'}=-1$ for all p,p'.

Conclusion

- Based on our key conjecture, exact expressions are proposed for the modular covariant partition functions of critical dense $A_1^{(1)}$ and dilute $A_2^{(2)}$ loop models on the square lattice. These are expressed as sesquilinear forms in (i) generic Verma characters for $\frac{\lambda}{\pi}\in(0,1)$ with general twists $\gamma\in\mathbb{R}$ and (ii) affine $u(1)$ characters for $\frac{\lambda}{\pi}\in\mathbb{Q}$ with zero twist ($\gamma=0$).

- These results extend the known results for the modular invariant partition function, based on Coulomb methods, to the full set of four combinations of periodic/antiperiodic boundary conditions with $h, v = 0, 1$. For $\gamma = 0$, the results involve generalized (half-integer) Bezout conjugate pairs.

- All these results apply equally to the 6-vertex and Izergin-Korepin 19-vertex models.

- Despite being described by $sl(2)$ and $sl(3)$ models respectively, all of the conformal data and partition functions of these dense and dilute loop models precisely coincide. This implies a very strong form of universality between these critical dense and dilute loop models as log CFTs.

- Specifically, these results imply a very strong form of universality between critical bond percolation on the square lattice [MDKP2017] and critical site percolation on the triangular lattice [MDKP2023] as log CFTs with $c=0$.

- Similarly, the coincidence of all these conformal results imply a very strong form of *universality* between critical dense [PR2007,PRV2010] and dilute polymers on the square lattice as log CFTs with $c=-2$.

Thank you for your attention!

Photo courtesy of Patrick Dorey!