A variety of partially solvable models: From closed spin chains to open spin chains

[Phys. Rev. B 109, 104307 (2024), Coming soon, Work in progress]

MPI2024 @Creswick, Jul 1-19

Chihiro Matsui (The University of Tokyo)

Collaborators: Naoto Tsuji (The University of Tokyo)

Hosho Katsura (The University of Tokyo)

Chiara Paletta (Ljubljana University)

Balazs Pozsgay (Eötvös Loránd University Budapest)

Outline

- What is “partial solvability”?
  - Definition of partial solvability
  - Thermalization & quantum many-body scars (QMBS)
- Closed partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Open partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Concluding remarks

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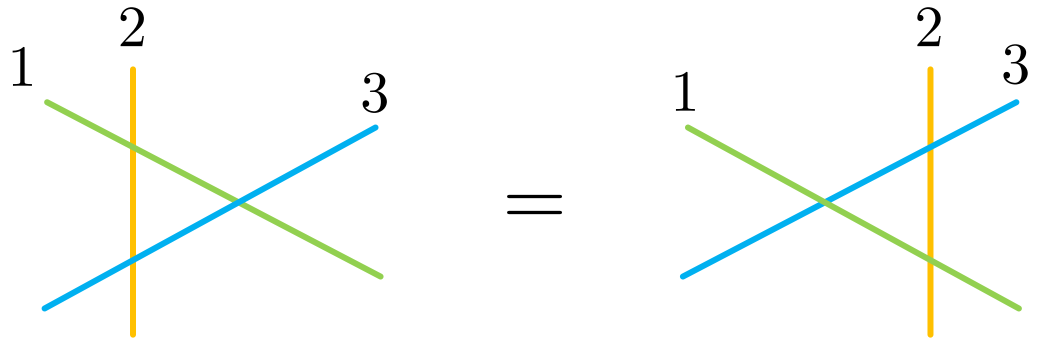
Integrability

- (Quantum) Integrable systems
  - No clear definition.
  - Often said to be "integrable" if the Yang-Baxter structure exists.

$$R_{12}(\lambda_1,\lambda_2)R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3) = R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3)R_{12}(\lambda_1,\lambda_2)$$

$$R_{12}, R_{23}, R_{13} \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$$



Commuting transfer matrices,
Many conserved charges,
Exact correlation functions & Form factors, etc.

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$$R_{12}, R_{23}, R_{13} \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$$

  - Never thermalize.

⇒ Violation of (strong) eigenstate thermalization hypothesis (ETH) ≒ Typicality

$$\lim_{N\to\infty} \langle E_a|X_{\mathrm{macro}}|E_a\rangle = \langle X_{\mathrm{macro}}\rangle_{\mathrm{MC}}, \quad \forall E_a \in (E-\delta E, E]$$

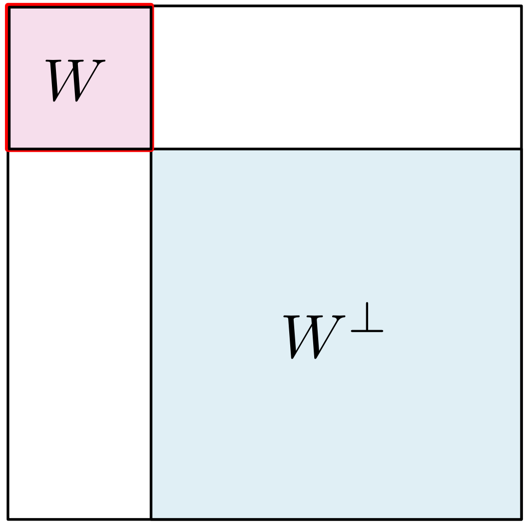
[Deutsch (1991), Srednicki (1994)]
[Iyoda et al. (2017)]

Partial Solvability

- Partially solvable systems
  - Hamiltonians with some solvable energy eigenstates (not all).
  - Hamiltonians with the block diagonal structure.

$$\mathcal{H} \simeq W \oplus W^{\perp}$$

Solvable (invariant) subspace



⇒ Solvability does not necessarily come from integrability.

e.g. Projector embeddings [Shiraishi et al. (2017)]
Restricted spectrum generating algebra (rSGA) [Moudgalya et al. (2018)]
[Vefek et al. (2017), Moudgalya et al. (2020)]
Hilbert space fragmentation (HSF) [Pai et al. (2019), Sala et al. (2020), Khemani et al. (2020)]

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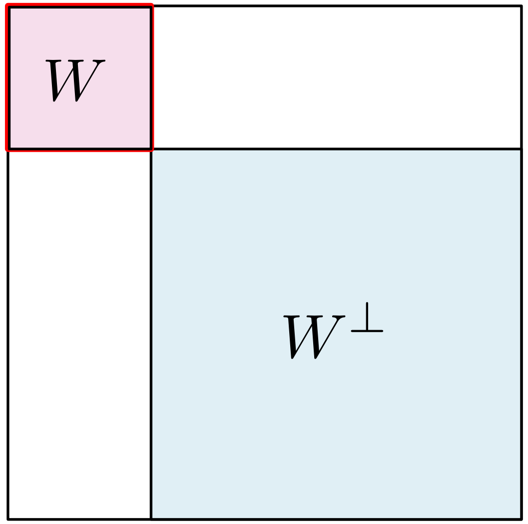
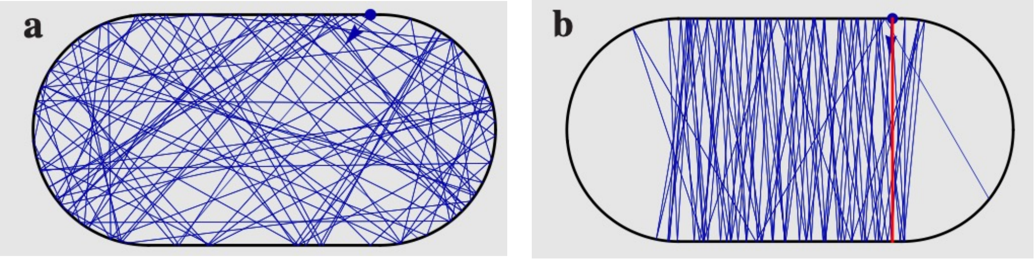


[Shiraishi et al. (2017), Turner et al. (2017), Bernien et al. (2017)]

  - Partially non-thermalize. “Quantum many-body scars (QMBS)”

    ⇒ Weakly violate ergodicity in Hilbert space. ≒ Principle of equal probability

e.g. Scar in stadium billiard [Serbyn et al. (2021)]



a. starting away from unstable periodic trajectory

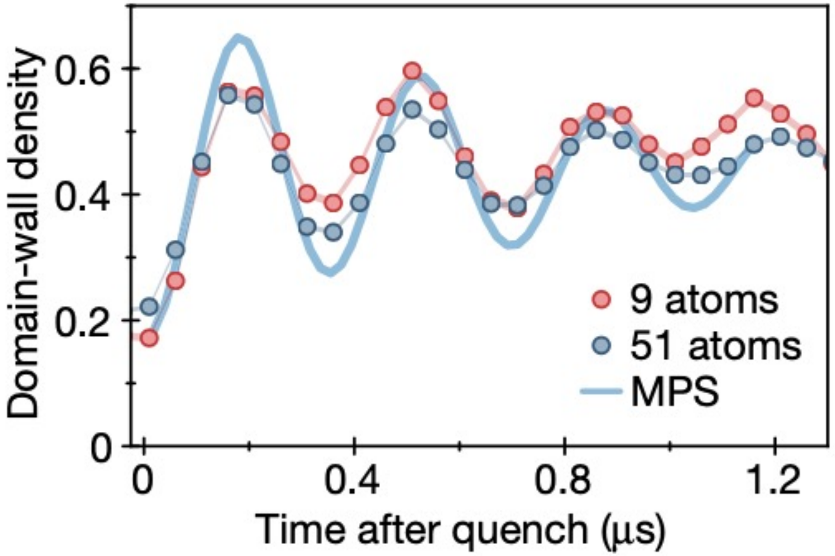
b. starting from near unstable periodic trajectory

Thermalization & QMBS

- QMBS exhibit
  - Persistent oscillation in local observables.
  - Relatively small entanglement entropy $\sim o(V)$ compared to those of thermal states $\sim O(V)$.
- Matrix product states
  - Have entanglement entropy estimated by their bond dimensions χ from above.

$$S_{\mathrm{EE}} := -\mathrm{tr}\left(\rho' \log \rho'\right) \leq \log \chi$$

  - With a finite bond dimension is a good benchmark for finding QMBS.



Domain-wall density after the quench from $|\mathbb{Z}_2\rangle = |\bullet \circ \bullet \circ \cdots\rangle$ on the Rydberg atom chain.

(●: Excited state; ○: Ground state)

[Bernien et al. (2017); Nature 551, 579 (Fig. 6b)]

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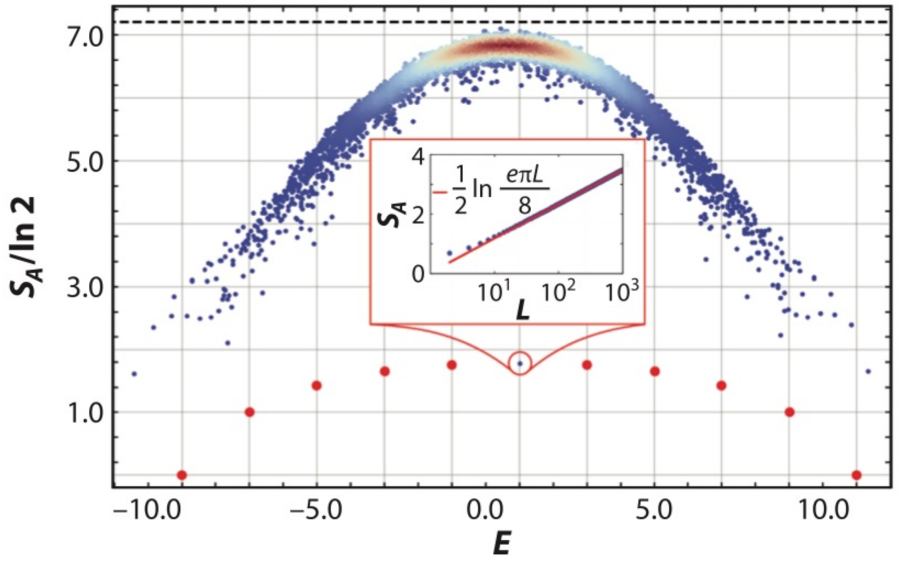

Half-chain entanglement entropy of the spin-1 XY model in zero-magnetization sector.

[Chandran et al. (2003); Ann. Rev. 14, 443 (Fig. 1c)]

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Restricted Spectrum Generating Algebra

- Restricted spectrum-generating algebra (rSGA) [Arno et al. (1988), Yang (1989)] [Moudgalya et al. (2018)]
  - Partial dynamical symmetry

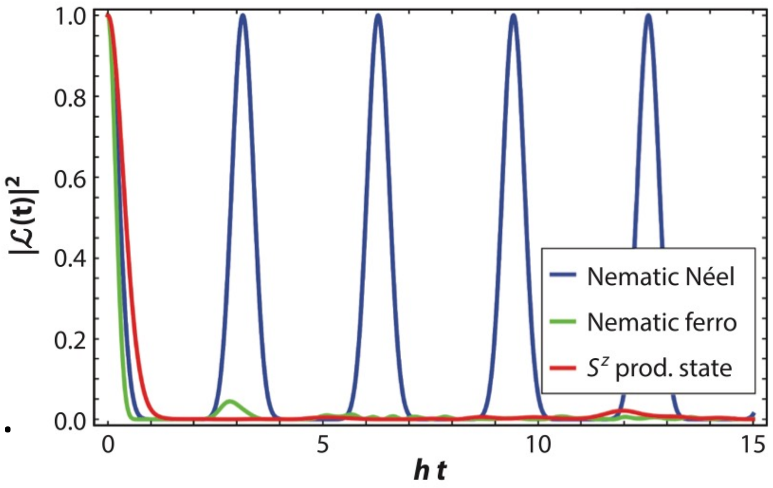
$$\exists Q, \quad \text{s.t.}\, [H,\, Q] - \mathcal{E}Q\big|_W = 0, \quad W \subset \mathcal{H}, \quad Q : \text{Local operator}$$

  - The solvable subspace is systematically constructed if $|\psi_0\rangle$ is an energy eigenstate:

$$H|\psi_0\rangle = \mathcal{E}_0|\psi_0\rangle \Rightarrow HQ^n|\psi_0\rangle = (\mathcal{E}_0 + n\mathcal{E})|\psi_0\rangle$$

  - $W = \mathrm{span}\,\{Q^n|\psi_0\rangle\}_n$ is the invariant subspace of H.

Strong revivals observed in dynamics of Loschmidt echo for the spin-1 XY from each initial state.



[Chandran et al. (2023); Ann. Rev. 14, 443 (Fig. 1d)]

Restricted Spectrum Generating Algebra

- Simple example: free fermion model

$$H = \sum_k \Lambda_k \eta_k^\dagger \eta_k$$

$$\{\eta_k, \eta_\ell^\dagger\} = \delta_{k.\ell}, \quad \{\eta_k, \eta_\ell\} = \{\eta_k^\dagger, \eta_\ell^\dagger\} = 0$$

No SGA in the entire Hilbert space
But holds in the subspace.
⇒ “Restricted” spectrum generating algebra (rSGA)

- “Spectrum generating algebra” (SGA)

$$[H, \eta_k^\dagger] = \Lambda_k \eta_k^\dagger$$

⇒ Provides the spectrum for a tower of states

$$\{|\mathrm{vac}\rangle, \eta_{k_1}^\dagger|\mathrm{vac}\rangle, \eta_{k_2}^\dagger\eta_{k_1}^\dagger|\mathrm{vac}\rangle, \ldots\}$$

- Energy eigenstates

$$H\eta_{k_1}^\dagger \cdots \eta_{k_n}^\dagger|\mathrm{vac}\rangle = (\Lambda_{k_1} + \cdots + \Lambda_{k_n})\eta_{k_1}^\dagger \cdots \eta_{k_n}^\dagger|\mathrm{vac}\rangle$$

Restricted Spectrum Generating Algebra

- Example of rSGA: perturbed spin-1 XY model [Schecter et al. (2019)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1} \;\in \mathrm{End}((\mathbb{C}^3)^N), \quad \mathbb{C}^3 = \mathrm{span}\{|0\rangle, |1\rangle, |2\rangle\}$$

$$h = \frac{J}{2}(S^x \otimes S^x + S^y \otimes S^y) + \frac{m}{2}(S^z \otimes \mathbf{1} + \mathbf{1} \otimes S^z)$$

  - Spin-1 operators

$$S^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^y = \frac{1}{\sqrt{2}i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

  - Non-integrable spin-1 chain. (Not all energy eigenstates are solvable.)

Restricted Spectrum Generating Algebra

- Example of rSGA: perturbed spin-1 XY model

  [Schecter et al. (2019)]

  - Trivial energy eigenstate

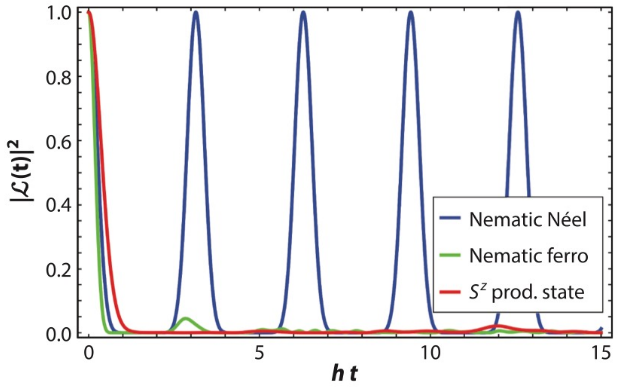
    $$H|\Omega\rangle = -hN|\Omega\rangle, \quad |\Omega\rangle = |22\dots2\rangle$$

  - Subspace of quasiparticle (bimagnon) excitations

    $$W = \mathrm{span}\{(Q^\dagger)^n|\Omega\rangle\}_n, \quad Q^\dagger = \sum_{x=1}^{N}(-1)^x(S_x^+)^2$$

    is the solvable subspace due to the spectrum generating algebra

    $$[H, Q^\dagger] - 2mQ^\dagger\Big|_W = 0.$$



[Chandran et al. (2023); Ann. Rev. 14, 443 (Fig. 1d)]

Energy

$\vdots$

$$H(Q^\dagger)^3|\Omega\rangle = (6m - hN)(Q^\dagger)^3|\Omega\rangle$$

$$H(Q^\dagger)^2|\Omega\rangle = (4m - hN)(Q^\dagger)^2|\Omega\rangle$$

$$HQ^\dagger|\Omega\rangle = (2m - hN)Q^\dagger|\Omega\rangle$$

$$H|\Omega\rangle = -hN|\Omega\rangle$$

Restricted Spectrum Generating Algebra

- Example of rSGA: AKLT-type model [Moudgalya et al. (2018), CM (2023)]

$$H=\sum_{x=1}^{N}\mathbf{1}\otimes\cdots\otimes\underset{x,x+1}{h}\otimes\cdots\otimes\mathbf{1}\ \in\mathrm{End}((\mathbb{C}^3)^N),\quad \mathbb{C}^3=\mathrm{span}\{|0\rangle,|1\rangle,|2\rangle\}$$

$$\begin{aligned}h=&\ \alpha(|00\rangle\langle 00|+|22\rangle\langle 22|)+\beta|11\rangle\langle 11|\\&+\frac{\alpha}{2}\sum_{a\in\{0,2\}}(\gamma|a1\rangle\langle a1|+|a1\rangle\langle 1a|+|1a\rangle\langle a1|+\gamma|1a\rangle\langle 1a|)\\&+\omega^2\beta(|02\rangle\langle 02|+|02\rangle\langle 20|+|20\rangle\langle 02|+|20\rangle\langle 20|)\\&-\omega\beta(|02\rangle\langle 11|+|11\rangle\langle 02|+|11\rangle\langle 20|+|20\rangle\langle 11|)\end{aligned}$$

: ALKT at $\frac{\beta}{\alpha}=\frac{2}{3},\ \gamma=1,\ \omega=-\frac{1}{2}.$

- Non-integrable spin-1 chain. (Not all energy eigenstates are solvable.)
- The ground state and some excitation states were known to be solvable for AKLT.

[Affleck et al. (1987), Arovas (1989)]

Restricted Spectrum Generating Algebra

- Example of rSGA: AKLT-type model [Moudgalya et al. (2018), CM (2023)]
  - The exact zero-energy state is written by the matrix product state:

$$|\psi_A\rangle = \sum_{\{m_1,\ldots,m_N\}\in\{0,1,2\}^N} \mathrm{tr}_a(K_a A_{m_1} A_{m_2} \cdots A_{m_N})\,|m_1, m_2, \ldots, m_N\rangle \in (\mathbb{C}^3)^N$$

$$= \mathrm{tr}_a(K_a \vec{A} \otimes_p \vec{A} \otimes_p \cdots \otimes_p \vec{A})$$

$$\vec{A} = \begin{pmatrix} a_0\sigma^+ \\ a_1\sigma^z \\ a_2\sigma^- \end{pmatrix}, \quad a_1^2/a_0a_2 = \omega,\ a_0, a_1, a_2 \in \mathbb{C}, \quad \sigma^+, \sigma^z, \sigma^-$$: Pauli matrices

  - $K_a \in \mathrm{End}(\mathbb{C}^2)$ is determined by the boundary condition.
    ($K_a = \mathbf{1}_a$ for the periodic boundary; $\mathrm{rank}\, K_a = 1$ for an open boundary)

Restricted Spectrum Generating Algebra

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$$|\psi_A\rangle = \mathrm{tr}_a(\vec{A} \otimes_p \vec{A} \otimes_p \cdots \otimes_p \vec{A})$$

$$\Leftarrow\ h(\vec{A} \otimes_p \vec{A}) = \vec{A}' \otimes_p \vec{A} - \vec{A} \otimes_p \vec{A}'$$

: Local divergence condition/
Frustration-free condition for $\vec{A}' = 0$

  - Quasiparticle-picture for the excitation states:

$$|\psi_{A,B^n}\rangle = (Q^\dagger)^n |\psi_A\rangle, \quad Q^\dagger := \sum_{x=1}^{N} e^{i\pi x}(S_x^+)^2$$

: Creates a quasiparticle with momentum π.

$$= \sum_{x_1,\ldots,x_n} e^{i\pi \sum_{j=1}^{n} x_j} \mathrm{tr}_a(\vec{A} \otimes_p \cdots \otimes_p \underset{x_1}{\vec{B}} \otimes_p \cdots \otimes_p \underset{x_n}{\vec{B}} \otimes_p \cdots \otimes_p \vec{A}), \quad \vec{B} := (S^+)^2 \vec{A}$$

$$\Leftarrow h(\vec{B} \otimes_p \vec{A} + e^{i\pi} \vec{A} \otimes_p \vec{B}) = \frac{\mathcal{E}}{2}(\vec{B} \otimes_p \vec{A} + e^{i\pi} \vec{A} \otimes_p \vec{B}) - (\vec{B} \otimes \vec{A}' - e^{i\pi} \vec{A}' \otimes_p \vec{B})$$

$$\vec{B} \otimes_p \vec{B} = 0, \quad (S^+)^2 \vec{B} = 0$$

: No double/adjacent occupations are allowed.

Restricted Spectrum Generating Algebra

- Example of rSGA: AKLT-type model [Moudgalya et al. (2018), CM (2023)]
  - Restricted spectrum-generating algebra:

$$\left.[H, Q^\dagger] - \mathcal{E}Q^\dagger\right|_W = 0$$

$$W = \mathrm{span}\left\{|\psi_A\rangle, Q^\dagger|\psi_A\rangle, (Q^\dagger)^2|\psi_A\rangle, \ldots, (Q^\dagger)^{\lfloor \frac{N}{2} \rfloor}|\psi_A\rangle\right\}$$

  - Embedded equally-spaced energy spectrum

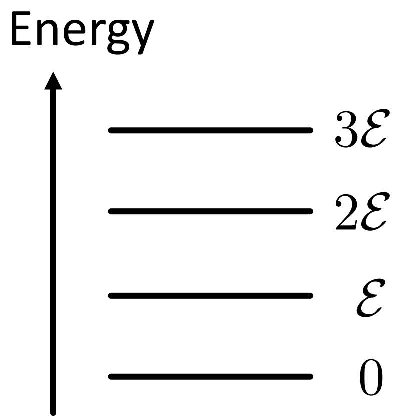
$$H|\psi_A\rangle = 0$$

$$HQ^\dagger|\psi_A\rangle = \mathcal{E}Q^\dagger|\psi_A\rangle$$

: Embedded equally-spaced spectrum
⇒ Identical & non-interacting quasiparticles

$$\vdots$$

$$H(Q^\dagger)^n|\psi_A\rangle = n\mathcal{E}(Q^\dagger)^n|\psi_A\rangle$$



Beyond rSGA

- Generalization of the AKLT-type model [CM (2023)]
  - Quasiparticle-excitation states

$$(Q^\dagger)^n|\psi_A\rangle, \quad Q^\dagger := \sum_{x=1}^{N} e^{i\pi x}(S_x^+)^2$$: Carrying momentum π

$$\Downarrow$$

$$Q^\dagger(k) := \sum_{x=1}^{N} e^{ikx}(S_x^+)^2$$: Carrying momentum k

  - Repulsive property is lost.

$$\vec{B} \otimes_p \vec{B} = 0, \quad (S^+)^2\vec{B} \neq 0$$

Beyond rSGA

- Deformed XXC model [CM (2023)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1}$$

$$h = \sum_{a \in \{0,2\}} \underbrace{(|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|)}_{\text{XXC model (integrable)}} + \alpha(|00\rangle\langle 00| + |22\rangle\langle 22|)$$

- $W = \mathrm{span}\,\{|\psi_{A,B^n}\rangle\}_n$ is solvable subspace of H.

$$|\psi_{A,B^n}\rangle = \sum_{1 \le x_1 < \cdots < x_n \le N} \sum_{P \in \mathfrak{S}_n} A_n(P)\, e^{i \sum_{j=1}^{n} k_{P(j)} x_j}\, \mathrm{tr}_a(\underset{|\uparrow\rangle}{\vec{A}} \otimes_p \cdots \otimes_p \underset{x_1}{\underset{|\downarrow\rangle}{\vec{B}}} \otimes_p \cdots \otimes_p \underset{x_n}{\underset{|\downarrow\rangle}{\vec{B}}} \otimes_p \cdots \otimes_p \underset{|\uparrow\rangle}{\vec{A}})$$

: Bethe-like state

$$e^{ik_j N} = (-1)^{n-1} \prod_{l=1, l \neq j}^{n} \frac{e^{i(k_j+k_l)} + 1 - 2e^{ik_j}}{e^{i(k_j+k_l)} + 1 - 2e^{ik_l}}, \quad \forall j = 1, \ldots, n$$

: Bethe-ansatz equations for s=1/2 XXX

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XXC model (integrable)

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$|\uparrow\rangle$ $|\downarrow\rangle$ $|\downarrow\rangle$ $|\uparrow\rangle$

$$|\psi_{A,B^n}\rangle = \sum_{1 \le x_1 < \cdots < x_n \le N} \sum_{P \in \mathfrak{S}_n} A_n(P)\, e^{i \sum_{j=1}^{n} k_{P(j)} x_j}\, \mathrm{tr}_a(\vec{A} \otimes_p \cdots \otimes_p \underset{x_1}{\vec{B}} \otimes_p \cdots \otimes_p \underset{x_n}{\vec{B}} \otimes_p \cdots \otimes_p \vec{A})$$

: Bethe-like state

$$\neq \prod_{j=1}^{n} Q^\dagger(k_j)|\psi_A\rangle \quad \Leftarrow \vec{B} = \begin{pmatrix} b_0 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_0 \end{pmatrix} \vec{A}, \quad \begin{pmatrix} b_0 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_0 \end{pmatrix} \vec{B} \neq 0$$

No reference state exits.

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XXC model (integrable)

- $W = \mathrm{span}\,\{|\psi_{A,B^n}\rangle\}_n$ is solvable subspace of H.

$$H|\psi_{A,B^n}\rangle = \mathcal{E}_n(\{k_j\})|\psi_{A,B^n}\rangle, \qquad \mathcal{E}_n(\{k_j\}) = \left(2\sum_{j=1}^{n}\cos k_j - \frac{n}{2}\right)$$

$$\Leftarrow h\vec{A} \otimes_p \vec{A} = h\vec{B} \otimes_p \vec{B} = 0$$

$$h\vec{A} \otimes_p \vec{B} = -\vec{A} \otimes_p \vec{B} + \vec{B} \otimes_p \vec{A}$$

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Embedded s=1/2 XXX spectrum (not equally-spaced)
⇒ Interacting quasiparticles

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Embedded s=1/2 XXX spectrum (not equally-spaced)
⇒ Interacting quasiparticles

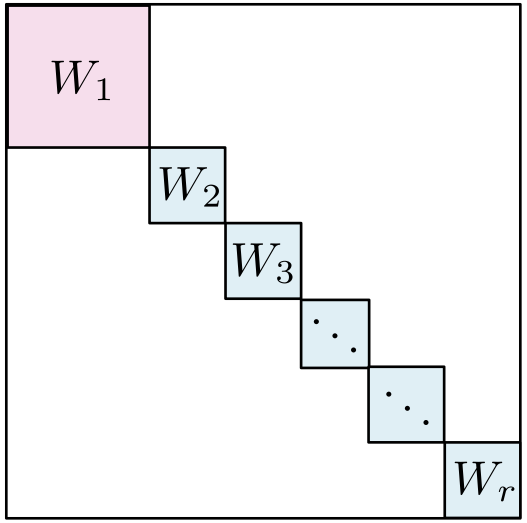
Why? ⇒ Hilbert-space fragmentation

Embedded Integrable Models

- Hilbert-space fragmentation (HSF; Krylov restricted thermalization)

[Retort et al. (2003), Pai et al. (2019)]

$$\mathcal{H} = \bigoplus_{\alpha=1}^{r} W_\alpha\,, \qquad W_\alpha = \mathrm{span}\,\{H^{n_\alpha}|\psi_\alpha\rangle\}_{n_\alpha}$$



  - Exponentially-many block diagonal structure.
  - Fragmented subspaces are not distinguished by obvious local symmetries of H.
  - Solvable subspaces are sometimes embedded (not always).

Embedded Integrable Models

- Simple example: Sato's model [Sato (1995)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1}$$

$$h = \sum_{\substack{a,b=0 \\ a\neq b}}^{2s} f(a,b)|ab\rangle\langle ba| + \sum_{r=0}^{2s} c(r)|rr\rangle\langle rr| + \sum_{\substack{a,b=0 \\ a\neq b}}^{2s} g(a,b)|ab\rangle\langle ab|$$

  - Non-integrable arbitrary spin-s chain.
  - The interactions only exchange the neighboring configurations.

$$\begin{aligned} H : |a_1, a_2, a_3, a_4\rangle \mapsto & f(a_1,a_2)|\underline{a_2, a_1}, a_3, a_4\rangle + f(a_2,a_3)|a_1, \underline{a_3, a_2}, a_4\rangle \\ & + f(a_3,a_4)|a_1, a_2, \underline{a_4, a_3}\rangle + f(a_4,a_1)|\underline{a_4}, a_2, a_3, \underline{a_1}\rangle \\ & + (g(a_1,a_2) + g(a_2,a_3) + g(a_3,a_4) + g(a_4,a_1))|a_1, a_2, a_3, a_4\rangle \end{aligned}$$

Embedded Integrable Models

- Simple example: Sato's model [Sato (1995)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1}$$

$$h = \sum_{\substack{a,b=0 \\ a\neq b}}^{2s} f(a,b)|ab\rangle\langle ba| + \sum_{r=0}^{2s} c(r)|rr\rangle\langle rr| + \sum_{\substack{a,b=0 \\ a\neq b}}^{2s} g(a,b)|ab\rangle\langle ab|$$

- Non-integrable arbitrary spin-s chain.
- The entries in each configuration never change by the interactions
  ⇒ Hilbert-space fragmentation (according to multisets of configuration entries).

$$\begin{array}{l} |m_1, m_2, \ldots, m_N\rangle \\ |m_2, m_1, \ldots, m_N\rangle \\ \quad\vdots \\ |\sigma(m_1), \sigma(m_2), \ldots, \sigma(m_N)\rangle, \quad \sigma \in \mathfrak{S}_N \end{array}$$

: All in the same invariant subspace.

Embedded Integrable Models

- Simple example: Sato's model [Sato (1995)]

$$H=\sum_{x=1}^{N}\mathbf{1}\otimes\cdots\otimes \underset{x,x+1}{h}\otimes\cdots\otimes\mathbf{1}$$

$$h=\sum_{\substack{a,b=0\\a\neq b}}^{2s} f(a,b)|ab\rangle\langle ba|+\sum_{r=0}^{2s}c(r)|rr\rangle\langle rr|+\sum_{\substack{a,b=0\\a\neq b}}^{2s} g(a,b)|ab\rangle\langle ab|$$

  - Non-integrable arbitrary spin-s chain.
  - The model is integrable in the subspaces given by

$$W^{\sigma}=(\mathrm{span}\,\{|0\rangle,|\sigma\rangle\})^{\otimes N},\quad \sigma=1,\dots,2s.$$

$|0\rangle \leftrightarrow |\uparrow\rangle$: Vacuum
$|\sigma\rangle \leftrightarrow |\downarrow\rangle$: Particles

$\Rightarrow$ $H\big|_W \sim H_{\mathrm{XXZ}}$ with $\Delta_{\sigma,0}=(g(0,\sigma)+g(\sigma,0)-c(\sigma))/f(\sigma,0)$

Anisotropy depending on σ.

Embedded Integrable Models

- Example of HSF: Deformed XXC model [CM (2023)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1}$$

$$h = \sum_{a\in\{0,2\}} \underbrace{(|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|)}_{\text{XXC model (integrable)}} + \alpha(|00\rangle\langle 00| + |22\rangle\langle 22|)$$

- Partially solvable spin-1 chain.
- Integrable at the (isotropic) XXC point $\alpha \to 0$. [Maassarani (1997, 1999), de Leeuw et al. (2023)]

$$R(\lambda) = \sum_{a,a'=0,2} \Big\{ (|aa'\rangle\langle a'a| + |11\rangle\langle 11|)\sinh(\lambda+\eta) + (|a1\rangle\langle 1a| + |1a\rangle\langle a1|)\sinh\eta + (x_a|a1\rangle\langle a1| + x_a^{-1}|1a\rangle\langle 1a|)\sinh\lambda \Big\}$$

Embedded Integrable Models

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  - The configuration of 0 & 2 never changes by the interactions.

    "Irreducible string (IS)" [Dhar et al. (1993), Barma et al (1994), Menon et al. (1997), Dhar (1997)]

  - Hilbert-space fragmentation occurs according to the IS.

e.g. $H : |10211210\rangle \mapsto (h^{10}_{10} + h^{21}_{21} + h^{12}_{12} + h^{21}_{21} + h^{10}_{10})|10211210\rangle$

$+ h^{10}_{01}|01211210\rangle + h^{21}_{12}|10121210\rangle + h^{12}_{21}|10212110\rangle$ (IS = 0 2 2 0)

$+ h^{21}_{12}|10211120\rangle + h^{10}_{01}|10211201\rangle$

Embedded Integrable Models

- Example of HSF: Deformed XXC model [CM (2023)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1}$$

$$h = \sum_{a \in \{0,2\}} \underbrace{(|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|)}_{\text{XXC model (integrable)}} + \alpha(|00\rangle\langle 00| + |22\rangle\langle 22|)$$

- Solvable in the subspace spanned by the alternating irreducible-string states.

[Work in progress with Pozsgay et al.]

$$W = \mathrm{span}\{|m_1, \ldots, m_N\rangle \,\big|\, (m_1, \ldots m_N) \underset{\mathrm{IS}}{=} (0,2,0,2,\ldots) \text{ or } (2,0,2,0,\ldots)\}$$

$$= P_{\mathrm{alt}} \mathcal{H}$$

$$P_{\mathrm{alt}} = \mathrm{tr}_{\mathrm{aux}}\Big(\otimes_{\mathrm{phys}} \left(\sigma^{+}_{\mathrm{aux}} \otimes |0\rangle\langle 0| + \mathbf{1}_{\mathrm{aux}} \otimes |1\rangle\langle 1| + \sigma^{-}_{\mathrm{aux}} \otimes |2\rangle\langle 2|\right) \Big)$$

Embedded Integrable Models

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$$h = \sum_{a\in\{0,2\}} (|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|) + \alpha(|00\rangle\langle 00| + |22\rangle\langle 22|) + h_{\text{pert}}$$

  - Stay partially-solvable by adding more perturbations: [Work in progress with Pozsgay et al.]

$$h_{\text{pert}} = \beta(|01\rangle\langle 01| + |21\rangle\langle 21| + |10\rangle\langle 10| + |12\rangle\langle 12|) + \gamma(|00\rangle\langle 22| + |22\rangle\langle 00|)$$

The β-term changes the anisotropy but does not violate the irreducible strings.
The γ-term violates the irreducible strings including 00 or 22. ⇒ Violates entire HSF.

Beyond Isolated Quantum Systems

- Non-thermal states in an isolated quantum system; QMBS.
  - Seem to be exactly solvable in the matrix product forms.
  - Often appear in a small invariant subspace, which is partially solvable.
  - Relatively small entanglement entropy.
- Partial solvability in open quantum systems?

Outline

- What is "partial solvability"?
  - Definition of partial solvability
  - Thermalization & quantum many-body scars (QMBS)
- Closed partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Open partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Concluding remarks

Solvability of Open Quantum Systems

- Lindblad master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t))\,, \quad \mathcal{L}(\rho) = -i[H,\, \rho] + \sum_\mu \varepsilon_\mu \mathcal{D}_\mu(\rho)$$

⇩

Steady state
as the fixed point of $\mathcal{L}$

$$\mathcal{D}_\mu(\rho) = 2A_\mu \rho A_\mu^\dagger - \{A_\mu^\dagger A_\mu,\, \rho\}$$: Dissipation terms

Quantum jump operator (A_μ)

- Is the most general CPTP map under the assumptions:
  Markovian time evolution & Direct product initial state.

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Quantum jump operator (A_{μ})

- Sometimes has the exactly solvable steady state.

Fully solvable ⇒ Free-fermionization, Bethe-ansatz solvability, Triangularization, etc.
[Prosen (2008), Medvedyeva et al. (2016), Buca et al. (2020)]

Partially solvable ⇒ Bulk solvability + "a good choice of dissipators" [Prosen (2013), Karevski et al. (2013)]

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⇩

Steady state
as the fixed point of $\mathcal{L}$

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Non-Hermitization of rSGA Hamiltonian [Tindall et al. (2020)]

Partial bulk solvability [Work in progress with Tsuji]

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Partial bulk solvability [Work in progress with Tsuji]

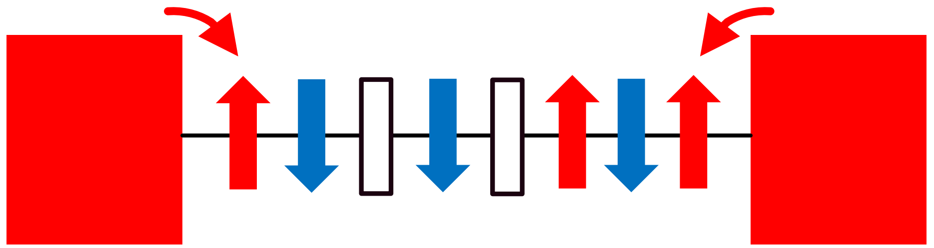
- Can partial solvability be robust against the boundary dissipators?

rSGA-induced solvable eigenmodes

- System coupled to boundary dissipators

$$\mathcal{L}(\rho) = -i[H,\, \rho] + \sum_\mu \gamma_\mu \mathcal{D}_\mu(\rho)$$

$$\mathcal{D}_\mu(\rho) = 2A_\mu \rho A_\mu^\dagger - \{A_\mu^\dagger A_\mu,\, \rho\}$$



- System with rSGA

$$[H,\, Q^\dagger] - \mathcal{E}Q^\dagger\big|_W = 0, \quad W = \mathrm{span}\{(Q^\dagger)^n|\psi_A\rangle\}_n$$

$Q^\dagger = \sum_x e^{i\pi x} q_x^\dagger$ ⇒ Quasiparticle excitations carrying momentum π

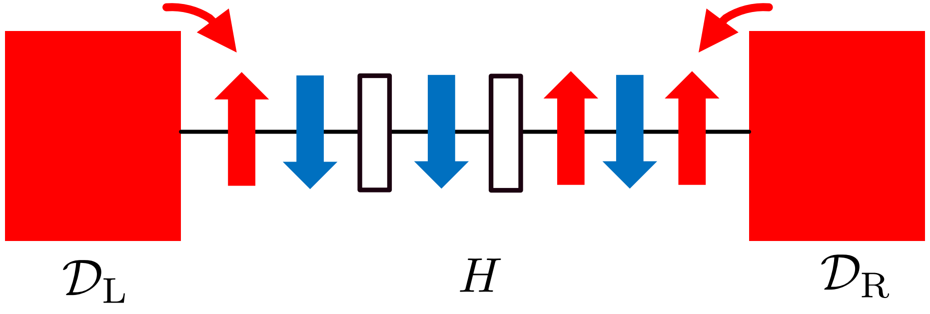
- Quasiparticle baths at the edges

$A_{\mathrm{L}} = q_1^\dagger, \quad A_{\mathrm{R}} = q_N^\dagger$ ⇒ Doping quasiparticles at the boundaries

rSGA-induced solvable eigenmodes

- System coupled to boundary dissipators

$$\mathcal{L}(\rho) = -i[H,\,\rho] + \sum_\mu \gamma_\mu \mathcal{D}_\mu(\rho)$$
$$\mathcal{D}_\mu(\rho) = 2A_\mu \rho A_\mu^\dagger - \{A_\mu^\dagger A_\mu,\,\rho\}$$



  - A pure state $|\psi\rangle$ becomes the steady state if

$[H,\,|\psi\rangle\langle\psi|] = 0$ ⇒ Energy eigenstate

$\mathcal{D}_\mu(|\psi\rangle\langle\psi|) = 0, \quad \forall\mu.$ ⇒ Dark state

cf. Dark state [Baumgartner et al. (2008)]

$[H,\,|\psi\rangle\langle\psi|] = 0, \quad A_\mu|\psi\rangle = 0, \, \forall\mu$

- Some solvable bulk energy eigenstates satisfy the dark state conditions.

rSGA-induced solvable eigenmodes

- Example: s=1 spin chains with rSGA + spin-2 magnon baths
  - s=1 spin chain with rSGA (e.g. AKLT model)

$$[H,\, Q^\dagger] - \mathcal{E} Q^\dagger \big|_{W^{(v_\mathrm{R}, v_\mathrm{L})}} = 0, \quad Q^\dagger := \sum_{x=1}^{N} e^{i\pi x} (S_x^+)^2$$

$$W^{(v_\mathrm{L}, v_\mathrm{R})} = \mathrm{span}\{(Q^\dagger)^n |\psi_A^{(v_\mathrm{L}, v_\mathrm{R})}\rangle\}_n$$

$$|\psi_A^{(v_\mathrm{L}, v_\mathrm{R})}\rangle = \langle \underline{v_\mathrm{L}} | \vec{A} \otimes_p \cdots \otimes_p \vec{A} | \underline{v_\mathrm{R}} \rangle, \qquad \vec{A} = \begin{pmatrix} a_0 \sigma^+ \\ a_1 \sigma^z \\ a_2 \sigma^- \end{pmatrix}, \quad a_0, a_1, a_2 \in \mathbb{C}$$

Boundary vectors $\in V_a = \mathrm{span}\{|0\rangle,\, |1\rangle\}$

$\Rightarrow$ Four degenerate zero-energy states.

  - Spin-2 magnon baths at the edges

$$A_\mathrm{L} = (S_1^+)^2, \quad A_\mathrm{R} = (S_N^+)^2$$

$\Rightarrow$ Doping spin-2 magnons at the boundaries

rSGA-induced solvable eigenmodes

- Example: s=1 spin chains with rSGA + spin-2 magnon baths

[In preparation with Tsuji]

  - The subspace $W^{(0,1)}$ consists of the dark states.

$$[H,\, (Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n] = 0$$

⇒ Eigenstates of the Hamiltonian

$$\mathcal{D}_{(S_1^+)^2}\left((Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right) = 0$$

$$\mathcal{D}_{(S_N^+)^2}\left((Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right) = 0$$

⇒ Dissipators are irrelevant.
(Robust eigenstates against boundary dissipators)

  - Any density matrix diagonal in $W^{(0,1)}$ becomes the steady states.

$$\mathcal{L}\left(\sum_n p_{nn}(Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right) = 0, \quad \sum_n p_{nn} = 1, \quad p_{nn} > 0,\ \forall n$$

rSGA-induced solvable eigenmodes

- Example: s=1 spin chains with rSGA + spin-2 magnon baths

[In preparation with Tsuji]

  - Short proof

$$\mathcal{D}_{(S^+)^2}((p_{mn})) = \begin{pmatrix} p_{22} & 0 & -\frac{1}{2}p_{02} \\ 0 & 0 & -\frac{1}{2}p_{12} \\ -\frac{1}{2}p_{20} & -\frac{1}{2}p_{21} & -p_{22} \end{pmatrix}$$

$$_a\langle v_\mathrm{L}|\{\vec{A}\otimes\vec{A}^{\dagger_p},\ \vec{A}\otimes\vec{B}^{\dagger_p},\ \vec{B}\otimes\vec{A}^{\dagger_p},\ \vec{B}\otimes\vec{B}^{\dagger_p}\}^{\otimes N}|v_\mathrm{R}\rangle_a$$

$$\vec{A} = \begin{pmatrix} a_0\sigma^+ \\ a_1\sigma^z \\ a_2\sigma^- \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} a_2\sigma^- \\ 0 \\ 0 \end{pmatrix}$$

Set the boundary vectors s.t.

$$_a\langle v_\mathrm{L}|\sigma^- = 0, \quad \sigma^-|v_\mathrm{R}\rangle_a = 0 \qquad \Rightarrow \mathcal{D}_{(S_1^+)^2}\left((Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right) = 0$$

$$\Leftrightarrow\ _a\langle v_\mathrm{L}| = {}_a\langle 0|, \quad |v_\mathrm{R}\rangle_a = |1\rangle_a \qquad \mathcal{D}_{(S_N^+)^2}\left((Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right) = 0$$

rSGA-induced solvable eigenmodes

- Example: s=1 spin chains with rSGA + spin-2 magnon baths

[In preparation with Tsuji]

  - Other solvable eigenmodes in $W^{(0,1)} \otimes (W^{(0,1)})^*$

$$[H,\, (Q^\dagger)^m|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n] = \sum_{n,m}(m-n)\mathcal{E}(Q^\dagger)^m|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n$$

⇒ Eigenstates of the Hamiltonian

$$\mathcal{D}_{(S_1^+)^2}\left((Q^\dagger)^m|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right) = 0$$

$$\mathcal{D}_{(S_N^+)^2}\left((Q^\dagger)^m|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right) = 0$$

⇒ Dissipators are irrelevant.
(Robust eigenstates against boundary dissipators)

  - Persistent oscillation emerges.

$$|\Psi(t=0)\rangle = \sum_n a_n (Q^\dagger)^n|\psi_A^{(0,1)}\rangle \in W^{(0,1)}$$

$$\Rightarrow \lim_{t\to\infty}\rho(t) = e^{\mathcal{L}t}\rho(0) = \sum_{n,m} e^{-i(m-n)\mathcal{E}t}(Q^\dagger)^m|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n$$

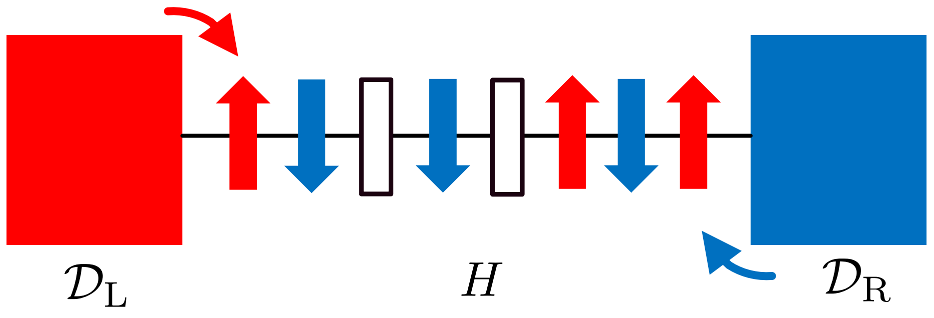
$$\lim_{t\to\infty}\langle O(t)\rangle = \sum_{n\le m} 2\cos((m-n)\mathcal{E}t)a_m a_n \,\mathrm{Re}\, O_{nm}$$

HSF-induced solvable eigenmodes

- System coupled to boundary dissipators

$$\mathcal{L}(\rho) = -i[H,\, \rho] + \sum_\mu \gamma_\mu \mathcal{D}_\mu(\rho)$$

$$\mathcal{D}_\mu(\rho) = 2A_\mu \rho A_\mu^\dagger - \{A_\mu^\dagger A_\mu,\, \rho\}$$



- Thermofield double vector expression

$$\rho = \sum_{m,n} \rho_{m,n} |m\rangle\langle n| \mapsto \sum_{m,n} \rho_{m,n} |m\rangle \otimes |n\rangle = |\rho\rangle\rangle$$

  - Evolution of the density matrix

$$\frac{d}{dt}|\rho(t)\rangle\rangle = -i\widetilde{H}|\rho(t)\rangle\rangle \in V \otimes V^*$$

Can $\widetilde{H}$ be an integrable Hamiltonian in a certain subspace?

$$\widetilde{H} = H \otimes \mathbf{1} - \mathbf{1} \otimes {}^t H + i \sum_\alpha \gamma_\alpha \left((A_\alpha \otimes A_\alpha^*) - \frac{1}{2}(A_\alpha^\dagger A_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes {}^t A_\alpha A_\alpha^*) \right)$$

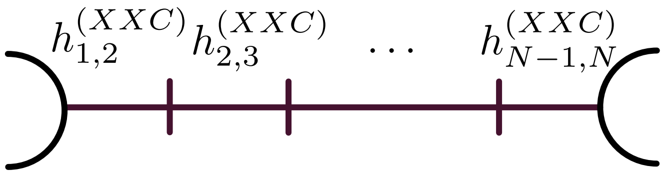
: Non-Hermitian effective Hamiltonian

HSF-induced solvable eigenmodes

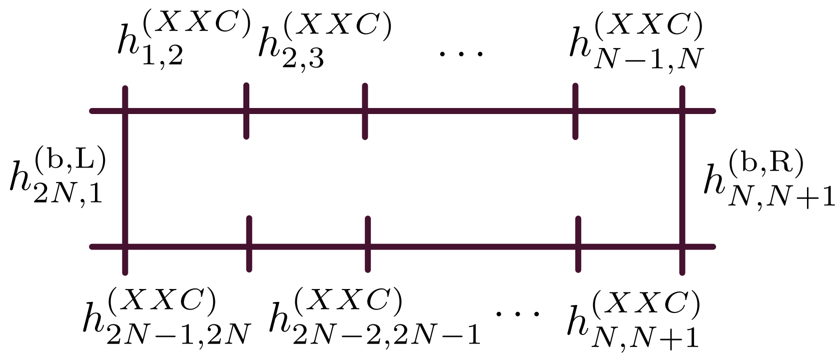
[In preparation with Tsuji]

- Example: XXC Hamiltonian coupled to boundary dissipators
  - Spin-1 XXC Hamiltonian ⇒ HSF by config. of 0 & 2
  - Boundary dissipators ⇒ violate integrability

$$A_{\mathrm{L},+} = (S_1^+)^2, \quad A_{\mathrm{L},-} = (S_1^-)^2, \quad A_{\mathrm{R},+} = (S_N^+)^2, \quad A_{\mathrm{R},-} = (S_N^-)^2$$



⇩ thermofield double



  - Effective non-Hermitian Hamiltonian

$$\widetilde{H}_{XXC} = \sum_{n=1}^{N-1} h_{n,n+1}^{(XXC)} + h_{N,N+1}^{(\mathrm{b,R})} + \sum_{n=N+1}^{2N} h_{n,n+1}^{(XXC)} + h_{2N,1}^{(\mathrm{b,L})}$$

⇒ Two XXC chains coupled at the boundaries.

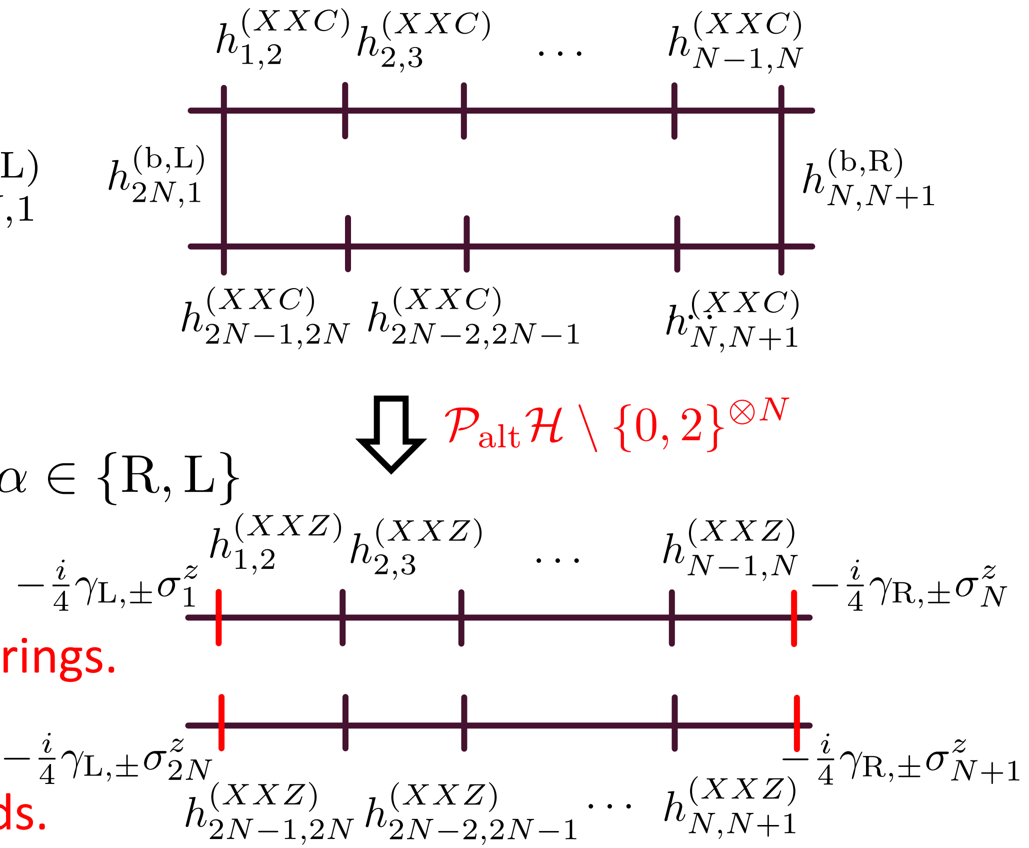
HSF-induced solvable eigenmodes

[In preparation with Tsuji]

- Example: XXC Hamiltonian coupled to boundary dissipators
  - Effective non-Hermitian Hamiltonian

$$\widetilde{H}_{XXC} = \sum_{n=1}^{N-1} h^{(XXC)}_{n,n+1} + h^{(\mathrm{b,R})}_{N,N+1} + \sum_{n=N+1}^{2N} h^{(XXC)}_{n,n+1} + h^{(\mathrm{b,L})}_{2N,1}$$

$$h^{(\mathrm{b},\alpha)} = i\gamma_{\alpha,+}\left(|00\rangle\langle 22| - \frac{1}{2}(|2\rangle\langle 2| \otimes \mathbf{1} + \mathbf{1} \otimes |2\rangle\langle 2|)\right) + i\gamma_{\alpha,-}\left(|22\rangle\langle 00| - \frac{1}{2}(|0\rangle\langle 0| \otimes \mathbf{1} + \mathbf{1} \otimes |0\rangle\langle 0|)\right), \quad \alpha \in \{\mathrm{R}, \mathrm{L}\}$$



Two terms irrelevant in the subspace of alternating irreducible strings.

⇒ Two decoupled XXZ chains

The other terms work as the (imaginary) boundary magnetic fields.

HSF-induced solvable eigenmodes

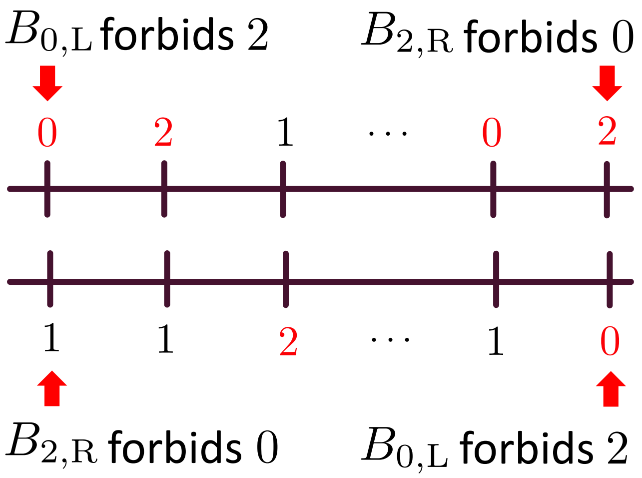
[In preparation with Tsuji]

- Example: XXC Hamiltonian coupled to boundary dissipators
  - Four integrable subspaces

$$P_{02,02} = \mathrm{tr}_{\mathrm{aux}}(B_{0,\mathrm{L}}\mathcal{P}_{\mathrm{alt}}B_{2,\mathrm{R}}) \otimes \mathrm{tr}_{\mathrm{aux}}(B_{0,\mathrm{L}}\mathcal{P}_{\mathrm{alt}}B_{2,\mathrm{R}})$$
$$P_{20,20} = \mathrm{tr}_{\mathrm{aux}}(B_{2,\mathrm{L}}\mathcal{P}_{\mathrm{alt}}B_{0,\mathrm{R}}) \otimes \mathrm{tr}_{\mathrm{aux}}(B_{2,\mathrm{L}}\mathcal{P}_{\mathrm{alt}}B_{0,\mathrm{R}})$$
$$P_{00,22} = \mathrm{tr}_{\mathrm{aux}}(B_{0,\mathrm{L}}\mathcal{P}_{\mathrm{alt}}B_{0,\mathrm{R}}) \otimes \mathrm{tr}_{\mathrm{aux}}(B_{2,\mathrm{L}}\mathcal{P}_{\mathrm{alt}}B_{2,\mathrm{R}})$$
$$P_{22,00} = \mathrm{tr}_{\mathrm{aux}}(B_{2,\mathrm{L}}\mathcal{P}_{\mathrm{alt}}B_{2,\mathrm{R}}) \otimes \mathrm{tr}_{\mathrm{aux}}(B_{0,\mathrm{L}}\mathcal{P}_{\mathrm{alt}}B_{0,\mathrm{R}})$$

: Full projectors onto integrable subspaces

e.g.)



Entire system accepts only alternating 0 & 2.

$$\mathcal{P}_{\mathrm{alt}} = \otimes_{\mathrm{phys}} \left(\sigma^{+}_{\mathrm{aux}} \otimes |0\rangle\langle 0| + \mathbf{1}_{\mathrm{aux}} \otimes |1\rangle\langle 1| + \sigma^{-}_{\mathrm{aux}} \otimes |2\rangle\langle 2|\right)$$: Bulk projectors

$$B_{0,\mathrm{L}} = (|0\rangle\langle 0| + |1\rangle\langle 1|)(\otimes\mathbf{1})^{N-1}, \quad B_{2,\mathrm{L}} = (|2\rangle\langle 2| + |1\rangle\langle 1|)(\otimes\mathbf{1})^{N-1},$$
$$B_{0,\mathrm{R}} = (\mathbf{1}\otimes)^{N-1}(|0\rangle\langle 0| + |1\rangle\langle 1|), \quad B_{2,\mathrm{R}} = (\mathbf{1}\otimes)^{N-1}(|2\rangle\langle 2| + |1\rangle\langle 1|)$$

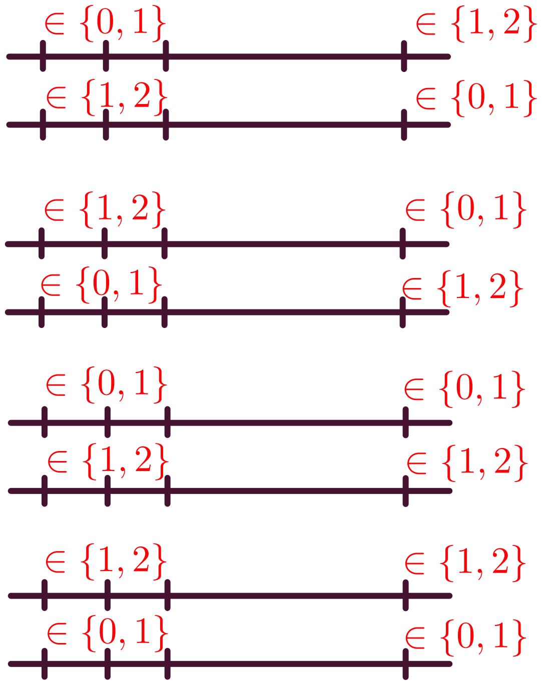
: Boundary projectors

HSF-induced solvable eigenmodes

[In preparation with Tsuji]

- Example: XXC Hamiltonian coupled to boundary dissipators
  - Four integrable subspaces (spin-1/2 XXZ + imaginary boundary magnetic fields)

Projectors



Boundary terms

$P_{02,02}$: $$-\frac{i}{4}\gamma_{+,\mathrm{L}}\sigma_1^z - \frac{i}{4}\gamma_{-,\mathrm{R}}\sigma_N^z - \frac{i}{4}\gamma_{+,\mathrm{R}}\sigma_{N+1}^z - \frac{i}{4}\gamma_{-,\mathrm{L}}\sigma_{2N}^z$$

$P_{20,20}$: $$-\frac{i}{4}\gamma_{-,\mathrm{L}}\sigma_1^z - \frac{i}{4}\gamma_{+,\mathrm{R}}\sigma_N^z - \frac{i}{4}\gamma_{-,\mathrm{R}}\sigma_{N+1}^z - \frac{i}{4}\gamma_{+,\mathrm{L}}\sigma_{2N}^z$$

$P_{00,22}$: $$-\frac{i}{4}\gamma_{+,\mathrm{L}}\sigma_1^z - \frac{i}{4}\gamma_{+,\mathrm{R}}\sigma_N^z - \frac{i}{4}\gamma_{-,\mathrm{R}}\sigma_{N+1}^z - \frac{i}{4}\gamma_{-,\mathrm{L}}\sigma_{2N}^z$$

$P_{22,00}$: $$-\frac{i}{4}\gamma_{-,\mathrm{L}}\sigma_1^z - \frac{i}{4}\gamma_{-,\mathrm{R}}\sigma_N^z - \frac{i}{4}\gamma_{+,\mathrm{R}}\sigma_{N+1}^z - \frac{i}{4}\gamma_{+,\mathrm{L}}\sigma_{2N}^z$$

HSF-induced solvable eigenmodes

[In preparation with Tsuji]

- Example: XXC Hamiltonian coupled to boundary dissipators
  - Eigenstates of effective Hamiltonian

$$\widetilde{H} \underset{\mathcal{P}_{\mathrm{alt}}\mathcal{H}\setminus\{0,2\}^{\otimes N}}{\longmapsto} H^{(+)}_{XXZ}(\gamma_{\mathrm{L}},\gamma_{\mathrm{R}}) \otimes \mathbf{1} - \mathbf{1} \otimes H^{(-)}_{XXZ}(\gamma_{\mathrm{L}},\gamma_{\mathrm{R}})$$

$$H^{(\pm)}_{XXZ}(\gamma_{\mathrm{L}},\gamma_{\mathrm{R}}) = \frac{1}{2}\sum_{x=1}^{N-1}\left(\sigma^+_x\sigma^-_{x+1} + \sigma^-_x\sigma^+_{x+1} + \frac{1}{2}\cosh\eta\cdot\sigma^z_x\sigma^z_{x+1}\right) \pm \frac{i}{4}\gamma_{\mathrm{L}}\sigma^z_1 \pm \frac{i}{4}\gamma_{\mathrm{R}}\sigma^z_N$$

⇒ Derived by Bethe ansatz for spin-1/2 XXZ.

$$R_{ij}(\lambda) = \sinh\left(\lambda + \frac{\eta}{2}\right)\cosh\frac{\eta}{2}\cdot\mathbf{1}_{ij} + \cosh\left(u + \frac{\eta}{2}\right)\sinh\frac{\eta}{2}\cdot\sigma^z_i\sigma^z_j + \sinh\eta\cdot\left(\sigma^+_i\sigma^-_j + \sigma^-_i\sigma^+_j\right)$$

$$K(\lambda,\xi) = \sinh\xi\cosh\lambda\cdot\mathbf{1} + \cosh\xi\sinh\lambda\cdot\sigma^z,$$

$$\gamma_{\mathrm{L}} = -2i\sinh\eta\coth\xi_{\mathrm{L}}, \quad \gamma_{\mathrm{R}} = 2i\sinh\eta\coth\xi_{\mathrm{R}}$$

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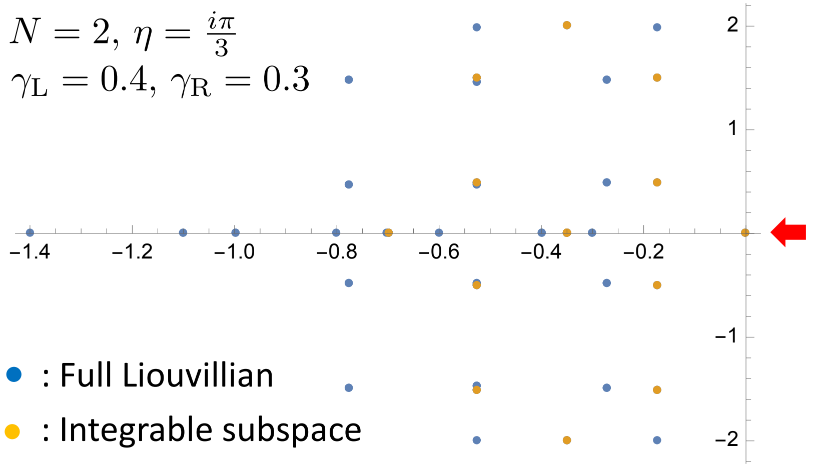

Full Liouvillian has degenerate steady states including the fully polarized state.

The steady state in the integrable subspace is given by the product state $\rho_{\rm ss} = (|1\rangle\langle 1|)^{\otimes N}$.

No persistent oscillation unlike the rSGA-induced solvable steady states.

Outline

- What is "partial solvability"?
  - Definition of partial solvability
  - Thermalization & quantum many-body scars (QMBS)
- Closed partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Open partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Concluding remarks

Concluding remarks

- Closed partially solvable models
  - Are described by Hamiltonians with the block diagonal structure, which is induced by the extra algebraic structure in the solvable subspace.
    e.g. rSGA, Yang-Baxter equation, etc.
  - Embedded integrability due to HSF is not violated by a certain kinds of site-dependent perturbations.
- Open partially solvable models
  - Have solvable eigenmodes although their Liouvillians are non-integrable.
  - Solvable energy eigenstates can be robust against a certain choice of dissipators.
  - Embedded integrability by HSF admits a new class of partially solvable boundary dissipative systems.

Future works

- From the phenomenological viewpoints,
  - Overlap between the initial state & solvable eigenmodes?
    ⇒ Needs determinant formula for boundary cases.
  - Can we find the solvable eigenmodes not based on the dark state conditions?
    ⇒ Affected by dissipators, but solvable.
- From the mathphys aspects,
  - Is it possible to extend the notion of partial integrability to QFT models?
  - What is the algebraic structure behind the XXC & related models?

[de Leeuw et al. (2023)]