Superintegrable chiral Potts, free parafermions and the spin-1 biquadratic model

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Outline of this talk

1) Superintegrable chiral Potts model

   ◇ coupled Temperley-Lieb algebra

   ◇ pictorial representation

   ◇ Baxterization

2) Free parafermions

   ◇ exceptional points, non-Hermitian aspects

3) spin-1 biquadratic model

   ◇ ferromagnetic end of the spectrum

Baxter2020: Frontiers in Intregrability, Canberra Feb 11-14 2020

Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory

Canberra July 10-14, 1989

BBQ July 15, 1989

$Z(N)$ spin chains

Building blocks are the $N \times N$ ('shift' and 'clock') matrices

$$\begin{aligned} (\tau)_{\ell m} &= \delta_{\ell, m+1} \pmod{N} \\ \sigma &= \mathrm{diag}\left(1, \omega, \omega^2, \ldots, \omega^{N-1}\right) \end{aligned}$$

with $\omega = \mathrm{e}^{2\pi i/N}$. For $N = 3$,

$$\tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}.$$

With 1 the identity, they satisfy

$$\tau^N = \sigma^N = 1, \qquad \tau^\dagger = \tau^{N-1}, \qquad \sigma^\dagger = \sigma^{N-1},$$

$$\sigma\tau = \omega\tau\sigma.$$

Some well studied Yang-Baxter integrable N-state quantum spin chains are of the form

$$H = -\sum_{j=1}^{L}\sum_{n=1}^{N-1} a_n \left(\lambda\,\tau_j^n + \sigma_j^n \sigma_{j+1}^{N-n}\right)$$

$$\tau_j = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \tau \otimes 1 \otimes \cdots \otimes 1$$
$$\sigma_j = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma \otimes 1 \otimes \cdots \otimes 1$$

where 1, τ and σ are $N \times N$ matrices, τ and σ occur in position j.

special cases

- N-state quantum Potts model

$$\boxed{a_n = 1} \tag{1}$$

- Fateev-Zamolodchikov $Z(N)$ model

$$\boxed{a_n = \frac{1}{\sin(\pi n/N)}} \tag{2}$$

- N-state superintegrable chiral Potts model

$$\boxed{a_n = \frac{2}{1-\omega^{-n}}} \tag{3}$$

Each model reduces to the quantum Ising model for $N = 2$.

Models (1) and (2) are equivalent for $N = 3$.

Model (3) still something of an enigma..

Potts and Temperley-Lieb

Recall the N-state quantum Potts representation of the TL algebra

$$
\begin{aligned}
e_{2j-1} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \tau_j^n && j = 1, \ldots, L \\
e_{2j} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left(\sigma_j \sigma_{j+1}^\dagger\right)^n && j = 1, \ldots, L-1
\end{aligned}
$$

with

$$
\begin{aligned}
e_j^2 &= \sqrt{N}\, e_j \\
e_j e_{j\pm 1} e_j &= e_j \\
e_j e_i &= e_i e_j \qquad |i-j| > 1
\end{aligned}
$$

Potts model hamiltonian

$$
H_{\mathrm{P}} = -\sum e_j,
$$

Ongoing decades of fun with TL models

- mappings between models, faithful representations etc
- combinatorics
- stochastic processes
- ...

Several known generalisations of TL algebra

Some examples

- multi-coloured TL algebra

  Grimm and Pearce (1992)
  Bisch and Jones (1997)

  aka k-colour Fuss-Catalan algebras

- 'blob' algebra

  Martin and Saleur (1994)

Power of the algebraic/pictorial approach

It is known that the TL algebra, along with the pictorial representation, can be used to derive the full eigenspectrum of the TL Hamiltonian, in that case via the Bethe Ansatz

- Levy (1990) (1991)
- Martin and Saleur (1994)
- de Gier and Pyatov (2004)
- Nepomechie and Pimenta (2016)

Superintegrable chiral Potts (SICP) chain

$$H_{\mathrm{CP}} = -\sum_{j=1}^{L}\sum_{n=1}^{N-1}\left(\lambda\,\alpha_n\,\tau_j^n + \overline{\alpha}_n\left(\sigma_j\sigma_{j+1}^\dagger\right)^n\right)$$

$$\alpha_n = \frac{\mathrm{e}^{\mathrm{i}(2n-N)\phi/N}}{\sin(\pi n/N)}, \qquad \overline{\alpha}_n = \frac{\mathrm{e}^{\mathrm{i}(2n-N)\overline{\phi}/N}}{\sin(\pi n/N)}$$

- The chiral Potts model has an R-matrix when

$$\lambda\cos\phi = \cos\overline{\phi}.$$

- The special values $\phi = \overline{\phi} = \frac{\pi}{2}$ define the superintegrable case.
- H_{SICP} admits an infinite set of commuting conserved charges.
- H_{SICP} only solved for periodic bc's.
  (N-state free parafermions only solved for open bc's)

H_{SICP} can be written in terms of a coupled TL algebra!

[$N = 3$ case, J Fjelstad and T Månsson, JPA 45, 155208 (2012)]

For general N there are $N-1$ generators $e_j^{(k)}$ which satisfy

$$
\begin{aligned}
\left(e_j^{(k)}\right)^2 &= Q\, e_j^{(k)} \\
e_j^{(k)} e_{j\pm 1}^{(\ell)} e_j^{(k)} &= e_j^{(k)} \\
e_i^{(k)} e_j^{(\ell)} &= e_j^{(\ell)} e_i^{(k)} \qquad |i-j| > 1 \\
e_j^{(k)} e_j^{(\ell)} = e_j^{(\ell)} e_j^{(k)} &= 0 \qquad\qquad k \neq \ell
\end{aligned}
$$

with $Q = \sqrt{N}$.

For $N = 2$ this reduces to the single TL generator e_j.

For $N = 3$ we label the generators by $e_j = e_j^{(1)}$ and $f_j = e_j^{(2)}$.

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Letters

Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **53** (2020) 36LT01 (10pp)

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Letter

A coupled Temperley–Lieb algebra for the superintegrable chiral Potts chain

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Abstract

The Hamiltonian of the N-state superintegrable chiral Potts (SICP) model is written in terms of a coupled algebra defined by $N-1$ types of Temperley–Lieb generators. This generalises a previous result for $N=3$ obtained by Fjelstad and Månsson (2012 *J. Phys. A: Math. Theor.* **45** 155208). A pictorial representation of a related coupled algebra is given for the $N=3$ case

In general we can write

$$e_{2j-1}^{(k)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left(\omega^k \tau_j\right)^n \qquad j = 1, \dots, L$$

$$e_{2j}^{(k)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left(\omega^k \sigma_j \sigma_{j+1}^\dagger\right)^n \qquad j = 1, \dots, L-1$$

for $k = 1, \dots, N-1$. Here $\omega = e^{2\pi \mathrm{i}/N}$.

Then, for periodic bc's

$$\boxed{H_{\mathrm{SICP}} = \frac{2}{\sqrt{N}} \sum_{j=1}^{L} \sum_{k=1}^{N-1} (N-k) \left(\lambda e_{2j-1}^{(k)} + e_{2j}^{(k)}\right) - (\lambda+1)(N-1)L}$$

And for open bc's

$$\begin{aligned} H_{\mathrm{SICP}} &= -(N-1)(L(\lambda+1)-1) \\ &\quad + \frac{2}{\sqrt{N}} \sum_{j=1}^{L} \sum_{k=1}^{N-1} \lambda(N-k) e_{2j-1}^{(k)} \\ &\quad + \frac{2}{\sqrt{N}} \sum_{j=1}^{L-1} \sum_{k=1}^{N-1} (N-k) e_{2j}^{(k)} \end{aligned}$$

The generators $e_j^{(k)}$ also satisfy additional cubic relations.

For the $N = 3$ case

$$\begin{aligned}
f_j e_{j\pm1} e_j &= \pm \mathrm{i} \left(\omega^{\mp1} e_{j\pm1} e_j - f_{j\pm1} e_j \right) + \omega^{\pm1} e_j \\
&= \pm \mathrm{i} \left(\omega^{\mp1} f_j e_{j\pm1} - f_j f_{j\pm1} \right) + \omega^{\pm1} f_j \\
e_j e_{j\pm1} f_j &= \mp \mathrm{i} \left(\omega^{\pm1} e_{j\pm1} f_j - f_{j\pm1} f_j \right) + \omega^{\mp1} f_j \\
&= \mp \mathrm{i} \left(\omega^{\pm1} e_j e_{j\pm1} - e_j f_{j\pm1} \right) + \omega^{\mp1} e_j \\
f_j f_{j\pm1} e_j &= \omega^{\pm1} f_j e_{j\pm1} e_j \\
e_j f_{j\pm1} f_j &= \omega^{\mp1} e_j e_{j\pm1} f_j
\end{aligned}$$

For $N = 4$ with $e_j = e_j^{(1)}$, $f_j = e_j^{(2)}$ and $g_j = e_j^{(3)}$, a typical cubic relation is of the type

$$f_1 e_2 e_1 = \frac{1}{2}(1 - \mathrm{i}) e_2 e_1 - \frac{1}{2}(1 + \mathrm{i}) g_2 e_1 - \mathrm{i} f_2 e_1 + \mathrm{i}\, e_1 \,.$$

A different way to write the $N = 3$ SICP hamiltonian

This originates from the staggered nature of the coupled operators e_j and f_j between odd and even sites:

$$H_{\mathrm{SICP}} = -\frac{2}{\sqrt{3}} \sum_{j=1}^{L} \left[\lambda \left(e_{2j-1} - f_{2j-1} \right) + \left(e_{2j} - f_{2j} \right) \right].$$

Using the definitions, with $(\omega - \omega^2)/\sqrt{3} = \mathrm{i}$, gives

$$\begin{aligned} e_{2j-1} - f_{2j-1} &= \mathrm{i} \left(\tau_j - \tau_j^2 \right) \\ e_{2j} - f_{2j} &= \mathrm{i} \left[\sigma_j \sigma_{j+1}^\dagger - (\sigma_j \sigma_{j+1}^\dagger)^2 \right] \end{aligned}$$

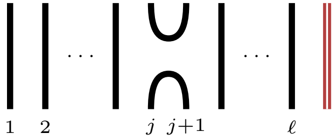
$$\boxed{H_{\mathrm{SICP}} = -\frac{2\,\mathrm{i}}{\sqrt{3}} \sum_{j=1}^{L} \left[\lambda \left(\tau_j - \tau_j^2 \right) + \sigma_j \sigma_{j+1}^\dagger - (\sigma_j \sigma_{j+1}^\dagger)^2 \right]}$$

The last terms $\sigma_L \sigma_{L+1}^\dagger$ and $(\sigma_L \sigma_{L+1}^\dagger)^2$ are omitted for open bc's.

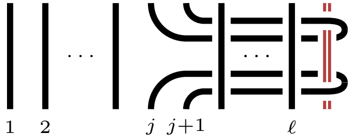
Pictorial representation

We gave a pictorial representation of the generators for $N = 3$:

$$e_j = e_j^{(1)} =$$



$$f_j = e_j^{(2)} =$$



The key feature of the pictorial representation is a pole around which loops can become entangled. Here we choose the position of the pole to be at one end of the chain. In the associated loop diagrams, closed (contractible) loops have weight Q, with $Q = \sqrt{3}$. The weight of closed (non-contractable) loops encircling the red line is zero.

Math. Res. Lett.
Volume 25, Number 6, 1911–1936, 2018

Extremal weight projectors

Hoel Queffelec and Paul Wedrich

We introduce a quotient of the affine Temperley-Lieb category that encodes all weight-preserving linear maps between finite-dimensional $\mathfrak{sl}_2$-representations. We study the diagrammatic idempotents that correspond to projections onto extremal weight spaces and find that they satisfy similar properties as Jones-Wenzl projectors, and that they categorify the Chebyshev polynomials of the first kind. This gives a categorification of the Kauffman bracket skein algebra of the annulus, which is well adapted to the task of categorifying the multiplication on the Kauffman bracket skein module of the torus.

1. Introduction

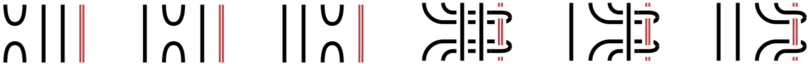
The Lie algebra $\mathfrak{sl}_2$ and its universal enveloping algebra $U(\mathfrak{sl}_2)$ are ubiqui-

SICP $N = 3$ example, generators e_j and f_j

The generators e_j are like the usual TL generators, with loops not encircling a line.

The generators f_j involve loops around the single red line.

Generators for the $L = 2$ site open chain:



e_1 e_2 e_3 f_1 f_2 f_3

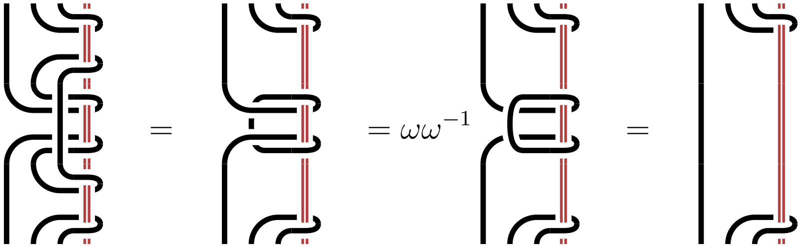
SICP $N = 3$ example, generators e_j and f_j

Algebraic relations can be proved via the diagrams.

We make use of the usual Kauffman-type relations.

The most interesting cubic relations are $f_1 f_2 f_1 = f_1$ and $f_2 f_1 f_2 = f_2$.

Proof of the relation $f_2 f_1 f_2 = f_2$



The knot can be resolved!

Key ingredients are crossing relations for loops encircling a red line.

Line crossing relations

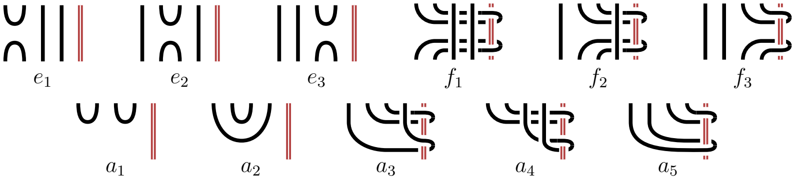
 $=$ ω  ,  $=$ ω 

For this example with $N = 3$ the parameter is $\omega = \mathrm{e}^{\mathrm{i}2\pi/3}$.

This value can be derived topologically.

The open $N = 3$ SICP chain with $L = 2$

Generators and basis states for the $L = 2$ site open chain



Act on the basis states $a_1, \ldots, a_5$ with the generators and resolve the diagrams to construct eigenvalues of H_{loop} to match eigenvalues of H_{SICP}.

Concluding remarks

We have given a pictorial representation for a coupled algebra which defines the $N = 3$ state SICP model, for which the weight of contractable loops is $Q = \sqrt{3}$ and the weight of closed (non-contractable) loops encircling a line is zero.

There are many points to follow up!

Some examples:

- We have only discussed open boundary conditions.
- Another known representation of this coupled algebra corresponds to the staggered spin-1/2 XX chain.
  It has two sets of generators e_i and f_i and $Q = 2$.
- q-deformation?
- The algebraic approach opens up a path towards Baxterisation of the SICP model.

But ...

There is a problem!

We found a problem with the diagrammatic proof for some of the algebraic commutation relations for the SICP model representation.

There is no problem for the staggered XX representation.

But ...

We finally fixed it!

Pictorial representation for $N = 3$ and also for general N.

A key ingredient is work by Jaffe and Liu.

This is the topic of the talk by Remy Adderton in Week 3.

Commun. Math. Phys. 352, 95–133 (2017)
Digital Object Identifier (DOI) 10.1007/s00220-016-2779-4

Communications in
Mathematical Physics

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Planar Para Algebras, Reflection Positivity

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Abstract: We define a *planar para algebra*, which arises naturally from combining planar algebras with the idea of $\mathbb{Z}_N$ para symmetry in physics. A subfactor planar para algebra is a Hilbert space representation of planar tangles with parafermionic defects that are invariant under para isotopy. For each $\mathbb{Z}_N$, we construct a family of subfactor planar para algebras that play the role of Temperley–Lieb–Jones planar algebras. The first example in this family is the parafermion planar para algebra (PAPPA). Based on this example, we introduce parafermion Pauli matrices, quaternion relations, and braided relations for parafermion algebras, which one can use in the study of quantum information. An important ingredient in planar para algebra theory is the string Fourier transform (SFT), which we use on the matrix algebra generated by the Pauli matrices. Two different reflections play an important role in the theory of planar para algebras. One is the adjoint operator; the other is the modular conjugation in Tomita–Takesaki

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J. Phys. A: Math. Theor. **48** (2015) 294001 (24pp) | doi:10.1088/1751-8113/48/29/294001

Discrete holomorphicity in the chiral Potts model

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Abstract

We construct lattice parafermions for the $Z(N)$ chiral Potts model in terms of quasi-local currents of the underlying quantum group. We show that the conservation of the quantum group currents leads to twisted discrete-holomorphicity (DH) conditions for the parafermions. At the critical Fateev–Zamolodchikov point the parafermions are the usual ones, and the DH conditions coincide with those found previously by Rajabpour and Cardy. Away from the critical point, we show that our twisted DH conditions can be understood as deformed lattice current conservation conditions for an underlying perturbed conformal field theory in both the general $N \geqslant 3$ and $N = 2$ Ising cases.

2) Free parafermions

Batchelor *et al. AAPPS Bulletin* (2023) 33:29
https://doi.org/10.1007/s43673-023-00105-3

AAPPS Bulletin

REVIEW ARTICLE **Open Access**

A brief history of free parafermions

Murray T. Batchelor[1*], Robert A. Henry[1] and Xilin Lu[1]

Abstract

In this article we outline the historical development and key results obtained to date for free parafermionic spin chains. The concept of free parafermions provides a natural N-state generalization of free fermions, which have long underpinned the exact solution and application of widely studied quantum spin chains and their classical counterparts. In particular, we discuss the Baxter-Fendley free parafermionic $Z(N)$ spin chain, which is a relatively simple non-Hermitian generalization of the Ising model.

Keywords Quantum spin chains, Free fermions, Free parafermions

1 Introduction

The concept of free fermions is fundamental to the celebrated exact solution of the two-dimensional Ising model in zero magnetic field [1–3] and its one-dimen- below, the Hamiltonian of this spin chain takes a similar form to that of the quantum Ising chain, but the spins have N allowed states, with Baxter's Hamiltonian reducing to the Ising model for $N = 2$. Baxter found that the

What are exceptional points?

Exceptional points are spectral singularities in the parameter space of a system in which two or more eigenvalues, and their corresponding eigenvectors, simultaneously coalesce.

Such degeneracies are peculiar features of nonconservative systems that exchange energy with their surrounding environment.

EPs are level degeneracies induced by non-Hermiticity.

They exhibit exotic topological phenomena associated with the winding of eigenvalues and eigenvectors.

A vast and highly active topic!

The Baxter-Fendley $Z(N)$ spin chain

A model that received **no attention** for a long time was found by Rodney Baxter in 1989.

R J Baxter, Phys Lett A **140**, 155 (1989); J Stat Phys **57**, 1 (1989)

For an L-site chain (with OBC) this model is defined as

$$\boxed{-H = \sum_{j=1}^{L} \tau_j + \lambda \sum_{j=1}^{L-1} \sigma_j^{\dagger} \sigma_{j+1}}$$

It reduces to the quantum Ising model for $N = 2$.

For $N = 3$ think of it as 'half' a Potts chain.

The eigenvalues of H have a simple form!

$$\boxed{-E = \omega^{p_1}\epsilon_1 + \omega^{p_2}\epsilon_2 + \cdots + \omega^{p_L}\epsilon_L}$$

for any choice of $p_k = 0, \ldots, N-1$. Recall $\omega = \mathrm{e}^{2\pi i/N}$.

- Initially a numerical observation.
- $N = 2$ is free fermions $\boxed{-E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_L}$
- Gives all N^L eigenvalues in the spectrum.
- The quasi-energies ϵ_k are known exactly.
- The model originates as the hamiltonian limit of the τ_2 model, a variant of the chiral Potts model.

- Fendley (2014) derived this result using a generalisation of the Jordan-Wigner transformation, namely the **Fradkin-Kadanoff transformation** to parafermionic operators originally introduced for the N-state clock models.

  P Fendley, J. Phys. A 47, 075001 (2014)

- Baxter (2014) and Au-Yang and Perk (2014,2016) applied Fendley's parafermionic approach to the more general τ_2 model with open boundaries.

  R J Baxter, J Phys A **47**, 315001 (2014)

  H Au-Yang and J H H Perk, J Phys A **47**, 315002 (2014); arXiv:1606.06319

- Fendley (2019) developed algebraic techniques which were also applied to multispin free fermion systems and adopted and generalised by Alcaraz and Pimenta (2020, 2021) to a class of multispin free fermion and free parafermion models.

The Baxter-Fendley hamiltonian is non-Hermitian, with complex energy eigenvalues for $N \geq 3$.

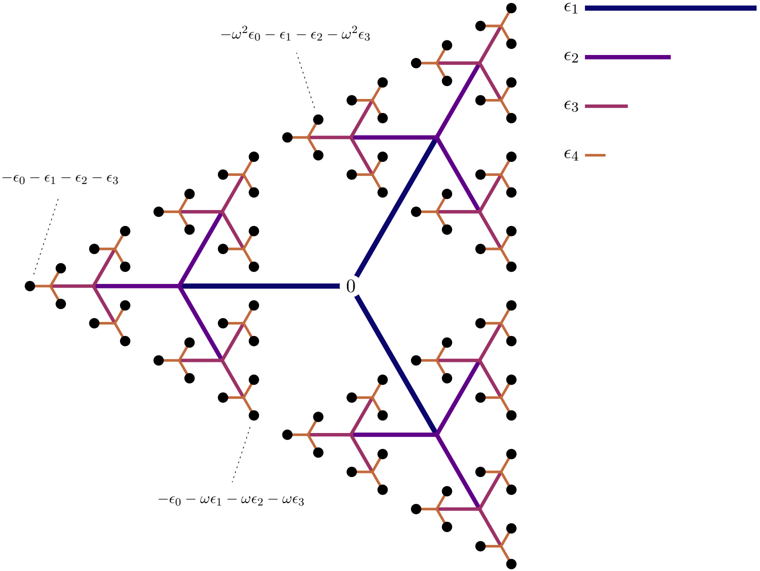
For any eigenvalue E, there are other eigenvalues $\omega E, \omega^2 E, \ldots$

This is the $Z(N)$ generalisation of the $E \leftrightarrow -E$ Ising symmetry (recall $\omega = -1$ for $N = 2$).

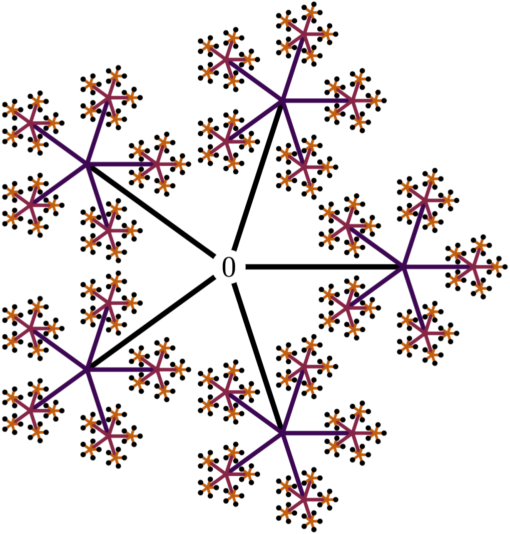
In general non-Hermitian hamiltonians describe the dynamics of physical systems that are not conservative.

The properties of the model are well worth exploring, being a rare example of an exactly solved non-Hermitian many-body system.

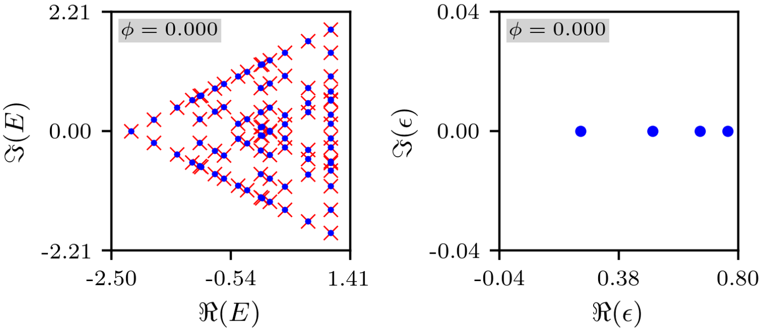
Free parafermion eigenspectrum ($N = 3,\ L = 4$)



$N = 5$



Free parafermion eigenspectrum ($N = 3,\ L = 4,\ \lambda = 1$)



(left) complex energy spectra obtained from the quasienergies (blue dots), and values obtained from exact diagonalisation of the full Hamiltonian (red crosses).

(right) quasienergies ϵ_j.

Motivation for the present work

Our previous work on this model showed unusual properties for $N \geq 3$ such as diverging correlation functions with system size L.

Z.-Z. Liu, R. A. Henry, MTB and H.-Q. Zhou, JSTAT 2019, 124002

Some ground-state expectation values for the free parafermion Z(N) spin chain

How to explain this behaviour?

The solution

F C Alcaraz, MTB and Z-Z Liu, J Phys A 50, 16LT03 (2017)

$$-H = \sum_{j=1}^{L} \tau_j + \lambda \sum_{j=1}^{L-1} \sigma_j \sigma_{j+1}^{\dagger}$$

$$-E = \sum_{j=1}^{L} \omega^{p_j} \epsilon_{k_j}, \quad p_j = 0, 1, \ldots, N-1, \quad \omega = \mathrm{e}^{2\pi i/N}$$

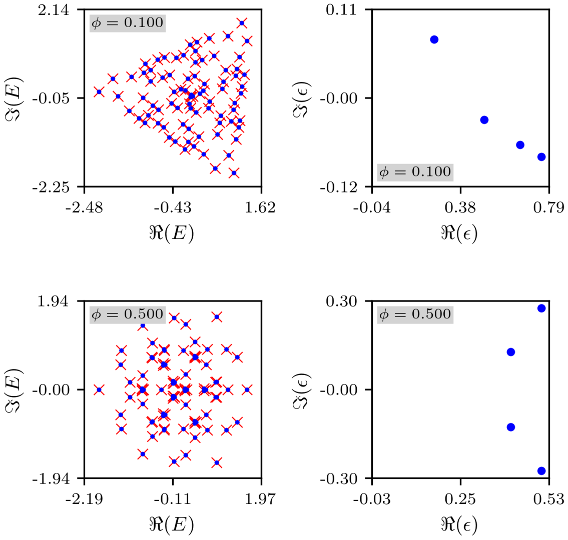
$$\boxed{\epsilon_k = \left(1 + \lambda^N + 2\lambda^{N/2} \cos k\right)^{1/N}}$$

k_j satisfy

$$\boxed{\sin(L+1)k = -\lambda^{N/2} \sin Lk}$$

for $\lambda = 1$, $k_j = \frac{2j\pi}{2L+1}$, $j = 1, \ldots, L$ and $\epsilon_k = \left(2 \cos \frac{k}{2}\right)^{2/N}$.

Free parafermion eigenspectrum ($N = 3$, $L = 4$, $\lambda = e^{2\pi i\phi/N}$)



Exceptional points

For real positive λ, the quasi-energies ϵ_j are always positive and distinct.

For **complex** λ, a pair of them may become equal at certain values of λ, which depend on N and L.

We call these *quasi-energy exceptional points*.

We call EPs in the energy spectrum *Hamiltonian exceptional points*.

$\Longrightarrow$ quasi-energy EPs give rise to Hamiltonian EPs.

Moreover, we can calculate them.

A quasi-energy EP will occur when

$$\sin(L+1)k = -\lambda^{N/2} \sin Lk$$

has a repeated root, meaning that both this equation and its derivative are satisfied.

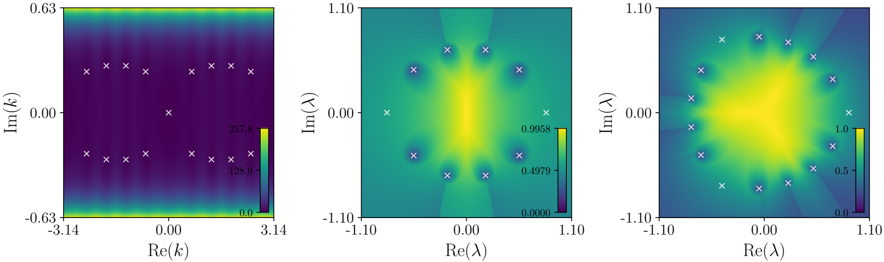
The EPs are pairs of values k_{EP} and λ_{EP} which satisfy these equations simultaneously.

In this way we obtain k_{EP} as the solution to

$$\sin(2L+1)k - (2L+1)\sin k = 0\,,$$

with the corresponding value λ_{EP} given by

$$\lambda = \left[\frac{-\sin(L+1)k}{\sin Lk}\right]^{2/N} \quad .$$

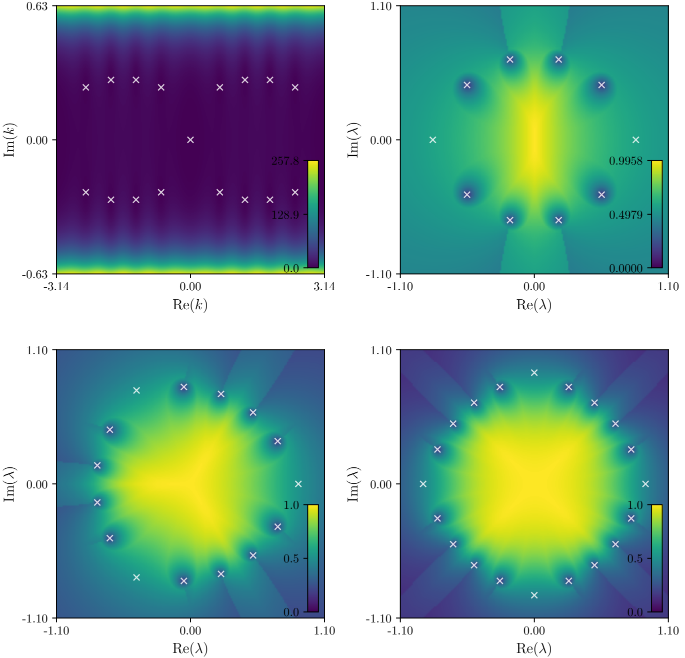


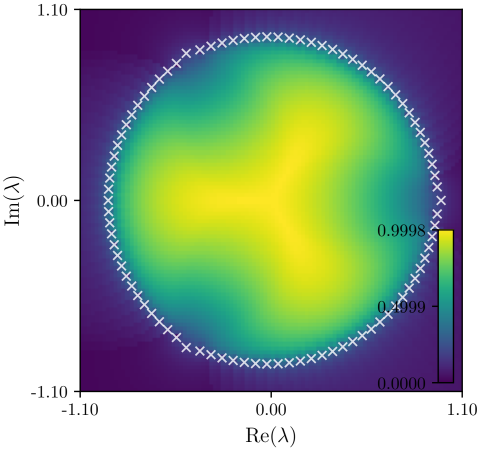
(left) k solutions for $L = 4$

(middle) difference between smallest and second-smallest quasi-energies for $N = 2$

(right) difference between smallest and second-smallest quasi-energies for $N = 3$

The corresponding values of λ_{EP} are also shown as crosses.

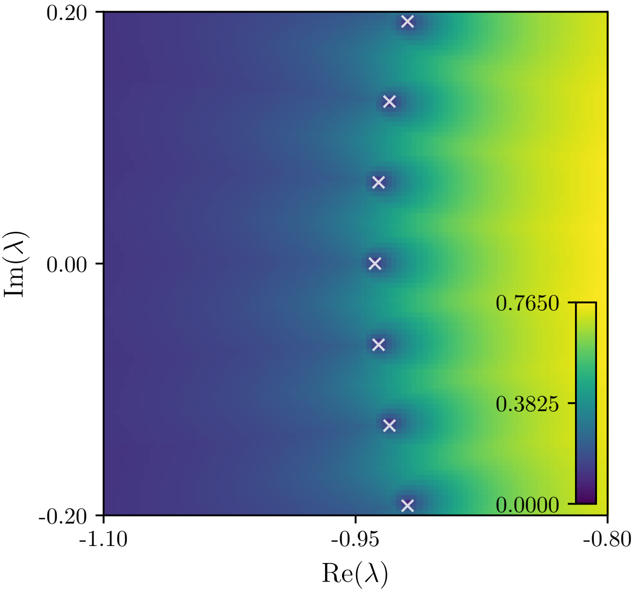

0.63
0.00
-0.63
Im(k)
-3.14
0.00
3.14
Re(k)
257.8
128.9
0.0
1.10
0.00
-1.10
Im(λ)
-1.10
0.00
1.10
Re(λ)
0.9958
0.4979
0.0000
1.0
0.5
0.0



$N = 3 \quad L = 50$

Can apply large L expansion results for k to show that λ_{EP} satisfies

$$\lambda^N = \cos\left(\frac{2\pi j}{L}\right) \pm \mathrm{i}\sin\left(\frac{2\pi j}{L}\right).$$



$N = 3 \quad L = 50$

Concluding remarks

- We have located the quasi-energy EPs in the complex plane.
- Numerical tests confirm they correspond to Hamiltonian EPs.
- And also confirm that the corresponding eigenvectors coalesce.
- For large L they are on the unit circle in the complex λ plane.
- There are other degeneracies in the energy eigenspectrum, but they are not EPs.
- Although in the complex plane, EPs can influence properties (such as correlations) along the real axis..

full details in SciPost Physics **15**, 016 (2023)

3) The spin-1 biquadratic model

The known integrable (isotropic) spin-1 chains

$$H = \sum_j \left[J_1(\mathbf{S}_j \cdot \mathbf{S}_{j+1}) + J_2(\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 \right]$$

- Uimin-Lai-Sutherland model (1970,1974,1975)

$$\boxed{J_2 = J_1}$$

- Takhtajan-Babujian model (1982)

$$\boxed{J_2 = -J_1}$$

The other well known (isotropic) spin-1 chains

$$H = \sum_j \left[J_1(\mathbf{S}_j \cdot \mathbf{S}_{j+1}) + J_2(\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 \right]$$

- spin-1 Heisenberg model

$$\boxed{J_2 = 0}$$

- Affleck-Kennedy-Lieb-Tasaki (AKLT) model (1987)

$$\boxed{J_2 = \frac{1}{3} J_1}$$

$J_2 = 0$ is key to the Haldane conjecture (1983)

The AKLT model has an exact valence bond groundstate

The spin-1 biquadratic model

$$\boxed{J_1 = 0}$$

$$H = \sum_j J_2 (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2$$

The model is ferromagnetic for $J_2 > 0$
and antiferromagnetic for $J_2 < 0$.

Parkinson (1987,1988) observed a partial mapping of states with the anisotropic spin-1/2 XXZ chain at $\Delta = -3/2$.

He also noted that if $h_j = (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 - 1$ then

$$\boxed{h_j^2 = 3h_j}$$

!!!

Exact results via Temperley-Lieb equivalence

PHYSICAL REVIEW B VOLUME 40, NUMBER 7 1 SEPTEMBER 1989

Spectrum of the biquadratic spin-1 antiferromagnetic chain

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(Received 13 March 1989)

An exact correspondence between the staggered biquadratic spin-1 antiferromagnetic chain and the quantum Hamiltonian limit of the nine-state Potts model is established for finite chains with free ends. For uniform interactions this equivalence is used (via a further exact mapping to a Bethe ansatz soluble *XXZ* model) to calculate exactly the infinite lattice values of the ground-state energy per site and the (nonzero) gap to the lowest-energy excited state. Periodic boundary conditions and the nature of the ground state as a function of bond alternation are also discussed.

Model independently solved by Andreas Klümper (1989,1990)

EUROPHYSICS LETTERS 15 August 1989

Europhys. Lett., 9 (8), pp. 815-820 (1989)

New Results for q-State Vertex Models and the Pure Biquadratic Spin-1 Hamiltonian.

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(received 17 April 1989; accepted 5 June 1989)

PACS. 75.10H – Ising and other classical spin models.
PACS. 75.10J – Heisenberg and other quantized localized spin models.

Abstract. – New exactly solvable $SO(q)$-invariant q-state vertex models are introduced. We employ a new method using inversion relations which enables us to determine directly the spectra of the transfer matrices. The results are applied to related quantum spin chains. A special case is the pure biquadratic spin-1 Hamiltonian which turns out to be noncritical. Various quantities are calculated, *e.g.*, the energy gap and the correlation length.

$$e_j = (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 - 1$$

$$\begin{aligned} e_j^2 &= 3\, e_j \\ e_j e_{j\pm1} e_j &= e_j \\ e_j e_i &= e_i e_j \qquad |i-j| > 1 \end{aligned}$$

In fact **three** TL equivalent models, each a rep of the TL algebra:

1) spin-1 biquadratic chain

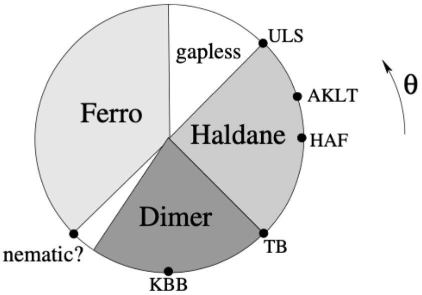
2) spin-1/2 XXZ chain at $\Delta = -3/2$

3) 9-state Potts chain

Used to obtain exact ground state energy, mass gap etc

The spin-1 bilinear-biquadratic model

$$H = \sum_j \left[\cos\theta \, (\mathbf{S}_j \cdot \mathbf{S}_{j+1}) + \sin\theta \, (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 \right]$$



phase diagram [from Mikeska & Kolezhuk, Lecture Notes in Physics (2004)]

Ferromagnetic magic

Groundstate degeneracies ↔ Fibonacci-Lucas sequences

L	OBC		PBC	
	$E_{gs}^{\mathrm{OBC}}(L)$	$\dim(\Omega_L^{\mathrm{OBC}})$	$E_{gs}^{\mathrm{PBC}}(L)$	$\dim(\Omega_L^{\mathrm{PBC}})$
2	0	8	0	8
3	0	21	0	18
4	0	55	0	47
5	0	144	0	123
6	0	377	0	322
7	0	987	0	843
8	0	2584	0	2207
10	0	17711	0	15127

$$\dim(\Omega_L^{\mathrm{OBC}}) = \frac{(3+\sqrt{5})^{L+1} - (3-\sqrt{5})^{L+1}}{2^{L+1}\sqrt{5}}, \qquad L \geq 2$$

$$\dim(\Omega_L^{\mathrm{PBC}}) = \frac{(3+\sqrt{5})^{L} + (3-\sqrt{5})^{L}}{2^{L}}, \qquad L \geq 3$$

Now in terms of the golden ratio $R = (\sqrt{5} - 1)/2$

$$\dim(\Omega_L^{\mathrm{OBC}}) = (R^{-2L-2} - R^{2L+2})/(R^{-2} - R^2), \quad L \geq 2$$
$$\dim(\Omega_L^{\mathrm{PBC}}) = R^{-2L} + R^{2L}, \quad L \geq 3$$

This leads to non-zero residual entropy

$$\boxed{S_r = -2 \log R}$$

see also

N Read and H Saleur, Nucl. Phys. B, 777, 263-315 (2007)

B Aufgebauer and A Klümper, J. Stat. Mech. P05018 (2010)

Y-T Oh, H Katsura, H-Y Lee and J H Han, Phys. Rev. B 96, 165126 (2017)

S Moudgalya and O I Motrunich, Phys. Rev. X 12, 011050 (2022)

Entanglement entropy

For ferromagnetic states it has been established that there is a connection between entanglement entropy and the counting rule for Nambu-Goldstone modes in Spontaneous Symmetry Breaking (SSB).

In particular, the entanglement entropy $S(n)$ for the highly degenerate ground states scales logarithmically, in the thermodynamic limit, with the block size n according to

$$S(n) = \frac{N_B}{2} \log_2 n + S_0.$$

The prefactor is precisely half the number N_B of type-B GMs.

Connecting with work by Castro-Alvaredo & Doyon (2012) from a field theoretic perspective, the above result leads to the identification of the number of type-B GMs with the fractal dimension d_f, namely $d_f = N_B$.

For this model we have constructed sequences of degenerate ground states generated from highest and generalized highest weight states to establish that the entanglement entropy indeed scales logarithmically with block size in the thermodynamic limit, with the prefactor indeed being half the number of type-B Goldstone modes.

The SSB pattern is from SU(3) to U(1) x U(1), with two type-B GMs.

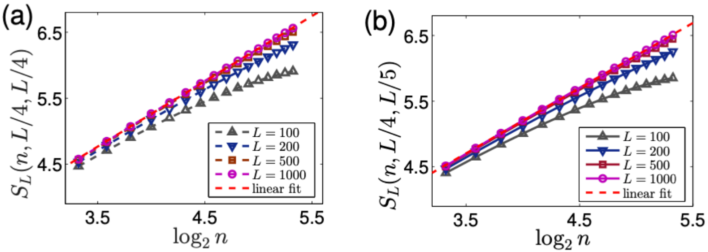


FIG. 1. (a) The entanglement entropy $S_L(n, M_1, M_2)$ vs $\log_2 n$ for the ferromagnetic spin-1 biquadratic model. Here $M_1 = M_2 = L/4$ for the indicated L values. (b) The entanglement entropy $S_L(n, M_1, M_3)$ vs $\log_2 n$. Here $M_1 = L/4$ and $M_3 = L/5$ for the indicated L values. The block size n is a multiple of two, and ranges from 10 to 40.

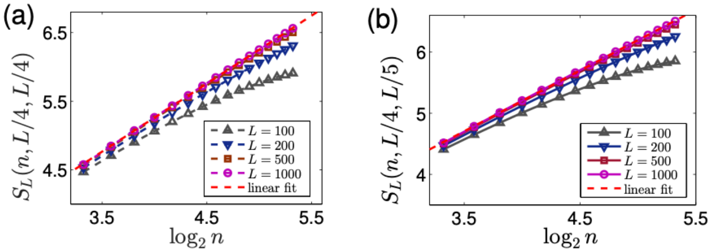
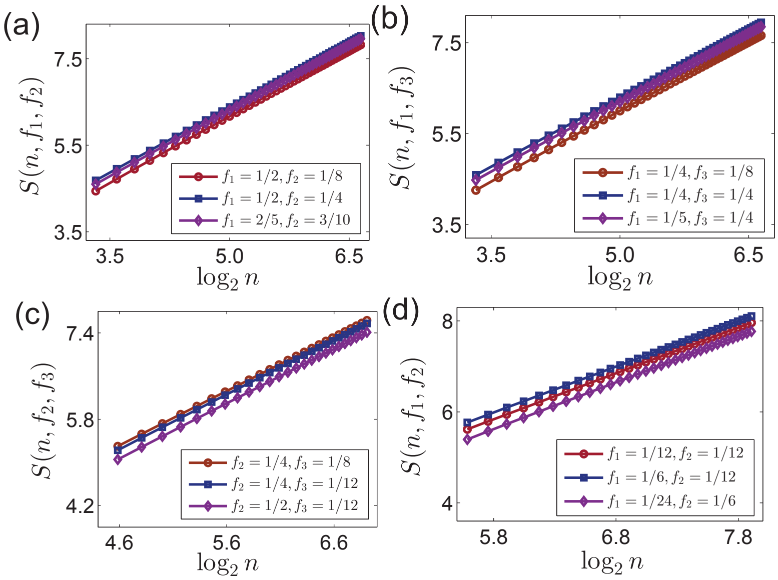


FIG. 2. (a) The entanglement entropy $S_L(n, M_2, M_3)$ vs $\log_2 n$. Here $M_2 = L/4$ and $M_3 = L/8$ for the indicated L values. A generalized highest weight state is chosen to be $|1110...1110\rangle$. The block size n is a multiple of four, and ranges from 12 to 60. (b) The entanglement entropy $S_L(n, M_1, M_2)$ vs $\log_2 n$. Here $M_1 = L/12$ and $M_2 = L/12$ for the indicated L values. A generalized highest weight state is chosen to be $|10{-}1{-}101...10{-}1{-}101\rangle$. The block size n is a multiple of six, and ranges from 12 to 120.



In each of the plots the prefactor is close to 1, within errors of 1.2% to 2%.

Concluding remarks

- The highly degenerate ferromagnetic groundstates of the spin-1 biquadratic model have a rich mathematical structure.
- The groundstate degeneracies for this model are asymptotically the golden spiral.
- They are also highly entangled, scale-invariant states, originating from spontaneous symmetry breaking from SU(3) to U(1) $\times$ U(1) with two type-B Goldstone modes.
- The entanglement entropy scales logarithmically with the block size in the thermodynamic limit, with the prefactor being half the number of type-B Goldstone modes.
- The latter in turn is identified to be the dimension of an abstract fractal in the Hilbert space.

full details

arXiv > cond-mat > arXiv:2302.13126

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Condensed Matter > Strongly Correlated Electrons

[Submitted on 25 Feb 2023 (v1), last revised 30 Nov 2023 (this version, v3)]

Goldstone modes and the golden spiral in the ferromagnetic spin-1 biquadratic model

Huan-Qiang Zhou, Qian-Qian Shi, Ian P. McCulloch, Murray T. Batchelor

Ferromagnetic ground states have often been overlooked in comparison to seemingly more interesting antiferromagnetic ground states. However, both the physical and mathematical structure of ferromagnetic ground states are particularly rich. We show that the highly degenerate and highly entangled ground states of the ferromagnetic spin-1 biquadratic model are scale invariant, originating from spontaneous symmetry breaking from $\mathrm{SU}(3)$ to $\mathrm{U}(1) \times \mathrm{U}(1)$ with two type-B Goldstone modes. The ground state degeneracies are characterized as the Fibonacci-Lucas sequences -- an ancient mathematical gem, under open and periodic boundary conditions, with the residual entropy being non-zero. This implies that the ground state degeneracies for this model are asymptotically the golden spiral. In addition, sequences of degenerate ground states generated from highest and generalized highest weight states are constructed to establish that the entanglement entropy scales logarithmically with the block size in the thermodynamic limit, with the prefactor being half the number of type-B Goldstone modes. The latter in turn is identified to be the fractal dimension.

Comments: 20 pages, 3 figures,1 table, additional references to earlier related work added, a minor change
Subjects: **Strongly Correlated Electrons (cond-mat.str-el)**; Mathematical Physics (math-ph)
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THANK YOU!