The Half-space Open Exclusion Process

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MATRIX, Mathematics & Physics of Integrability
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Outline

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1. Six-vertex model with boundaries
2. Partition function G

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3. Open boundary analogue of domain wall partition function

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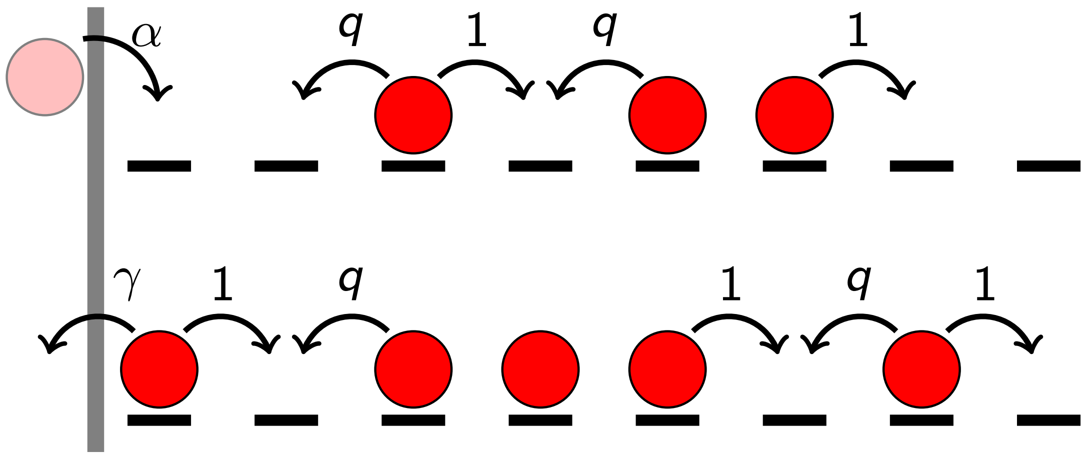
4. Limit to half-space open ASEP
5. Observables of the open ASEP

ASEP on the half-line

- We study the Asymmetric Simple Exclusion Process on the half-line



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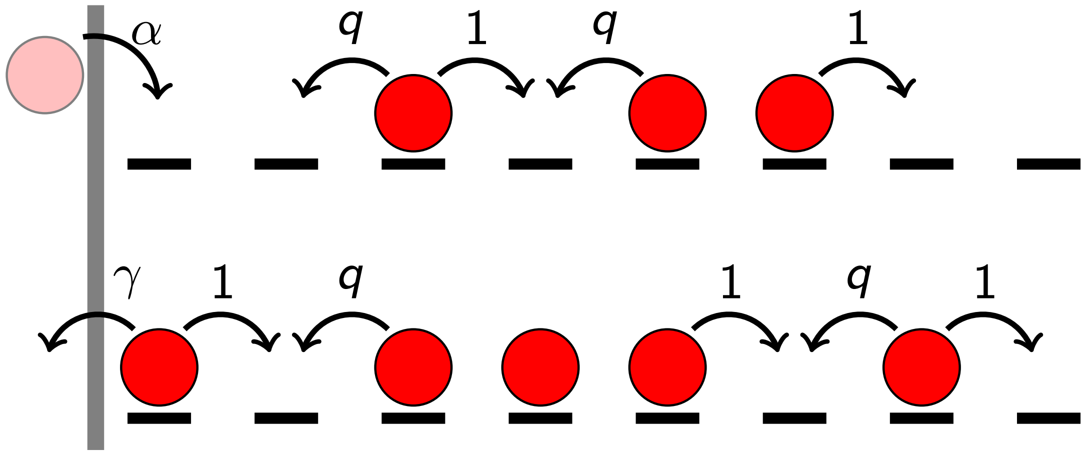

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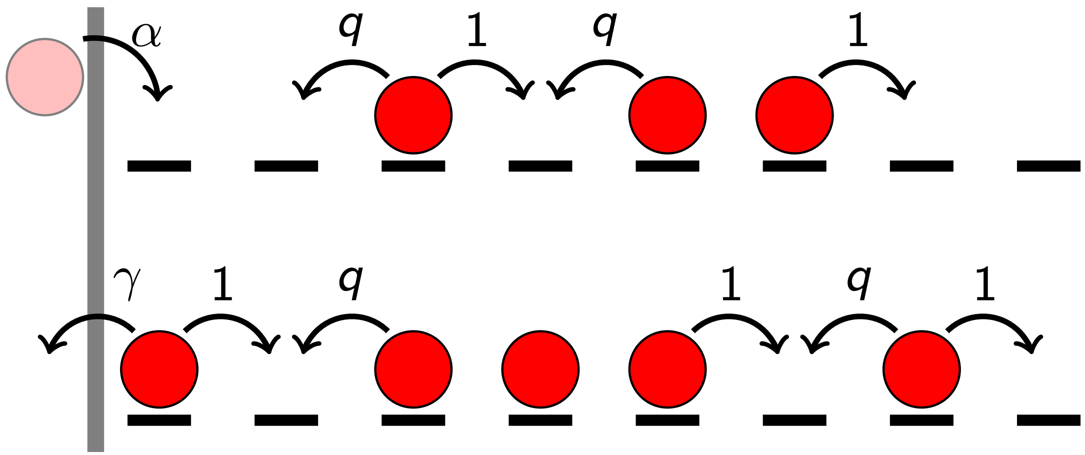

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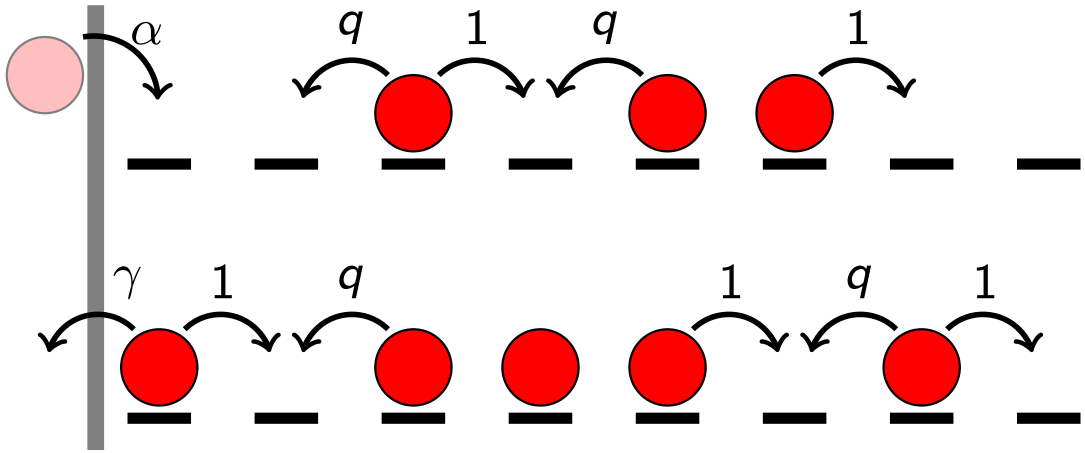

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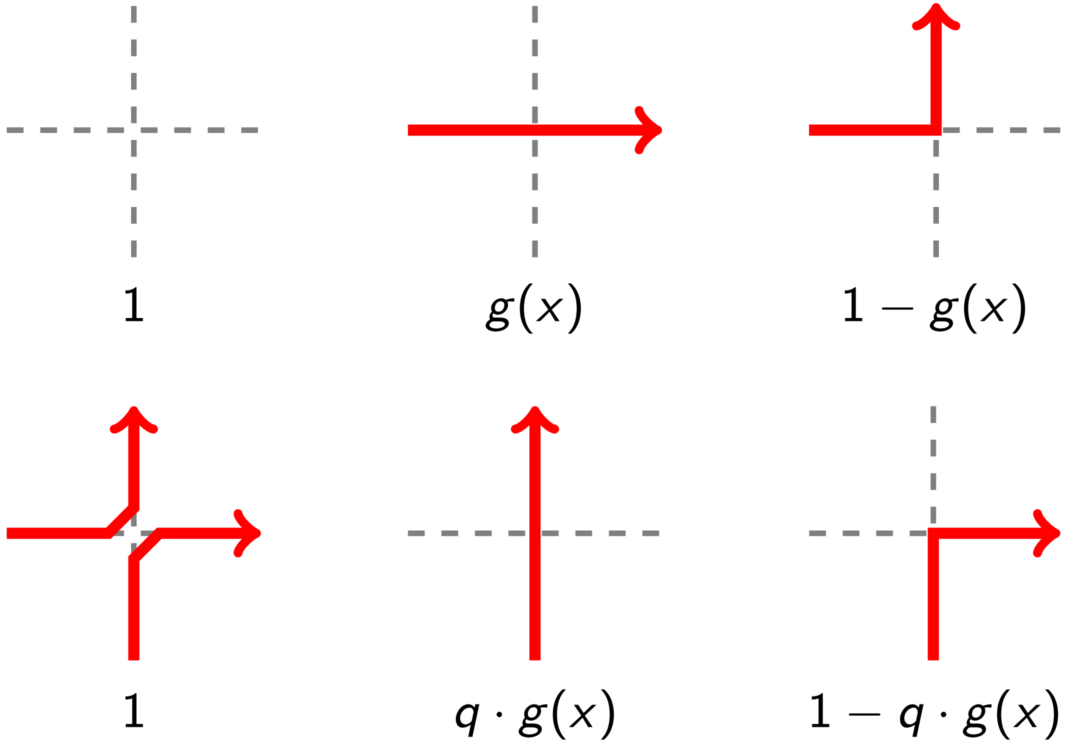
- We will recover this quantity as a reduction of an integrable vertex model.

Stochastic Six Vertex Model

- We study the six-vertex model with stochastic weights



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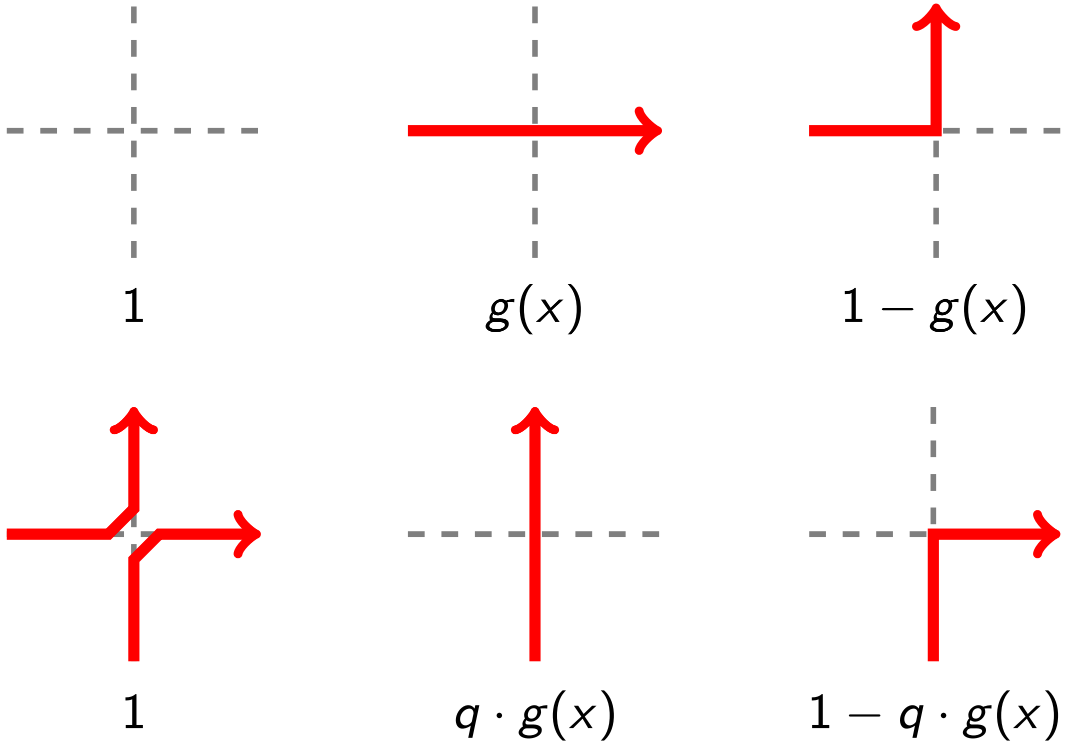

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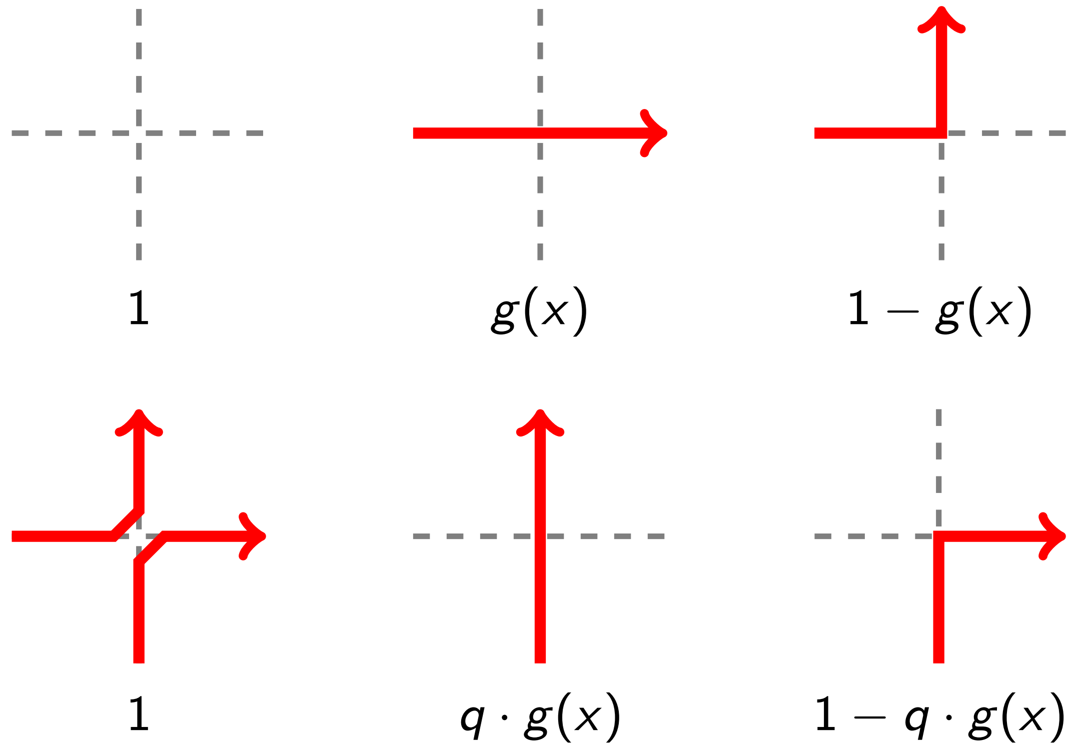

- $g(x) = \dfrac{1-x}{1-qx}$ is simple rational function.
- The classical partition function is computed by summing over connected path configurations

$$\mathcal{Z} = \sum_{\Omega} g(x)^{\#}(1-g(x))^{\#}(qg(x))^{\#}(1-qg(x))^{\#}.$$

Yang-Baxter Equation

- The R-matrix of the model satisfies the Yang-Baxter equation

$$R_{12}(y/x)R_{13}(z/x)R_{23}(z/y) = R_{23}(z/y)R_{13}(z/x)R_{12}(y/x)$$

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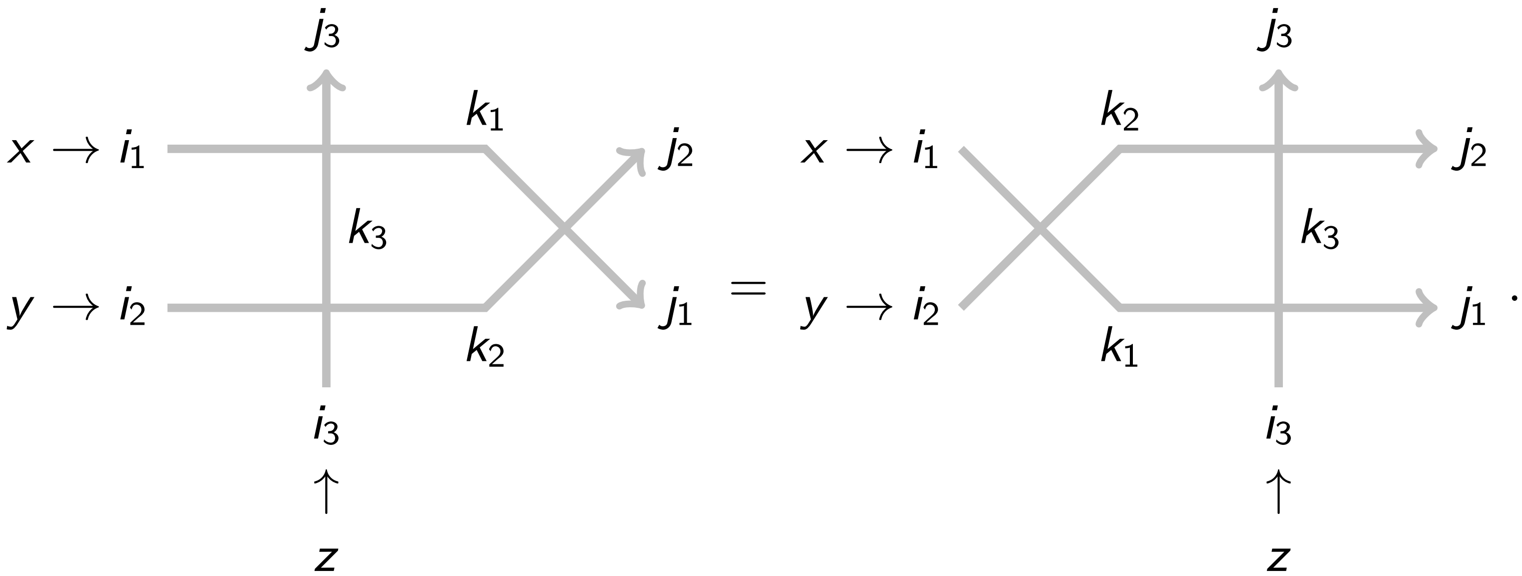

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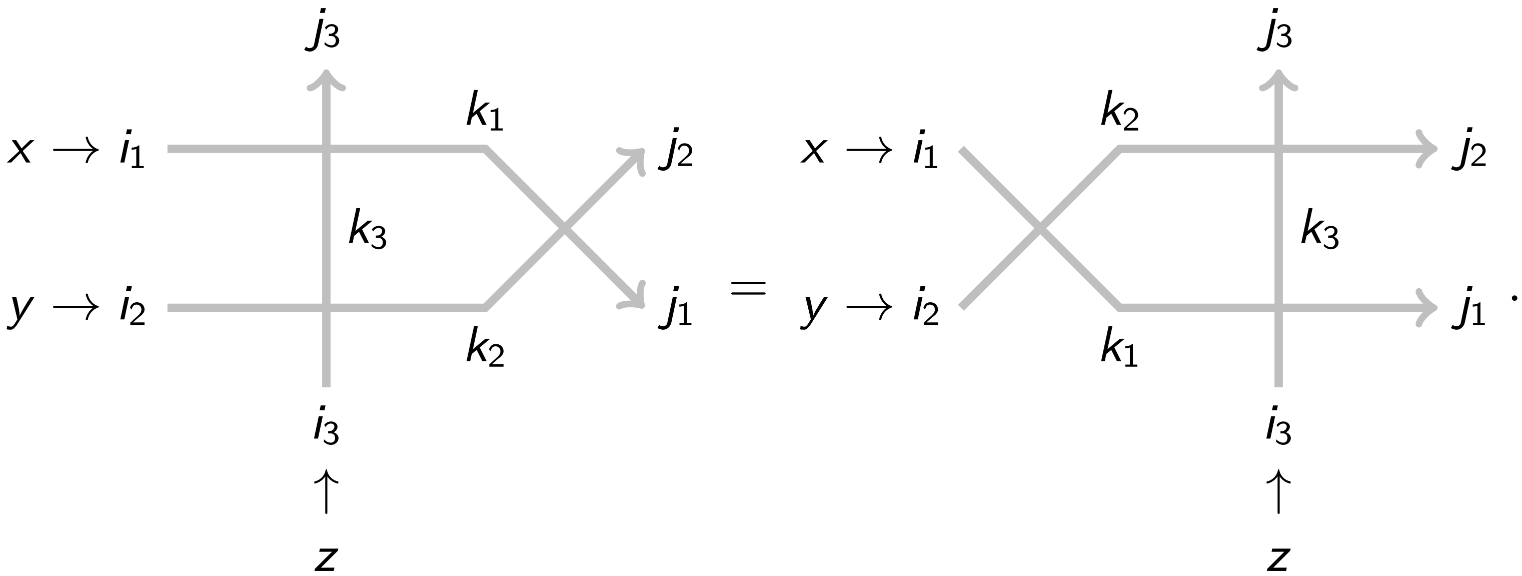
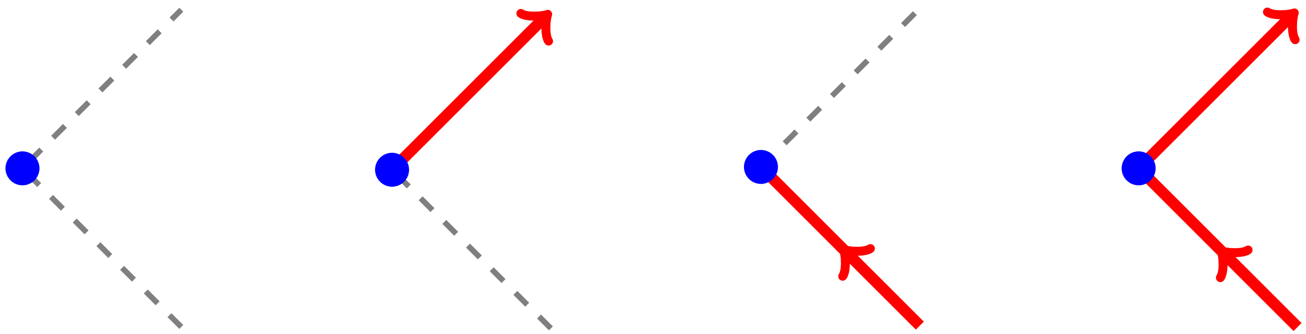
- 

- Proof by explicit check.

Boundary Vertices

- We introduce boundary vertices which depend on 2 parameters



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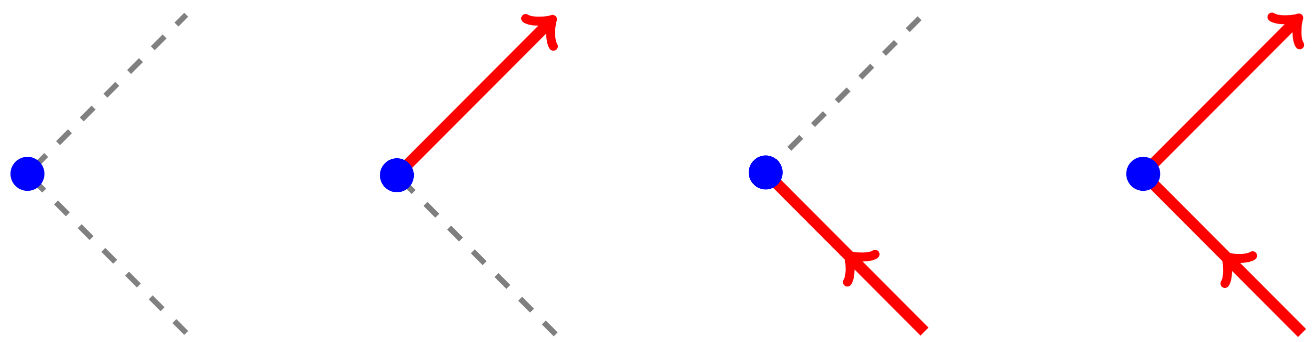

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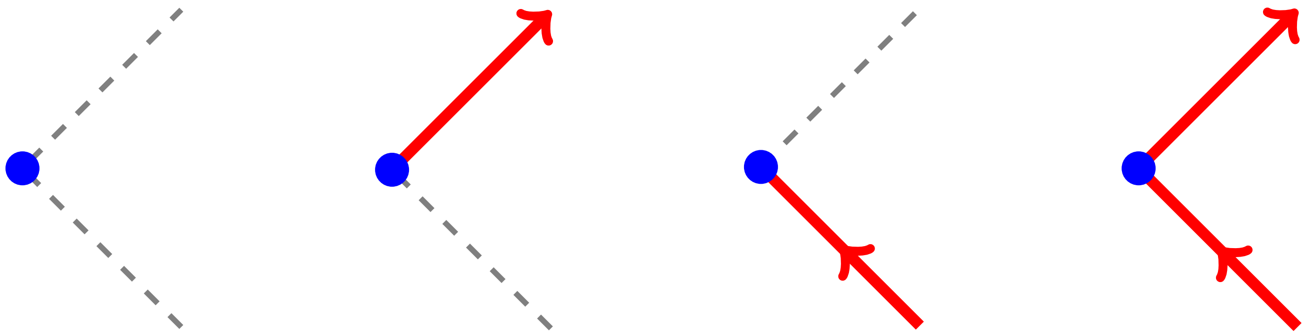

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- These weights are also stochastic.

Reflection Equation

- The boundary weights form the entries of a stochastic K-matrix which satisfies the reflection equation

$$R_{21}\left(\frac{x}{y}\right) K_1(x) R_{12}(xy) K_2(y) = K_2(y) R_{21}(xy) K_1(x) R_{12}\left(\frac{x}{y}\right).$$

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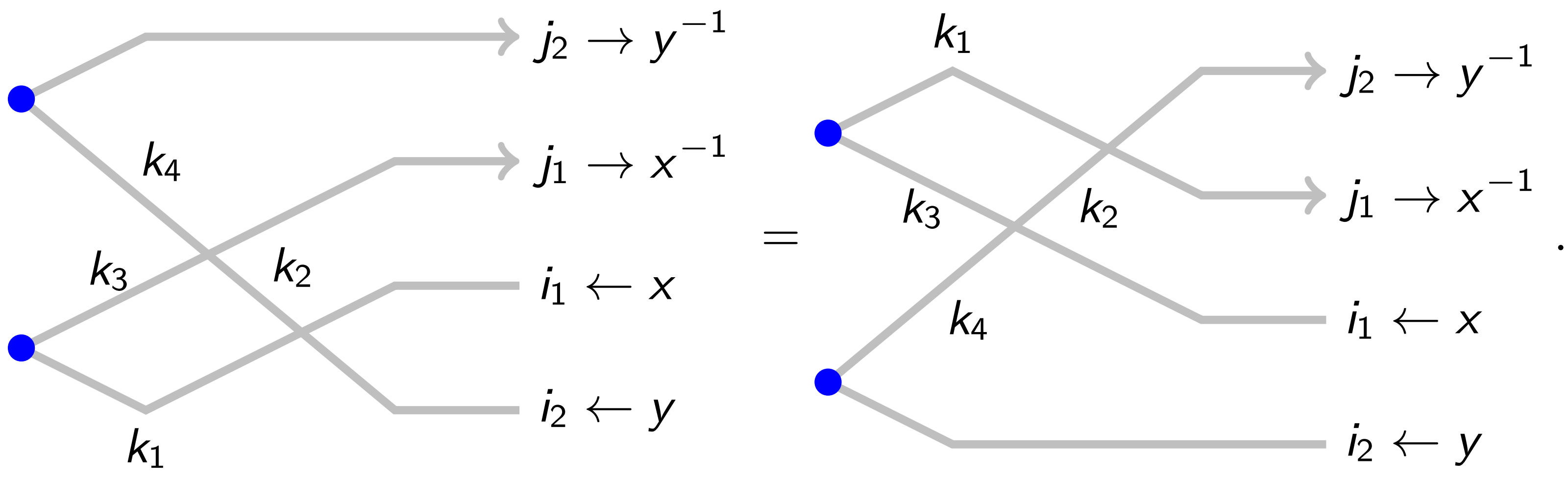
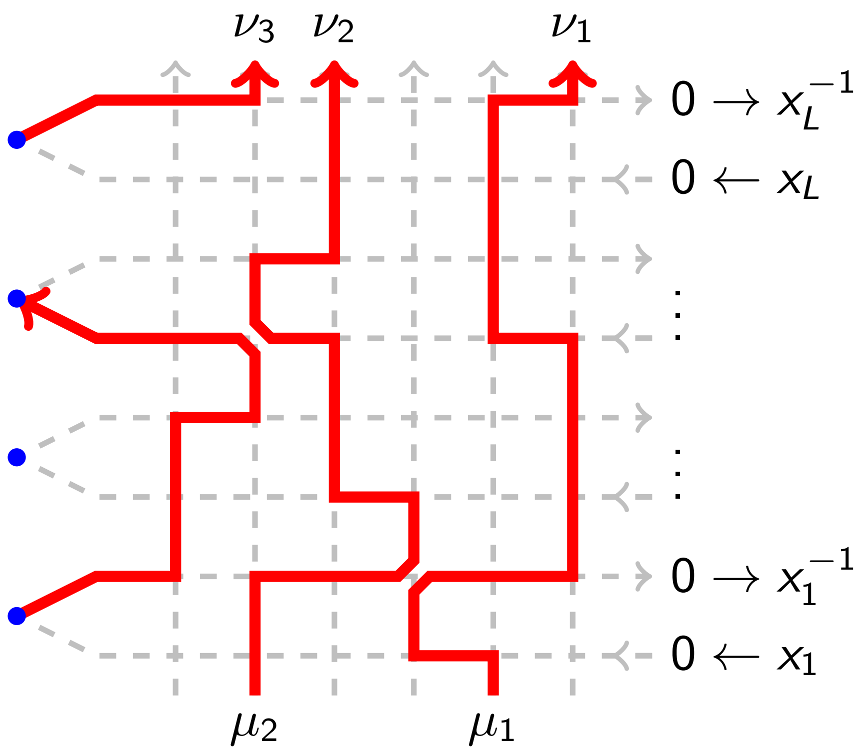
- Represented graphically



- We define the partition function indexed by configurations μ, ν

$G_{\nu/\mu}(x_1, \ldots, x_L) =$



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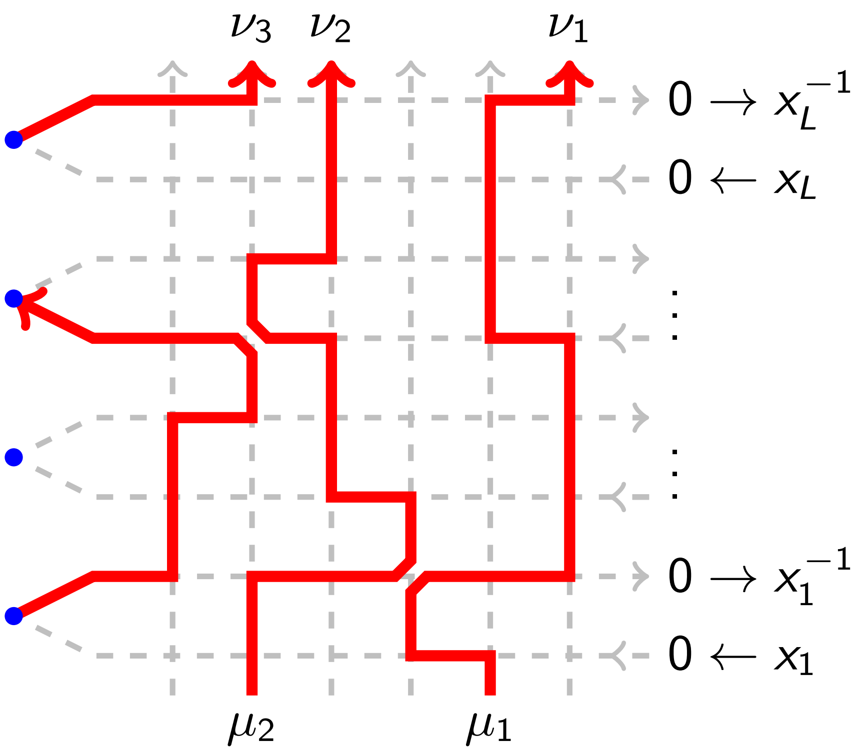

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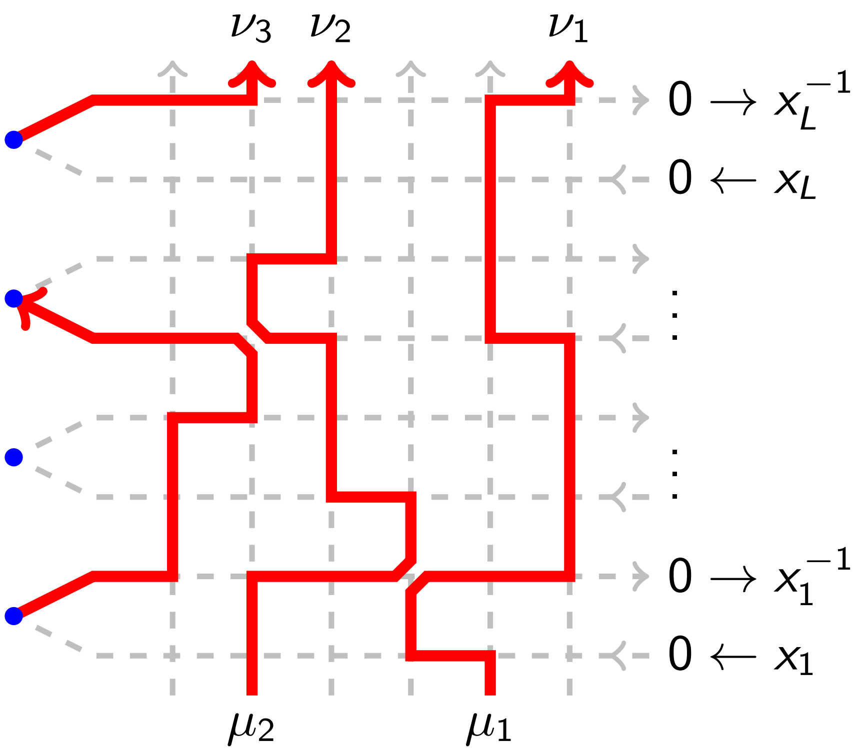

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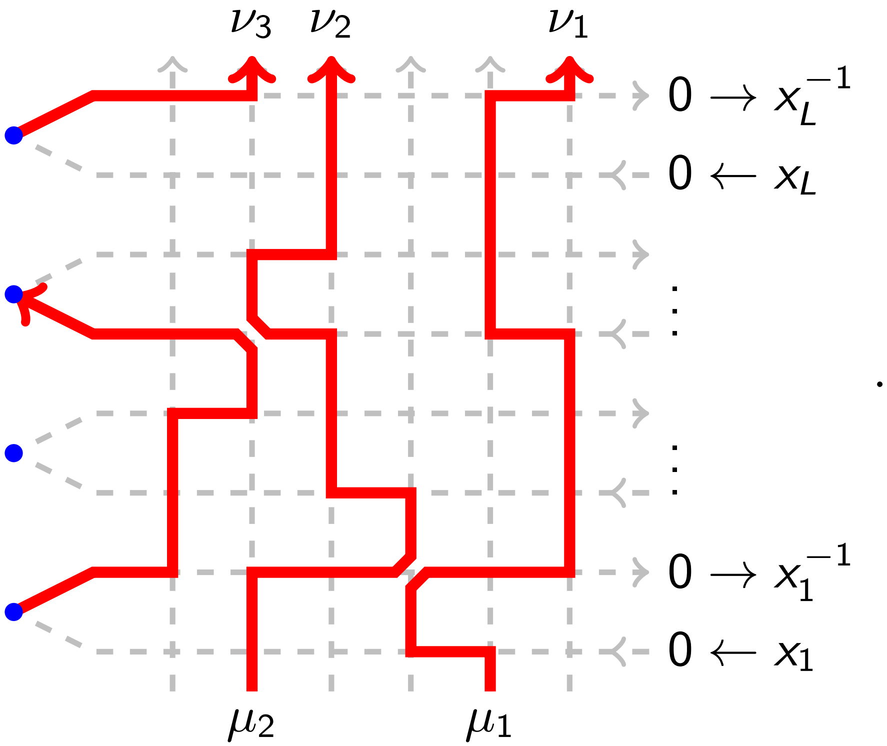

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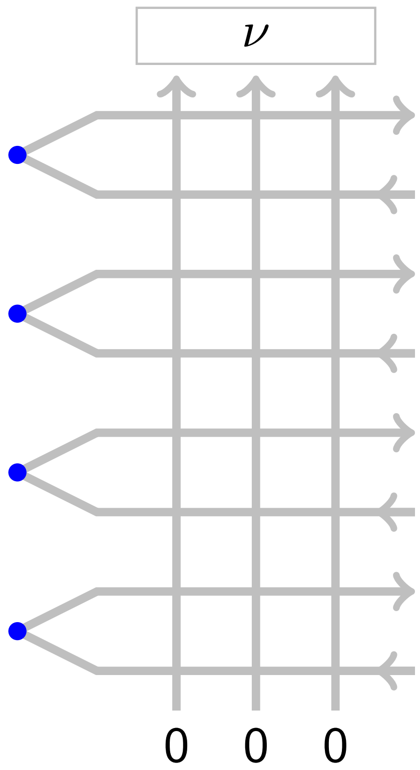
The function $G_{\nu/\mu}$ is a symmetric function in the x-alphabet.

- It is sufficient to show that $G_{\nu/\mu}(x_1, x_2) = G_{\nu/\mu}(x_2, x_1)$.

Evaluation

- When the bottom state is empty the partition function is given by the simplified diagram



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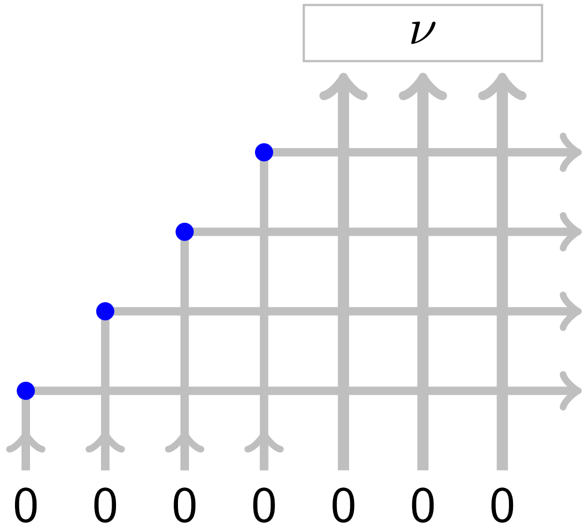

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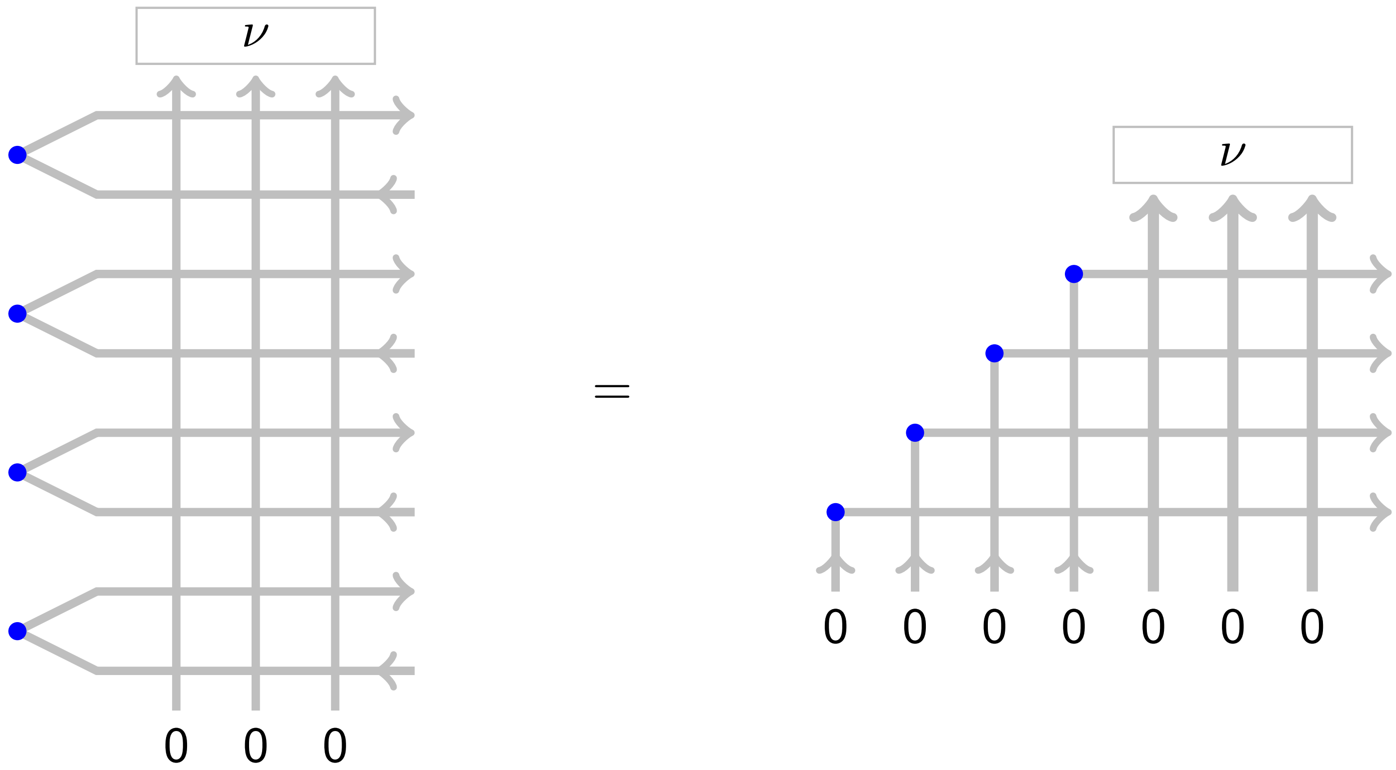

- Diagrammatic proof by Yang-Baxter application.

Triangular Partition Function

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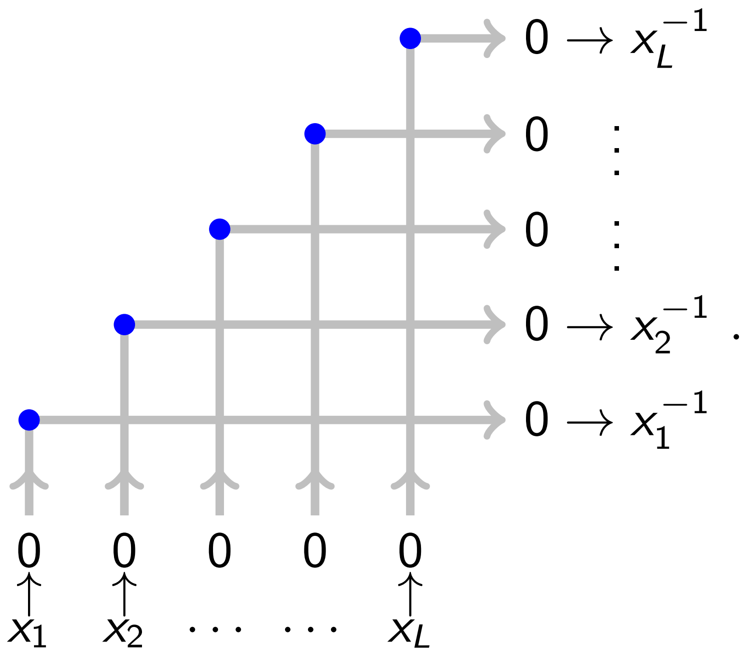

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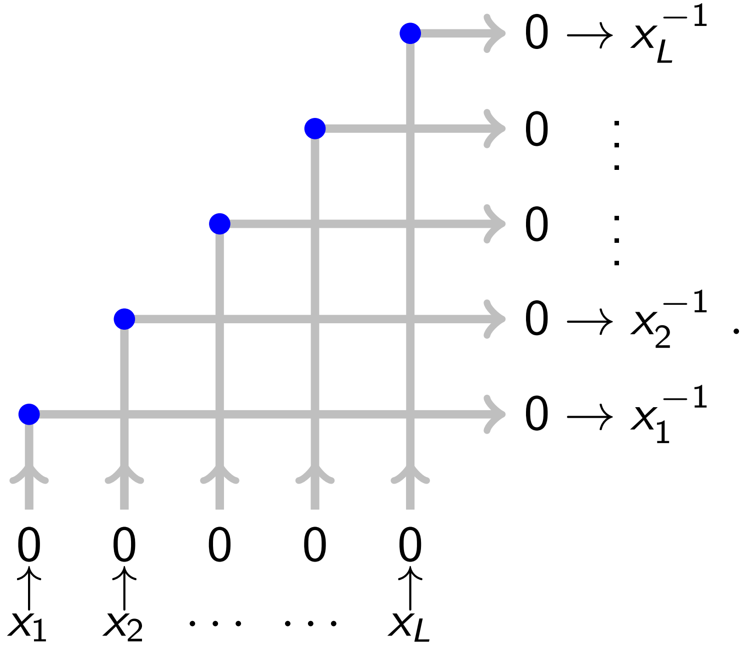
$Z_L(x_1 \ldots, x_L) =$ 

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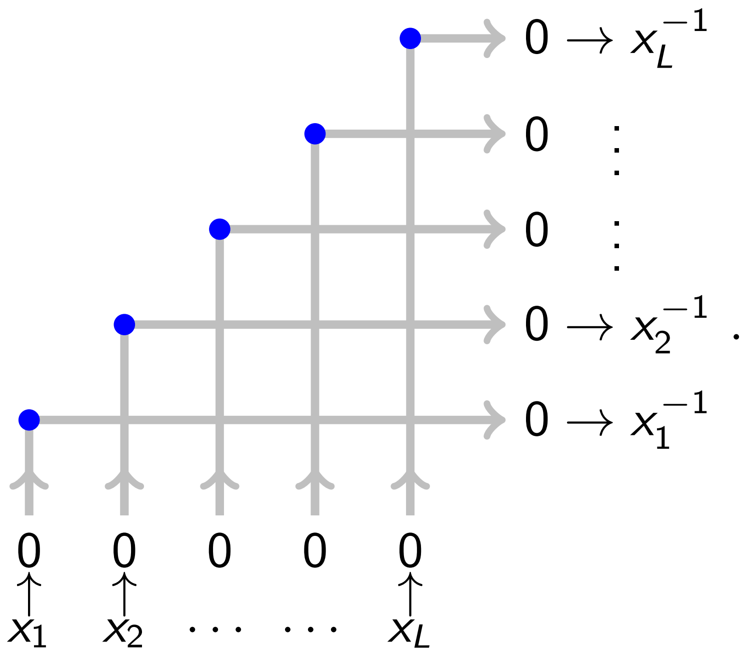

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Theorem

When both top and bottom configurations are empty

$$G_{\emptyset/\emptyset}(x_1, \ldots, x_L) = Z(x_1, \ldots, x_L).$$

Pfaffian formula

Theorem

The triangular partition function admits a Pfaffian formula

$$Z_L(x_1, \ldots, x_L) = \prod_{1 \le i < j \le L} \frac{1 - x_i x_j}{x_i - x_j} \cdot \mathrm{Pf}\left(\frac{x_i - x_j}{1 - x_i x_j} Q(x_i, x_j) \right)_{1 \le i,j \le L},$$

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- Shuffle product techniques are convenient to prove the Pfaffian satisfies the recursion relations.

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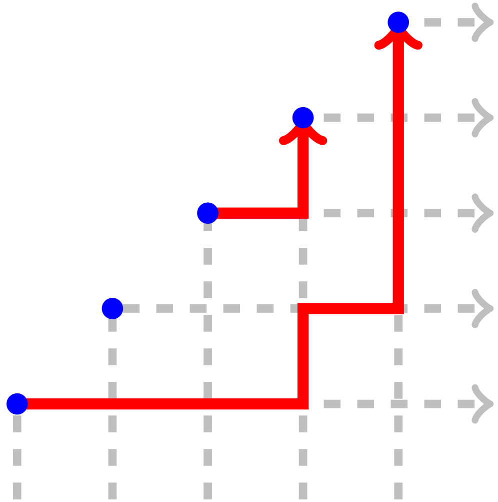

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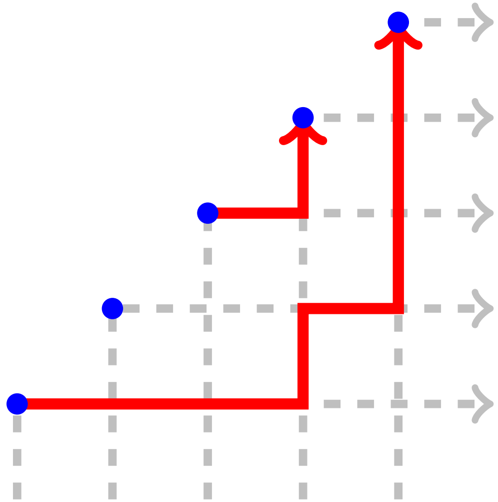

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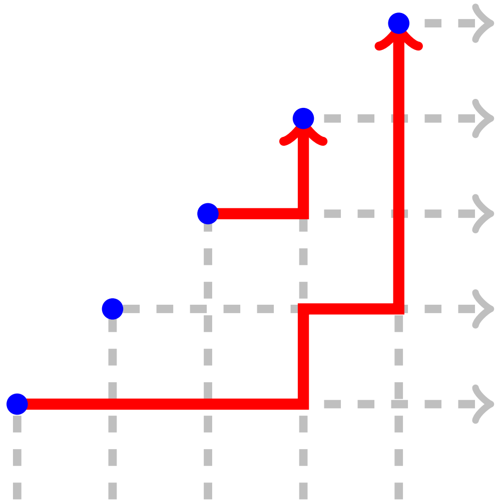

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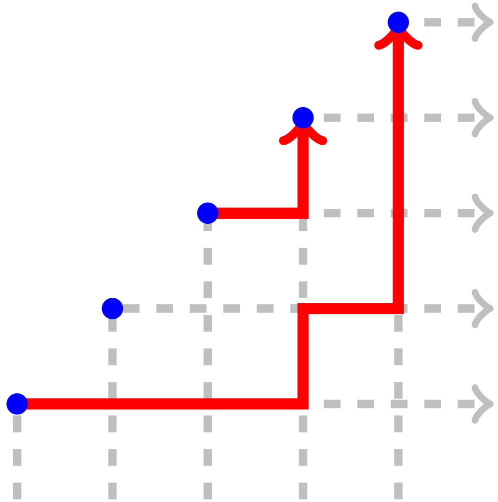

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- The enumeration was recently completed by Behrend–Fischer–Koutschan in 2023.

Integral Formula

Theorem

When the bottom configuration is empty, G has an integral formula

$$G_{\nu/\emptyset}(x_1,\ldots,x_L) = \oint_{\mathcal{C}} \frac{\mathrm{d}w_1}{2\pi\mathrm{i}} \cdots \oint_{\mathcal{C}} \frac{\mathrm{d}w_n}{2\pi\mathrm{i}} Z_{L+n}\left(x_1,\ldots,x_L,w_1^{-1},\ldots,w_n^{-1}\right) \prod_{i=1}^{n} f_{\nu_i}(w_i) \prod_{1\le i<j\le n}\left[\frac{w_j-w_i}{qw_j-w_i}\frac{1-qw_iw_j}{1-w_iw_j}\right]$$

The contours enclose simple poles at all points $w_i = x_1,\ldots,x_L$. n is the number of non-zero entries in ν.

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- Recursive proof over the entries in ν.

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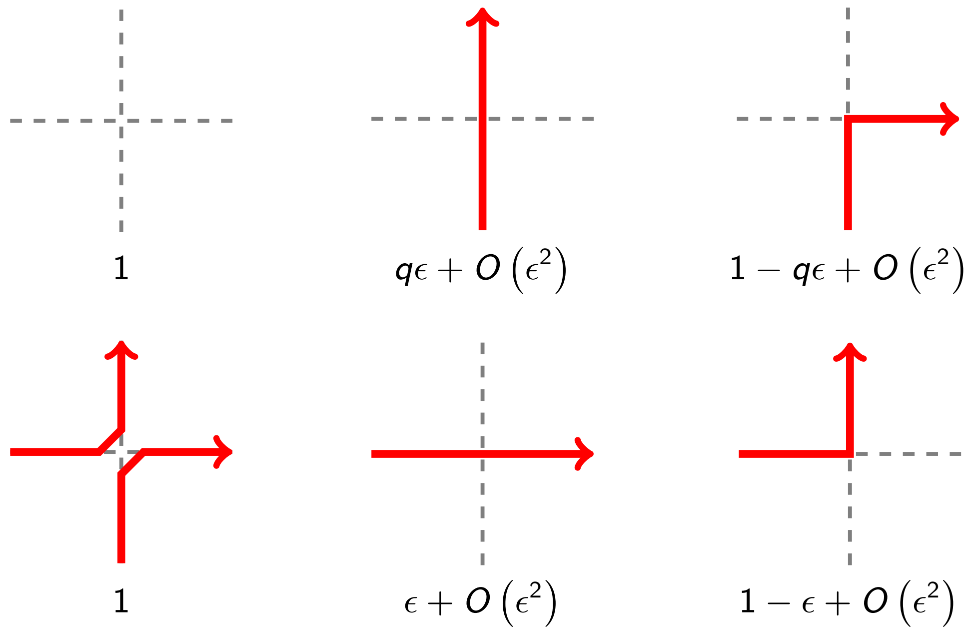

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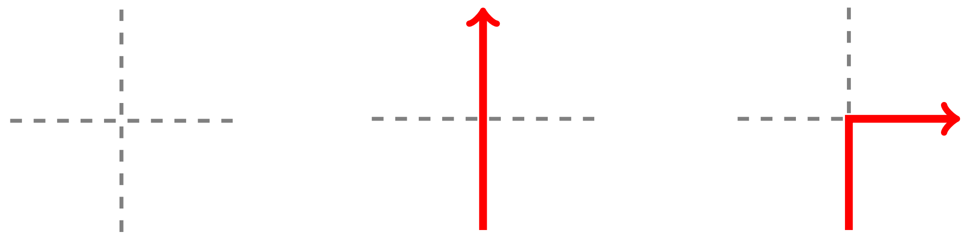

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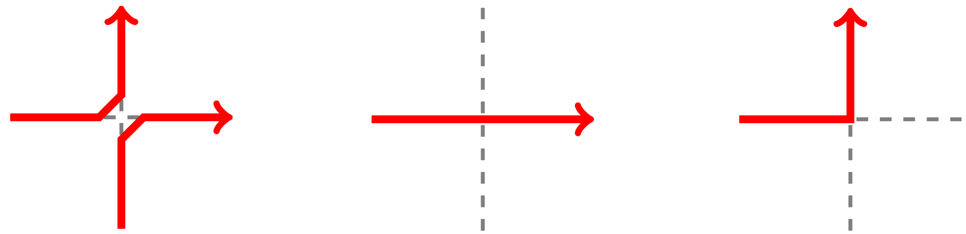

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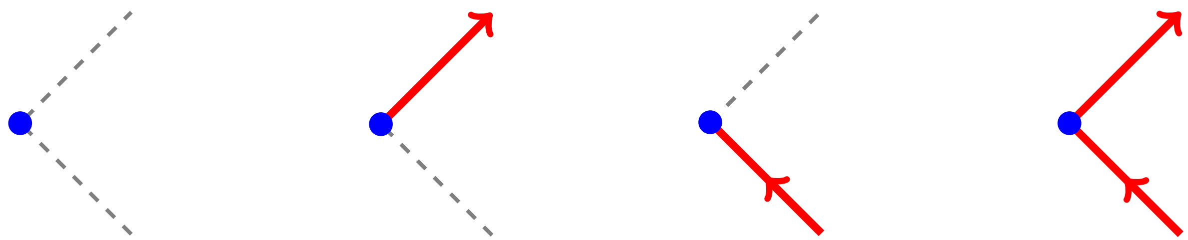
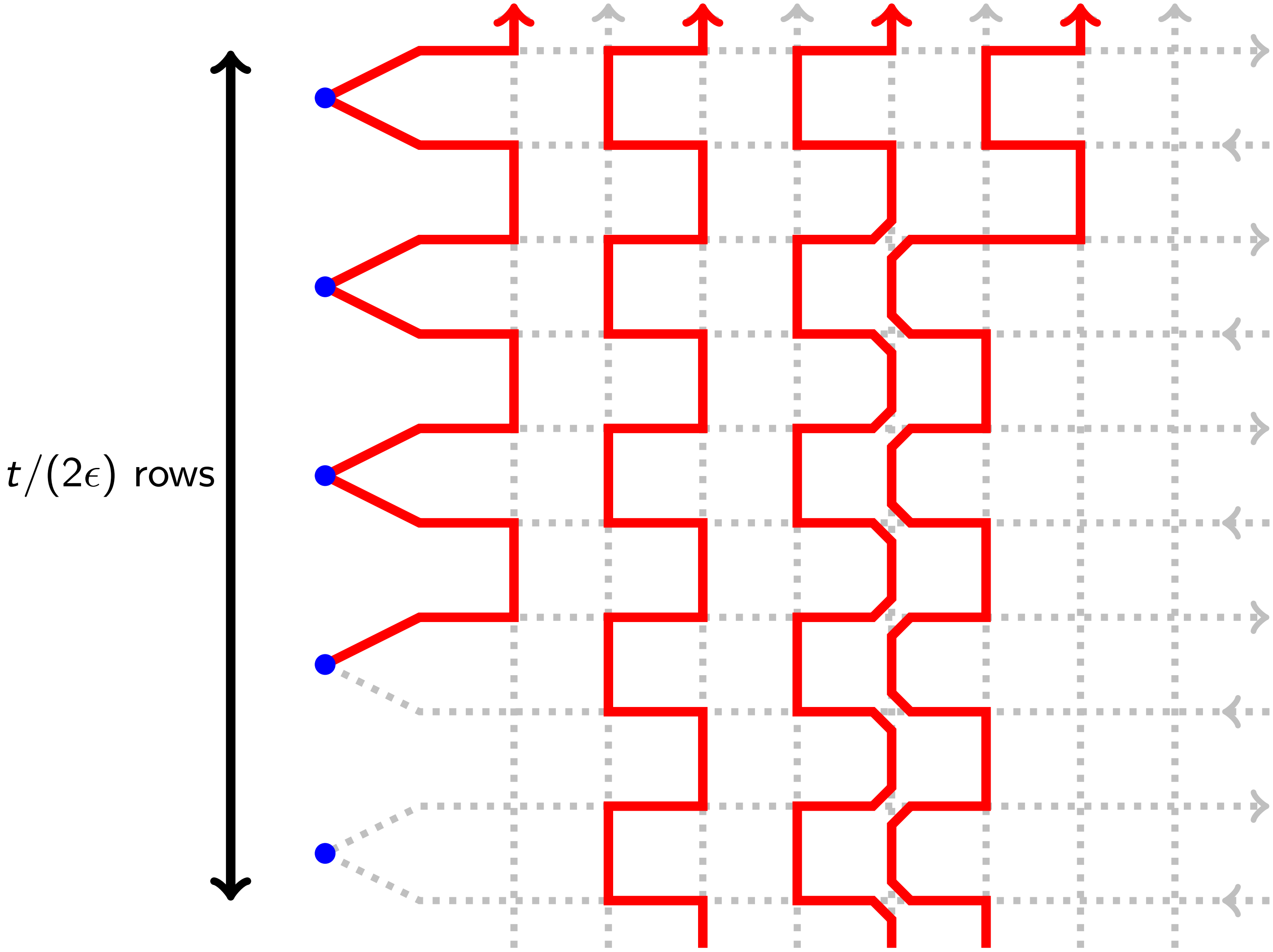


$1 - 2\alpha\epsilon + O\left(\epsilon^2\right) \qquad 2\alpha\epsilon + O\left(\epsilon^2\right) \qquad 2\gamma\epsilon + O\left(\epsilon^2\right) \qquad 1 - 2\gamma\epsilon + O\left(\epsilon^2\right)$



$t/(2\epsilon)$ rows

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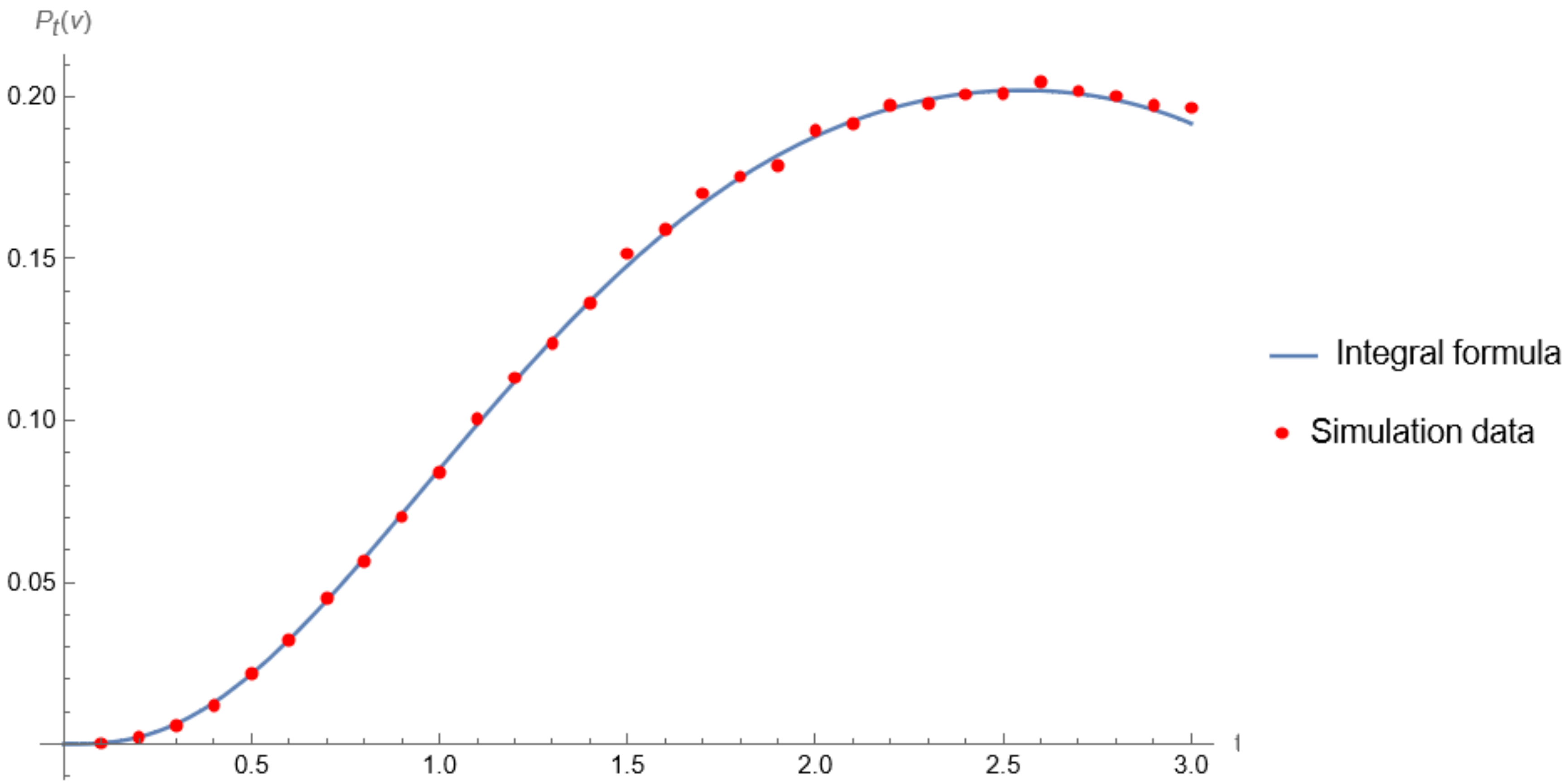

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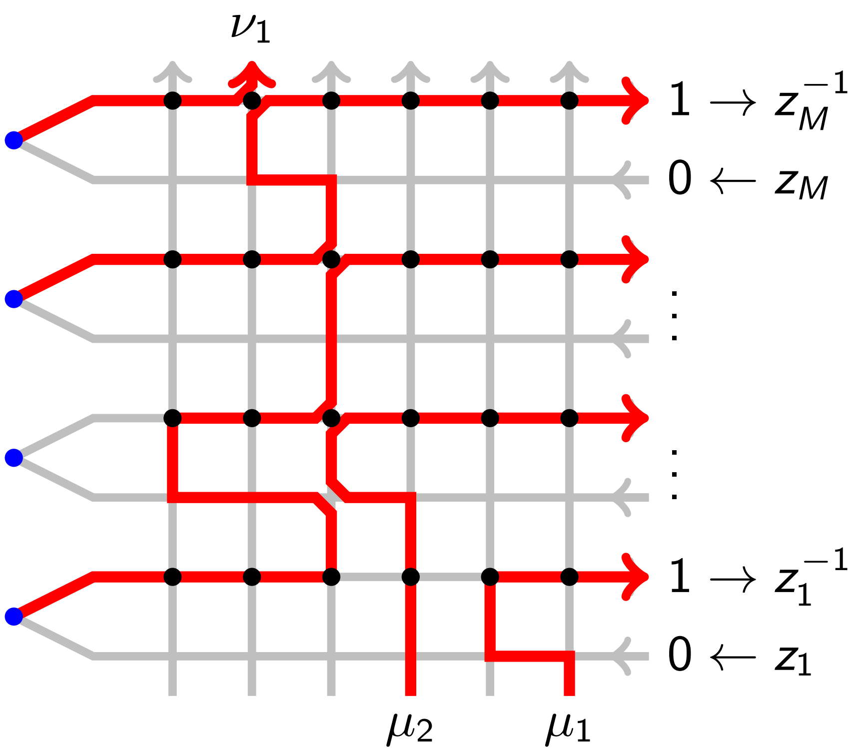
- This quantity is expected to obey a large deviation principle described by macroscopic fluctuation theory (MFT) at the diffusive scale with $q = 1$ (SSEP).

Dual Family

- We may also define the partition function

$F_{\mu/\nu}(z_1, \ldots, z_M) =$



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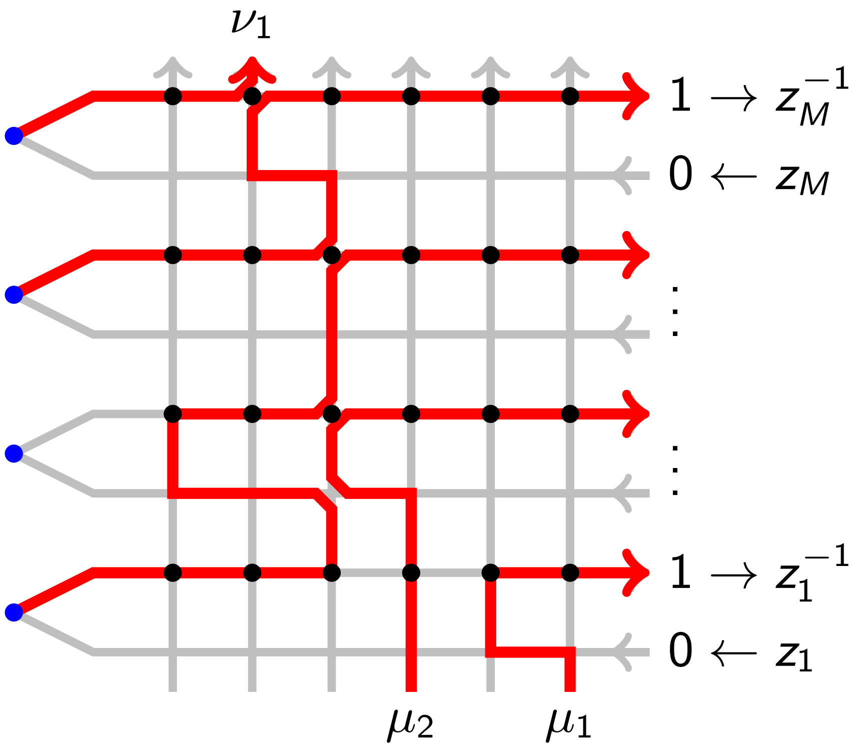
$F_{\mu/\nu}(z_1, \ldots, z_M) =$ 

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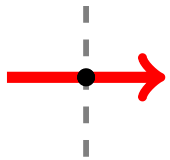 $= 1.$

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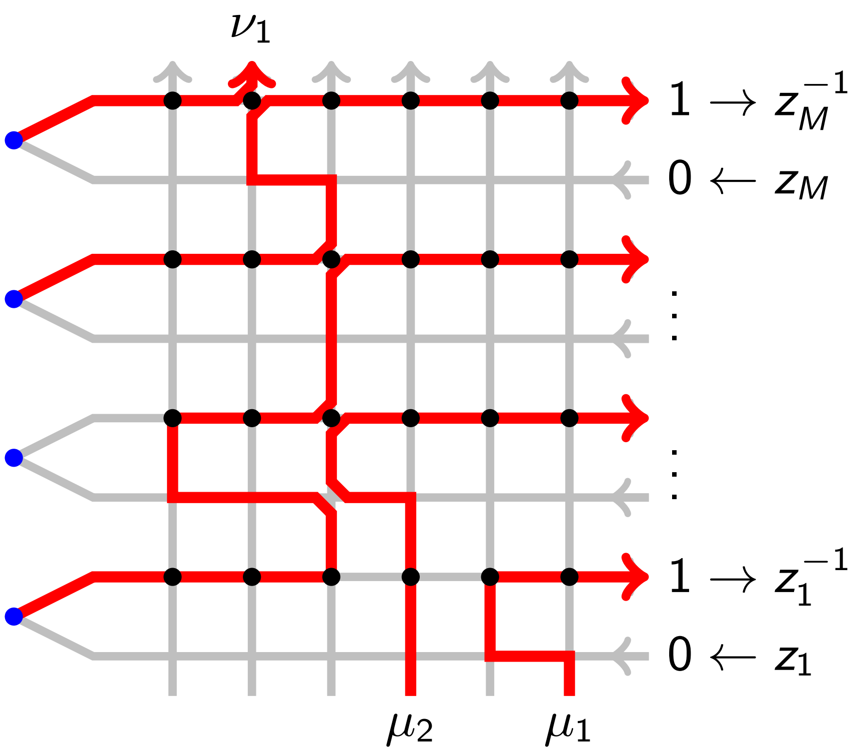

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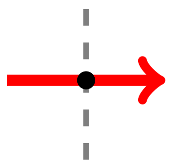


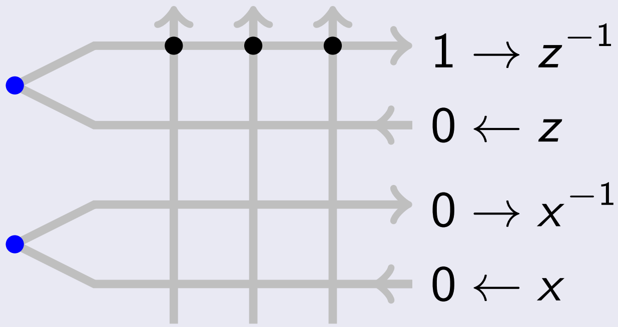
$$= 1.$$

- $F_{\mu/\nu}$ is a symmetric function in the z-alphabet.

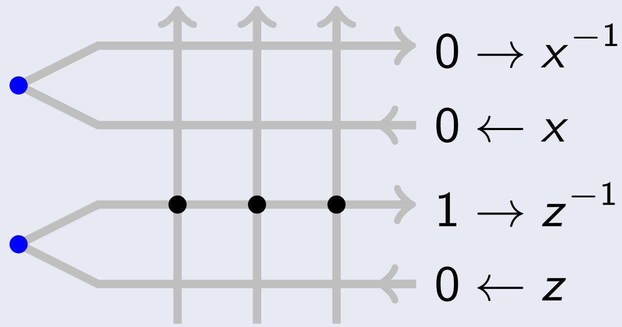
Cauchy Identity

Proposition

There is an exchange relation between double-row transfer matrices



$$= \frac{x - qz}{x - z} \frac{1 - xz}{1 - qxz} \cdot$$



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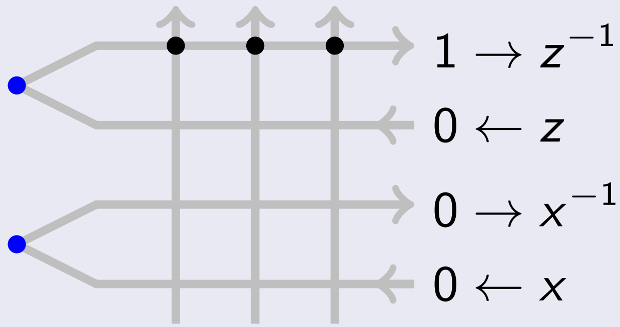

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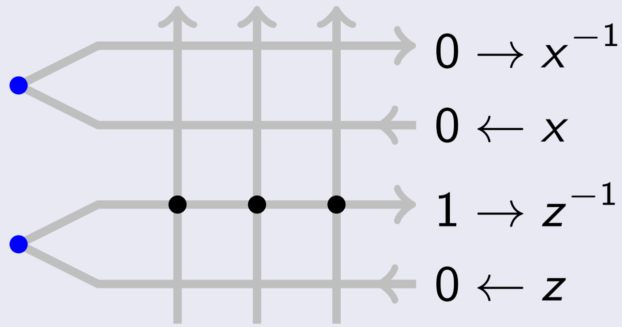

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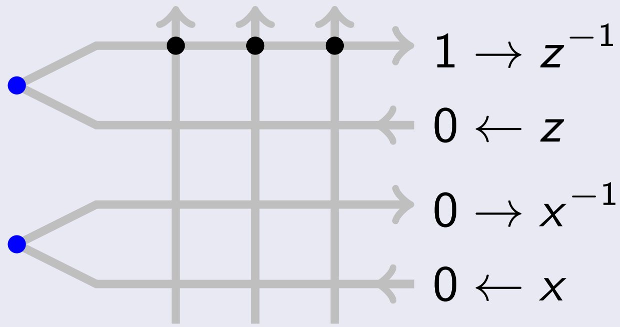

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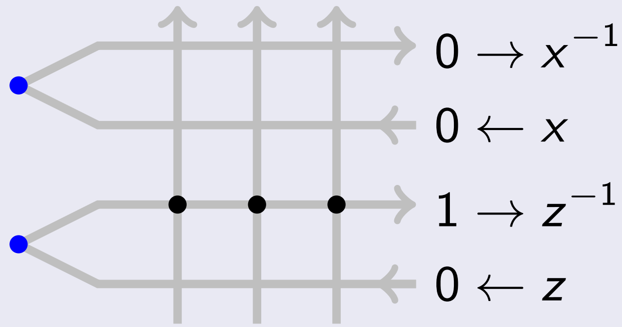

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Theorem

$$\sum_{\kappa} G_{\kappa/\mu}(x_1, \ldots, x_L) F_{\kappa/\nu}(z_1, \ldots, z_M)$$

$$= \prod_{i=1}^{M} \prod_{j=1}^{L} \left[\frac{x_j - qz_i}{x_j - z_i} \frac{1 - z_i x_j}{1 - qz_i x_j} \right] \sum_{\lambda} F_{\mu/\lambda}(z_1, \ldots, z_M) G_{\nu/\lambda}(x_1, \ldots, x_L),$$

where the left is an infinite sum while the right is a finite one.

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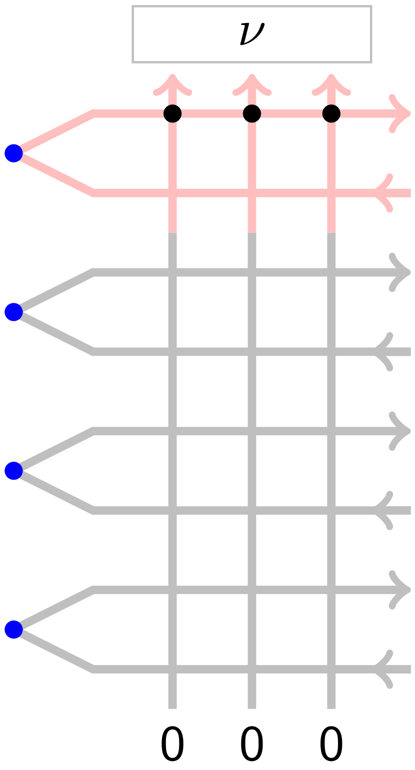
- 

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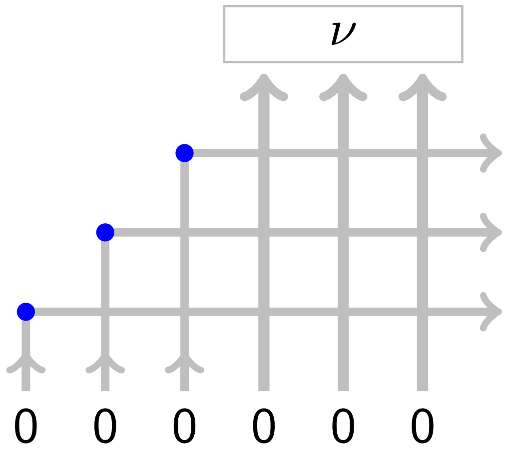

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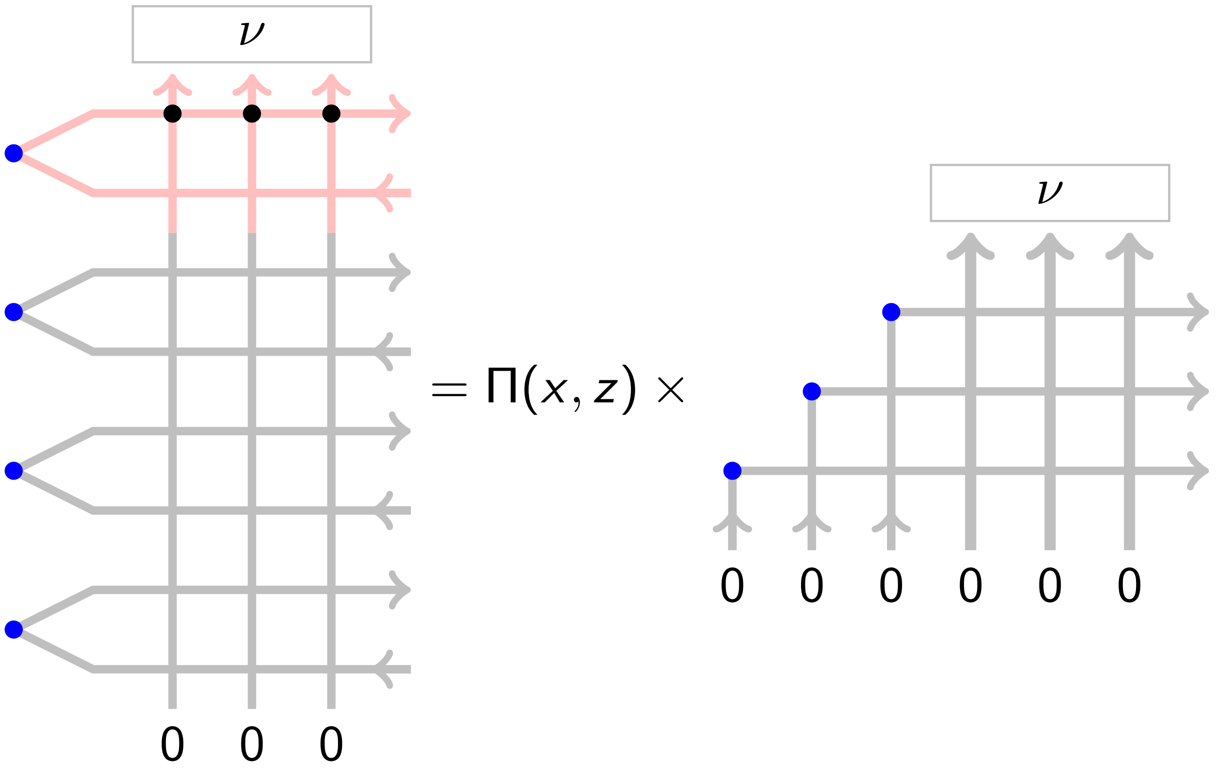
- 

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- This is effectively

$$\mathbb{E}[F_\nu] = \Pi(x, z) \cdot G_\nu.$$

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Theorem

When $\mu = \nu = \emptyset$, there is a Cauchy summation identity

$$\sum_{\kappa} G_\kappa(x_1,\dots,x_L)F_\kappa(z_1,\dots,z_M) = \prod_{i=1}^{M} h(z_i) \prod_{1\le i<j\le L} \frac{1-x_ix_j}{x_i-x_j} \prod_{i=1}^{M}\prod_{j=1}^{L}\left[\frac{x_j-qz_i}{x_j-z_i}\frac{1-z_ix_j}{1-qz_ix_j}\right] \operatorname{Pf}\left(\frac{x_i-x_j}{1-x_ix_j}Q(x_i,x_j)\right)_{1\le i,j\le L}.$$

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- Proof to come.

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