Off-diagonal approach to the exact solution of quantum integrable systems

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Outline

- Introduction
- The Heisenberg spin chains
  - With $U(1)$-symmetry.
  - Without $U(1)$-symmetry.
- Thermodynamic limit of the Heisenberg spin chain
  - With periodic boundary condition.
  - With non-diagonal boundary terms.
- Thermodynamic limit of other spin chains
  - The spin-1 Heisenberg spin chain.
  - The Izergin-Korepin model.
- Conclusion and Comments

Quantum integrable systems have many applications in

- String/ Gauge theories: AdS/CFT, Super-symmetric Yang-Mills theories...
- Statistical mechanics: The Ising model, the six-vertex models...
- Condensed Matter Physics: The super-symmetric $t - J$ Model, the Hubbard model...
- Mathematics: Quantum group, Representation theory, Algebraic Topology, ...

There are many methods to solve quantum integrable systems (The case of $T = 0$):

- The Coordinate Bethe Ansatz method (Bethe 1931)
- The Baxter's $T - Q$ relation method (Baxter 1970s)
- The Quantum Inverse Scattering (or Algebraic Bethe Ansatz) method (Faddeev's School 1979s) and its generalizations (such as the analytic Bethe Ansatz method (Reshetikhin 1983 and Mezincescu & Nepomechie 1992, the separation of variables method (Sklyanin 1985, the Lyon's group 2013), the q-Onsager algebra approach (Baseilhac 2006))
- The off-diagonal Bethe Ansaz method (Wang's school 2013s)
- The modified algebraic Bethe Ansatz (Belliard et. al 2013s)

$\vdots$

The Hamiltonian of the closed Heisenberg chain is

$$H=\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x+\sigma_k^y\,\sigma_{k+1}^y+\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+1}^{\alpha}=\sigma_1^{\alpha},\quad \alpha=x,\,y,\,z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar\frac{\partial}{\partial t}h_i=[H,h_i]=0,\qquad i=1,\ldots.$$

and

$$[h_i,h_j]=0.$$

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u)=\sum_{i=0} h_i u^i.$$

Then

$$[t(u),\,t(v)]=0,\qquad H\propto \frac{\partial}{\partial u}\ln t(u)|_{u=0}+const,$$

or

$$H\propto h_0^{-1}h_1+const,$$
$$h_0\,\sigma_i^{\alpha}\,h_0^{-1}=\sigma_{i+1}^{\alpha}.$$

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0N}(u-\theta_N)\ldots R_{01}(u-\theta_1),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \begin{pmatrix} u+\eta & & & \\ & u & \eta & \\ & \eta & u & \\ & & & u+\eta \end{pmatrix}.$$

The transfer matrix is $t(u) = trT(u) = A(u) + D(u)$.

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v)=R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{1}$$

The above fundamental relation leads to the following so-called RLL relation between the monodromy matrix

$$R_{0\,0'}(u-v)\,T_0(u)\,T_{0'}(v)=T_{0'}(v)\,T_0(u)\,R_{0\,0'}(u-v). \tag{2}$$

This leads to

$$[t(u),t(v)]=0, \tag{3}$$

which ensures the integrability of the Heisenberg chain with periodic boundary condition.

The eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ can be parameterized by some parameters $\{\lambda_1, \cdots, \lambda_M | M = 0, \cdots, N\}$ as follows (H. Bethe, Z. Phys. 71, 205 (1931)):

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}, \tag{4}$$

where

$$a(u) = \prod_{l=1}^{N}(u - \theta_j + \eta) = d(u+\eta), \quad Q(u) = \prod_{j=1}^{M}(u - \lambda_j),$$

the parameters $\{\lambda_i\}$ should satisfy Bethe ansatz equations,

$$\prod_{k\neq j}^{M} \frac{\lambda_j - \lambda_k + \eta}{\lambda_j - \lambda_k - \eta} = \prod_{l=1}^{N} \frac{\lambda_j - \theta_l + \eta}{\lambda_j - \theta_l}, \qquad j = 1, \ldots, M.$$

It can be proven that the transfer matrix $t(u)$ satisfies the relations:

$$t(\theta_j)\, t(\theta_j - \eta) = a(\theta_j) d(\theta_j - \eta), \quad j = 1, \cdots, N, \tag{5}$$

$$t(u) = 2u^N \times \mathrm{id} + \cdots, \quad u \to \infty. \tag{6}$$

This leads to that the eigenvalues $\Lambda(u)$ satisfy the corresponding relations:

$$\Lambda(\theta_j)\, \Lambda(\theta_j - \eta) = a(\theta_j) d(\theta_j - \eta), \quad j = 1, \cdots, N, \tag{7}$$

$$\Lambda(u) = 2u^N + \cdots, \quad u \to \infty. \tag{8}$$

The Hamiltonian of the Heisenberg chain with parallel boundary fields is

$$H=\sum_{k=1}^{N-1}\left(\sigma_k^x\sigma_{k+1}^x+\sigma_k^y\sigma_{k+1}^y+\sigma_k^z\sigma_{k+1}^z\right)+\frac{\eta}{p}\sigma_1^z+\frac{\eta}{q}\sigma_N^z+N.$$

The system is **integrable**, i.e., the corresponding transfer matrix $t(u)$ can be constructed by the R-matrix and the associated K-matrices

$$t(u)=tr(K^+(u)\mathcal{T}(u))=tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),$$

where the K-matrices $K^{\pm}(u)$ are the diagonal K-matrices

$$K^-(u)=\begin{pmatrix} p+u & \\ & p-u \end{pmatrix},\quad K^+(u)=\begin{pmatrix} q+u+\eta & \\ & q-u-\eta \end{pmatrix}.$$

The K-matrix $K^-(u)$ satisfies the reflection equation (RE)

$$\begin{aligned}R_{12}(u_1-u_2)K_1^-(u_1)R_{21}(u_1+u_2)K_2^-(u_2)\\ =K_2^-(u_2)R_{12}(u_1+u_2)K_1^-(u_1)R_{21}(u_1-u_2),\end{aligned}$$

while the dual K-matrix $K^+(u)$ satisfies the following dual RE

$$\begin{aligned}R_{12}(u_2-u_1)K_1^+(u_1)R_{21}(-u_1-u_2-2)K_2^+(u_2)\\ =K_2^+(u_2)R_{12}(-u_1-u_2-2)K_1^+(u_1)R_{21}(u_2-u_1).\end{aligned}$$

The eigenvalues of the associated transfer matrix is also given in terms of a $T-Q$ relation (E.K. Sklyanin, J. Phys. A 21, 2375 (1988))

$$\Lambda(u)=a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)},$$

where

$$a(u)=\frac{2(u+\eta)}{2u+\eta}(u+p)(u+q)\prod_{l=1}^{N}(u-\theta_l+\eta)(u+\theta_l+\eta),$$

$$d(u)=a(-u-\eta),\quad Q(u)=\prod_{j=1}^{M}(u-\lambda_j)(u+\lambda_j+\eta).$$

The Hamiltonian of the Heisenberg chain with unparallel boundary fields is

$$H = \sum_{k=1}^{N-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right) + \frac{\eta}{p}\sigma_1^z + \frac{\eta}{q}(\sigma_N^z + \xi \sigma_N^x) + N. \tag{9}$$

The system is **integrable**, i.e., the corresponding transfer matrix $t(u)$ can be constructed by the R-matrix and the associated K-matrices

$$t(u) = tr(K^+(u)\mathcal{T}(u)) = tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),$$

where the K-matrices $K^{\pm}(u)$ are the diagonal K-matrices

$$K^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix}, \quad K^+(u) = \begin{pmatrix} q+u+\eta & \xi(u+\eta) \\ \xi(u+\eta) & q-u-\eta \end{pmatrix}.$$

$$H = \eta \frac{\partial}{\partial u} \ln t(u)|_{u=0,\{\theta_j\}=0}.$$

Without losing generality, we set $\eta = i$.

The eigenvalue $\Lambda(u)$ satisfies the properties (J. Cao et al, Nucl. Phys. B 875 (2013), 152-165):

$$\Lambda(-u-\eta) = \Lambda(u), \tag{10}$$

$$\Lambda(0) = 2pq\prod_{l=1}^{N}(1-\theta_l)(1+\theta_l), \tag{11}$$

$$\lim_{u\to\infty} \Lambda(u) = 2u^{2N+2} + \cdots, \tag{12}$$

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta) = -\frac{\Delta_q(\theta_j)}{(2\theta_j+\eta)(2\theta_j-\eta)}, \quad j=1,\cdots,N, \tag{13}$$

where the quantum determinant $\Delta_q(u)$ is given by

$$\Delta_q(u) = 4(u^2-1)(p^2-u^2)((1+\xi^2)u^2-q^2)\prod_{l=1}^{N}((u-1)^2-\theta_j^2)((u+1)^2-\theta_j^2).$$

The eigenvalue $\Lambda(u)$ of the corresponding transfer matrix is given in terms of an inhomogeneous $T-Q$ relation

$$\begin{aligned}\Lambda(u) &= a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)}\\ &\quad +2\left[1-(1+\xi^2)^{\frac{1}{2}}\right]u(u+\eta)\frac{a(u)d(u)}{Q(u)},\end{aligned} \tag{14}$$

where

$$\begin{aligned}a(u) &= \frac{2(u+\eta)}{2u+\eta}(u+p)[(1+\xi^2)^{\frac{1}{2}}u+q]\prod_{l=1}^{N}(u-\theta_l+\eta)(u+\theta_l+\eta),\\ d(u) &= a(-u-\eta),\quad Q(u)=\prod_{j=1}^{N}(u-\lambda_j)(u+\lambda_j+\eta).\end{aligned}$$

The roots of $Q(u)$ satisfy the BAEs

$$\frac{a(\lambda_j)}{d(\lambda_j)}+\frac{Q(\lambda_j+\eta)}{Q(\lambda_j-\eta)}=-2[1-(1+\xi^2)^{\frac{1}{2}}]\lambda_j(\lambda_j+\eta)\frac{a(\lambda_j)}{Q(\lambda_j-\eta)},\quad j=1,\cdots,N. \tag{15}$$

The eigenvalue can be given in terms of a homogeneous $T-Q$ relation

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}, \tag{16}$$

where the roots of $Q(u)$ satisfy the Bethe ansatz equations (BAEs)

$$\frac{a(\lambda_j)}{d(\lambda_j)} = -\frac{Q(\lambda_j+\eta)}{Q(\lambda_j-\eta)}, \quad j = 1, \cdots, M. \tag{17}$$

BAEs $\Rightarrow$ TBA

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The eigenvalue $\Lambda(u)$ of the corresponding transfer matrix is given in terms of an inhomogeneous $T-Q$ relation

$$\begin{aligned}\Lambda(u) &= \frac{2(u+1)^{2N+1}}{2u+1}(u+p)\left[(1+\xi^2)^{\frac{1}{2}}u+q\right]\frac{Q(u-1)}{Q(u)} \\ &+\frac{2u^{2N+1}}{2u+1}(u-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(u+1)-q\right]\frac{Q(u+1)}{Q(u)} \\ &+2\left[1-(1+\xi^2)^{\frac{1}{2}}\right]\frac{[u(u+1)]^{2N+1}}{Q(u)},\end{aligned} \tag{18}$$

where the function $Q(u)$ can be parameterized as $Q(u)=\prod_{j=1}^{N}(u-\lambda_j)(u+\lambda_j+1)$.

- Bethe ansatz equations

$$\left(\frac{\lambda_j+1}{\lambda_j}\right)^{2N+1}\frac{(\lambda_j+p)\left[(1+\xi^2)^{\frac{1}{2}}\lambda_j+q\right]}{(\lambda_j-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(\lambda_j+1)-q\right]}=$$

$$-\frac{\left[1-(1+\xi^2)^{\frac{1}{2}}\right](2\lambda_j+1)(\lambda_j+1)^{2N+1}}{(\lambda_j-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(\lambda_j+1)-q\right]\prod_{l=1}^{N}(\lambda_j-\lambda_l-1)(\lambda_j+\lambda_l)}$$

$$-\prod_{l=1}^{N}\frac{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)}{(\lambda_j-\lambda_l-1)(\lambda_j+\lambda_l)},\quad j=1,\cdots,N. \tag{19}$$

- The eigenvalue of the Hamiltonian

$$E=\sum_{j=1}^{N}\frac{2}{\lambda_j(\lambda_j+1)}+N-1+\frac{1}{p}+\frac{(1+\xi^2)^{\frac{1}{2}}}{q}. \tag{20}$$

We define the contribution of the inhomogeneous term to the ground state energy as

$$E_{inh} = E_{hom} - E_{true}. \tag{21}$$

Here E_{hom} is the ground state energy of the Heisenberg chain calculated by the homogeneous $T - Q$ relation

$$\begin{aligned}\Lambda_{hom}(u) \quad &= \quad \frac{2(u+1)^{2N+1}}{2u+1}(u+p)\left[(1+\xi^2)^{\frac{1}{2}}u+q\right]\frac{Q(u-1)}{Q(u)} \\ &+ \frac{2u^{2N+1}}{2u+1}(u-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(u+1)-q\right]\frac{Q(u+1)}{Q(u)}. \end{aligned} \tag{22}$$

The singular property of the $T - Q$ relation (22) gives the following BAEs

$$\left(\frac{\mu_j - \frac{i}{2}}{\mu_j + \frac{i}{2}}\right)^{2N} \frac{(\mu_j - i\bar{p})\,(\mu_j - i\bar{q})}{(\mu_j + i\bar{p})\,(\mu_j + i\bar{q})} = \prod_{l\neq j}^{M} \frac{(\mu_j - \mu_l - i)(\mu_j + \mu_l - i)}{(\mu_j - \mu_l + i)(\mu_j + \mu_l + i)}, \tag{23}$$

where we have put $\lambda = i\mu - \frac{1}{2}$, $\bar{p} = p - \frac{1}{2}$ and $\bar{q} = q(1+\xi^2)^{-\frac{1}{2}} - \frac{1}{2}$. Note, E_{hom} is given by equation (20) with the constraint (23).

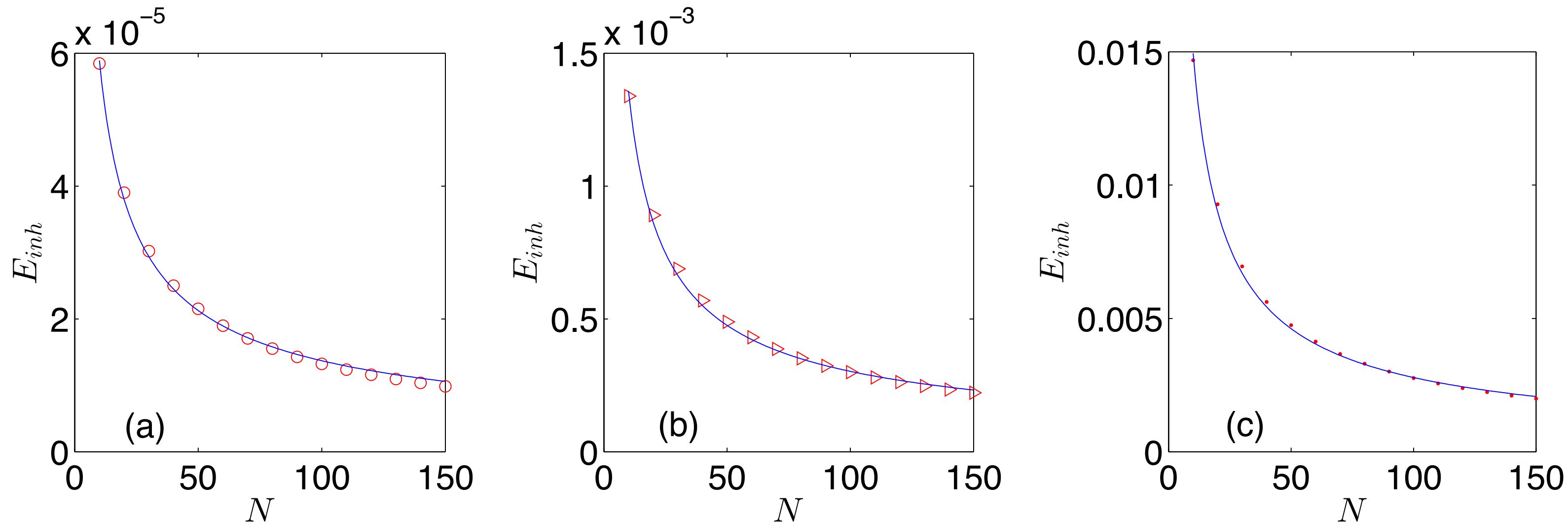

Figure 1: The contribution of the inhomogeneous term to the ground state energy E_{inh} versus the system size N. The data can be fitted as $E_{inh} = \gamma_1 N^{\beta_1}$. **Due to the fact $\beta_1 < 0$, when the N tends to infinity, the contribution of the inhomogeneous term tends to zero.** Here $p = 8$, $q = 4$, (a) $\xi = \frac{1}{8}$, $\gamma_1 = 0.000253$ and $\beta_1 = -0.6334$; (b) $\xi = \frac{5}{8}$, $\gamma_1 = 0.006096$ and $\beta_1 = -0.6521$; (c) $\xi = \frac{25}{8}$, $\gamma_1 = 0.080180$ and $\beta_1 = -0.7297$.

- Boundary energy

$$
\begin{aligned}
E_b(p,q,\xi) &= \lim_{N\to\infty}\left[E_0(N;p,q,\xi) - 2E_0^{periodic}(N)\right] \\
&= -2\int_0^\infty \frac{e^{-p\omega}}{1+e^{-\omega}}d\omega - 2\int_0^\infty \frac{e^{-\frac{q}{\sqrt{1+\xi^2}}\omega}}{1+e^{-\omega}}d\omega \\
&\quad +\pi - 2\ln 2 - 1 + \frac{1}{p} + \frac{(1+\xi^2)^{\frac{1}{2}}}{q}. \qquad (24)
\end{aligned}
$$

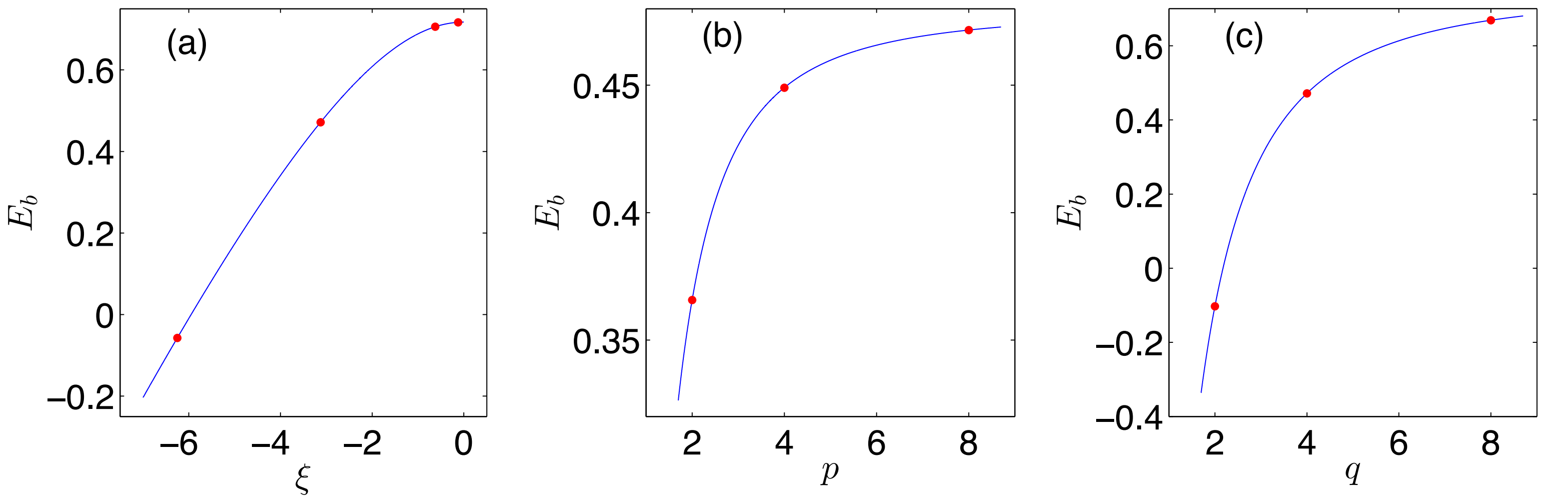

Figure 2: The boundary energies versus the boundary parameters. The blue curves are the ones calculated from equation (24), while the red points are the ones obtained from the Hamiltonian (9) with the BST algorithms. Here (a) $p = 8$ and $q = 4$; (b) $q = 4$ and $\xi = -\frac{25}{8}$; (c) $p = 8$ and $\xi = -\frac{25}{8}$.

When ξ is small, we can expand the boundary energy (24) with respect to ξ as

$$\begin{aligned} E_b(p,q,\xi) \quad \simeq \quad & \frac{1}{p} + \psi^{(0)}\left(\frac{p}{2}\right) - \psi^{(0)}\left(\frac{p+1}{2}\right) + \frac{1}{q} + \psi^{(0)}\left(\frac{q}{2}\right) - \psi^{(0)}\left(\frac{q+1}{2}\right) \\ & + \pi - 1 - 2\ln(2) + \xi^2 \left[\frac{1}{2q} - \frac{1}{4} q \psi^{(1)}\left(\frac{q}{2}\right) + \frac{1}{4} q \psi^{(1)}\left(\frac{q+1}{2}\right)\right] \\ & + \xi^4 \frac{\left[q^3 \psi^{(2)}\left(\frac{q}{2}\right) - q^3 \psi^{(2)}\left(\frac{q+1}{2}\right) + 6q^2 \psi^{(1)}\left(\frac{q}{2}\right) - 6q^2 \psi^{(1)}\left(\frac{q+1}{2}\right) - 4\right]}{32q} \\ & + O\left(\xi^6\right), \qquad (25) \end{aligned}$$

where $\psi^{(m)}(x)$ is the m-order derivative of digamma function. Up to the order ξ^2, our result coincides with that of R. Nepomechie, J. Phys. A 46 (2013), 442002.

The eigenvalue $\Lambda(u)$ satisfies the properties (J. Cao et al, Nucl. Phys. B 875 (2013), 152-165):

$$\Lambda(-u-\eta) = \Lambda(u), \tag{26}$$

$$\Lambda(0) = 2pq\prod_{l=1}^{N}(\eta-\theta_l)(\eta+\theta_l), \tag{27}$$

$$\lim_{u\to\infty} \Lambda(u) = 2u^{2N+2} + \cdots, \tag{28}$$

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta) = -\frac{\Delta_q(\theta_j)}{(2\theta_j+\eta)(2\theta_j-\eta)}, \quad j=1,\cdots,N. \tag{29}$$

In order to make the Hamiltonian (9) hermitian the boundary parameters have to be taken as follows:

$$p* = -p, \quad q^* = -q, \quad \xi^* = \xi,$$

which leads to

$$(t(u))^\dagger = t(-u^*), \quad \Lambda^*(u) = \Lambda(-u^*).$$

This fact allows us to give the decomposition of $\Lambda(u)$ for an eigenvalue of the transfer matrix

$$\begin{aligned}\Lambda(u) \quad = \quad & 2\prod_{j=1}^{M_1}(u-\mu_j+\frac{\eta}{2})(u+\mu_j+\frac{\eta}{2}) \\ & \times\prod_{j=1}^{M_2}(u-z_j+\frac{\eta}{2})(u+z_j+\frac{\eta}{2})(u-z_j^*+\frac{\eta}{2})(u+z_j^*+\frac{\eta}{2}) \\ & \times\prod_{j=1}^{M_b}(u-\eta\nu_j+\frac{\eta}{2})(u+\eta\nu_j+\frac{\eta}{2}), \end{aligned} \tag{30}$$

where μ_j and ν_j are real numbers, $z_j^* \neq (\pm)z_k$, and $M_b + M_1 + 2M_2 = N + 1$.

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For the ground state for a large even N, the corresponding $\Lambda(u)$ takes the decomposition

$$\begin{aligned}\Lambda(u) &= 2\,(u-\eta\nu+\frac{\eta}{2})(u+\eta\nu+\frac{\eta}{2}) \\ &\quad\times\prod_{j=1}^{\frac{N}{2}}(u-x_j-\frac{\eta}{2})(u+x_j+\frac{3\eta}{2})(u-x_j+\frac{3\eta}{2})(u+x_j-\frac{\eta}{2}).\end{aligned}$$

This implies that $z_j \simeq x_j \pm \eta$ for a large even N where x_j is a real number which corresponds to the position of the 2-string, and that ν is a real number which is the position of a boundary bound state.

The corresponding ground state energy $E_0(N; P, Q, \xi)$ is expressed in terms of roots as

$$\lim_{N\to\infty} E_0(N; P, Q, \xi) = \frac{1}{\frac{1}{4} - \nu^2} + \lim_{N\to\infty}\left\{\sum_{j=1}^{\frac{N}{2}}\left(\frac{3}{x_j^2 + \frac{9}{4}} - \frac{1}{x_j^2 + \frac{1}{4}}\right)\right\}$$

$$= \frac{1}{\frac{1}{4} - \nu^2} + \int_{-\infty}^{+\infty}\left(\frac{3}{x^2 + \frac{9}{4}} - \frac{1}{x^2 + \frac{1}{4}}\right)\rho(x)\,dx,$$

where $\rho(x)$ is the density of the distribution of roots.
Numerical study shows that inhomogeneous real parameters $\{\theta_j\}$ almost does not affect the imaginary parts of the roots $\{z_j\}$. Namely, $z_j \simeq x_j \pm \eta$. This fact allows us to derive a linear integral equation of the density of the roots with an auxiliary density $\sigma(\theta)$ of the inhomogeneities. Taking $\sigma(\theta) = \delta(0)$ or the homogeneous limit and the value of $\Lambda(0)$ allow us to determine $\rho(x)$ and the real value ν. Finally, we obtain the same expression (24) of the boundary energy $E_b(p, q, \xi)$.

The Hamiltonian of the spin-1 Heisenberg chain with unparallel boundary fields is

$$
\begin{aligned}
H = \; & \frac{1}{\eta}\sum_{j=1}^{N-1}\left[\vec{S}_j\cdot\vec{S}_{j+1}-(\vec{S}_j\cdot\vec{S}_{j+1})^2\right]+\frac{1}{\eta}\Big(3N+\frac{8}{3}\Big) \\
& +\frac{1}{p_-^2-\frac{1}{4}\left(1+\alpha_-^2\right)\eta^2}\Big[2p_-\left(\alpha_-\cos\phi_- S_1^x-\alpha_-\sin\phi_- S_1^y+S_1^z\right)-\eta\,(S_1^z)^2 \\
& -\frac{1}{2}\alpha_-^2\eta\left[\cos(2\phi_-)\left[(S_1^x)^2-\left(S_1^y\right)^2\right]-(S_1^z)^2\right]-\alpha_-\eta\cos\phi_-\left[S_1^xS_1^z+S_1^zS_1^x\right] \\
& +\frac{1}{2}\alpha_-^2\eta\sin(2\phi_-)\left[S_1^xS_1^y+S_1^yS_1^x\right]+\alpha_-\eta\sin\phi_-\left[S_1^yS_1^z+S_1^zS_1^y\right]+\eta I_1\Big] \\
& +\frac{1}{p_+^2-\frac{1}{4}\left(1+\alpha_+^2\right)\eta^2}\Big[2p_+\left(\alpha_+\cos\phi_+ S_N^x-\alpha_+\sin\phi_+ S_N^y-S_N^z\right)-\eta\,(S_N^z)^2 \\
& -\frac{1}{2}\alpha_+^2\eta\left[\cos(2\phi_+)\left[(S_N^x)^2-\left(S_N^y\right)^2\right]-(S_N^z)^2\right]+\alpha_+\eta\cos\phi_+\left[S_N^xS_N^z+S_N^zS_N^x\right] \\
& +\frac{1}{2}\alpha_+^2\eta\sin(2\phi_+)\left[S_N^xS_N^y+S_N^yS_N^x\right]-\alpha_+\eta\sin\phi_+\left[S_N^yS_N^z+S_N^zS_N^y\right]+\eta I_N\Big].
\end{aligned}
$$

The system is **integrable**, i.e., the corresponding transfer matrix $t(u)$ can be constructed by the R-matrix and the associated K-matrices

$$t^{(1,1)}(u) = tr_0 \left\{ K_0^{+(1)}(u) T_0^{(1,1)}(u) K_0^{-(1)}(u) \hat{T}_0^{(1,1)}(u) \right\},$$

The R-matrix $R^{(1,1)}$ defined in the tensor space $V_1 \otimes V_2$ is

$$R_{12}^{(1,1)}(u) = \left(\begin{array}{ccc|ccc|ccc}
c(u) & & & & & & & & \\
 & b(u) & & e(u) & & & & & \\
 & & d(u) & & g(u) & & f(u) & & \\
\hline
 & e(u) & & b(u) & & & & & \\
 & & g(u) & & a(u) & & g(u) & & \\
 & & & & & b(u) & & e(u) & \\
\hline
 & & f(u) & & g(u) & & d(u) & & \\
 & & & & & e(u) & & b(u) & \\
 & & & & & & & & c(u)
\end{array} \right),$$

with the non-zero entries

$$a(u) = u(u+\eta) + 2\eta^2, b(u) = u(u+\eta), c(u) = (u+\eta)(u+2\eta)$$
$$d(u) = u(u-\eta), e(u) = 2\eta(u+\eta), f(u) = 2\eta^2, g(u) = 2u\eta.$$

The corresponding K-matrices $K^{\pm(1)}(u)$ are the non-diagonal K-matrices

$$K_1^{+(1)}(u) = K_1^{-(1)}(-u-\eta)|_{(p_-,\alpha_-,\phi_-)\to(p_+,\alpha_+,\phi_+)},$$

$$K_1^{-(1)}(u) = (2u+\eta)\begin{pmatrix} x_1(u) & y_4'(u) & y_6'(u) \\ y_4(u) & x_2(u) & y_5'(u) \\ y_6(u) & y_5(u) & x_3(u) \end{pmatrix},$$

where the matrix elements are

$$x_1(u) = \left(p_- + u + \frac{\eta}{2}\right)\left(p_- + u - \frac{\eta}{2}\right) + \frac{\alpha_-^2}{2}\eta\left(u - \frac{\eta}{2}\right),$$

$$x_2(u) = \left(p_- + u - \frac{\eta}{2}\right)\left(p_- - u + \frac{\eta}{2}\right) + \alpha_-^2\left(u + \frac{\eta}{2}\right)\left(u - \frac{\eta}{2}\right),$$

$$x_3(u) = \left(p_- - u - \frac{\eta}{2}\right)\left(p_- - u + \frac{\eta}{2}\right) + \frac{\alpha_-^2}{2}\eta\left(u - \frac{\eta}{2}\right),$$

$$y_4(u) = \sqrt{2}\alpha_- e^{-i\phi_-} u\left(p_- + u - \frac{\eta}{2}\right), \quad y_4'(u) = \sqrt{2}\alpha_- e^{i\phi_-} u\left(p_- + u - \frac{\eta}{2}\right),$$

$$y_5(u) = \sqrt{2}\alpha_- e^{-i\phi_-} u\left(p_- - u + \frac{\eta}{2}\right), \quad y_5'(u) = \sqrt{2}\alpha_- e^{i\phi_-} u\left(p_- - u + \frac{\eta}{2}\right),$$

$$y_6(u) = \alpha_-^2 e^{-2i\phi_-} u\left(u - \frac{\eta}{2}\right), \quad y_6'(u) = \alpha_-^2 e^{2i\phi_-} u\left(u - \frac{\eta}{2}\right).$$

The eigenvalue $\Lambda^{(1,1)}(u)$ of the transfer matrix $t^{(1,1)}(u)$ satisfies the properties (J. Cao et al, JHEP 02 (2015), 036)

$$\Lambda^{(1,1)}(u) = \Lambda^{(1,1)}(-u-\eta), \quad \Lambda^{(\frac{1}{2},1)}(u) = \Lambda^{(\frac{1}{2},1)}(-u-\eta),$$

$$\Lambda^{(\frac{1}{2},1)}(0) = 2p_-p_+\prod_{l=1}^{N}(\theta_l+\frac{3}{2}\eta)(-\theta_l+\frac{3}{2}\eta),$$

$$\Lambda^{(\frac{1}{2},1)}(u)|_{u\to\infty} = 2(\alpha_-\alpha_+-1)u^{2N+2}+\cdots,$$

$$\Lambda^{(1,1)}(u)|_{u\to\infty} = 4[(1+\alpha_+^2)(1+\alpha_-^2)-4(\alpha_+\alpha_--1)^2]u^{4N+6}+\cdots,$$

$$\Lambda^{(1,1)}(u) = -4u(u+\eta)\Lambda^{(\frac{1}{2},1)}\left(u+\frac{\eta}{2}\right)\Lambda^{(\frac{1}{2},1)}\left(u-\frac{\eta}{2}\right)+4u(u+\eta)\delta^{(1)}\left(u+\frac{\eta}{2}\right),$$

$$\Lambda^{(1,1)}(\theta_j)\,\Lambda^{(\frac{1}{2},1)}(\theta_j-\frac{3\eta}{2}) = -4\theta_j\,(\theta_j+\eta)\,\delta^{(1)}(\theta_j-\frac{\eta}{2})\Lambda^{(\frac{1}{2},1)}(\theta_j+\frac{\eta}{2}), \quad j=1,\ldots,N,$$

where $\Lambda^{(\frac{1}{2},1)}(u)$ is the eigenvalue of a fundamental spin-$(\frac{1}{2},1)$ transfer matrix $t^{(\frac{1}{2},1)}(u)$.

This facts allows us to give the decomposition of $\Lambda^{(1,1)}(u)$ and $\Lambda^{(\frac{1}{2},1)}(u)$ for an eigenvalue of the transfer matrix

$$\Lambda^{(1,1)}(u) = \Lambda_0 \prod_{k=1}^{2N+3} \left(u - z_k^{(1)} + \frac{\eta}{2}\right)\left(u + z_k^{(1)} + \frac{\eta}{2}\right),$$

$$\Lambda^{(\frac{1}{2},1)}(u) = 2(\alpha_-\alpha_+ - 1) \prod_{l=1}^{N+1} \left(u - z_l + \frac{\eta}{2}\right)\left(u + z_l + \frac{\eta}{2}\right), \tag{31}$$

The energy spectrum of the Hamiltonian is determined by the zero roots $\{z_j^{(1)}\}$ as

$$E = -\sum_{j=1}^{2N+3} \frac{\eta}{(z_j^{(1)})^2 - \frac{\eta^2}{4}}. \tag{32}$$

Without losing generality, we set $\eta = 1$.

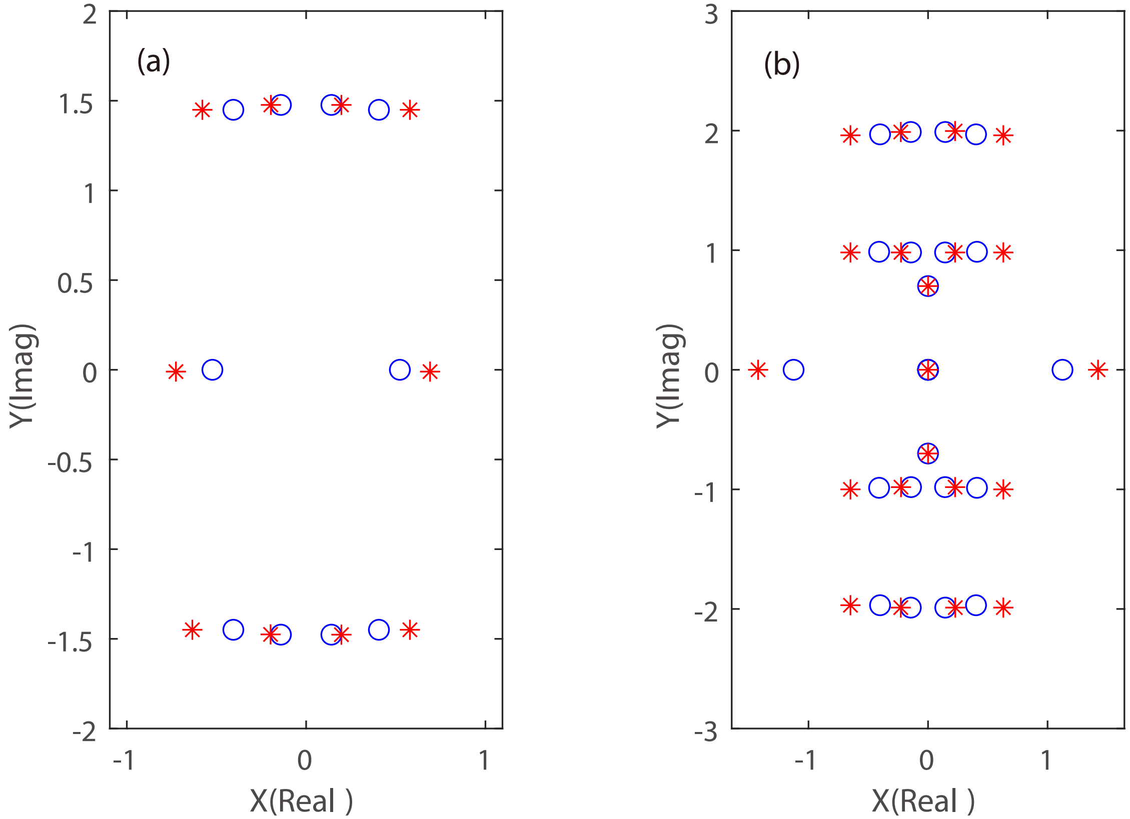

Figure 3: Exact numerical diagonalization results of the zero roots distributions at the ground state in parameter region E with $N = 4$, $p = 0.6$, $q = -0.2$. (a) the $\bar{z}$-roots of the eigenvalue $\Lambda^{\left(\frac{1}{2},1\right)}(u)$; (b) the $\bar{z}^{(1)}$-roots of the eigenvalue $\Lambda^{(1,1)}(u)$. The blue circles indicate the roots for $\{\bar{\theta}_j = 0|j = 1, \cdots, 2N\}$ and the red asterisks specify the roots with the inhomogeneity parameters $\{\bar{\theta}_j = 0.1(j - N - 0.5)|j = 1, \cdots, 2N\}$.

The corresponding ground state energy $E_0(N;p,q)$ is expressed in terms of roots as

$$\lim_{N\to\infty} E_0(N;p,q) = 4 + \frac{1}{\frac{1}{4}+z_0^2} + \frac{1}{\frac{1}{4}-(1+|q|)^2} + \int_{-\infty}^{+\infty}\left(\frac{5}{x^2+\frac{25}{4}} - \frac{1}{x^2+\frac{1}{4}}\right)\rho(x)\,dx,$$

where $p = \frac{p_+}{\sqrt{1+\alpha_+^2}} - \frac{1}{2}$, $q = -\frac{p_-}{\sqrt{1+\alpha_-^2}} - \frac{1}{2}$ and z_0 are real numbers.

Boundary energy $E_b(N;p,q)$ is then given by

$$E_b(N;p,q) = \lim_{N\to\infty}\left[E_0(N;p,q) - 2E_0^{periodic}(N)\right]$$

$$= \begin{cases} 2\pi - \frac{4}{3} + \frac{1}{p+1} - \frac{1}{p} + \frac{1}{q+1} - \frac{1}{q}, & p>0, q>0 \text{ or } \mathrm{q} < -1 \\ 2\pi - \frac{4}{3} + \frac{1}{p+1} - \frac{1}{p} + \frac{1}{q+1} - \frac{1}{q} + 2\pi\csc(q\pi), & p>0, -1<p<0 \end{cases}$$

The R-matrix of the Izergin-Korepin (IK) model corresponds to the twisted affine algebra $A_2^{(2)}$

$$R_{12}(u)=\left(\begin{array}{ccc|ccc|ccc}
h_3(u) & & & & & & & & \\
& h_2(u) & & e(u) & & & & & \\
& & h_4(u) & & g(u) & & f(u) & & \\
\hline
& \bar{e}(u) & & h_2(u) & & & & & \\
& & \bar{g}(u) & & h_1(u) & & g(u) & & \\
& & & & & h_2(u) & & e(u) & \\
\hline
& & \bar{f}(u) & & \bar{g}(u) & & h_4(u) & & \\
& & & & & \bar{e}(u) & & h_2(u) & \\
& & & & & & & & h_3(u)
\end{array}\right),$$

For the Izergin-Korepin model with unparallel boundary fields, the eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ satisfies the properties (K. Hao et al, JHEP 06 (2014), 128)

$$\Lambda(u) = \Lambda(-u + 6\eta + i\pi),$$

$$\Lambda(\pm\theta_j)\Lambda(\pm\theta_j + 6\eta + i\pi) = \frac{\delta_1(u)}{\varphi_1(2u)}\Big|_{u=\pm\theta_j}, \quad j = 1, \cdots, N,$$

$$\Lambda(\pm\theta_j)\Lambda(\pm\theta_j + 4\eta) = \frac{\delta_2(u) \times \Lambda(u + 2\eta + i\pi)}{\varphi_2(-2u + 8\eta)}\Big|_{u=\pm\theta_j}, \quad j = 1, \cdots, N,$$

$$\lim_{u\to\pm\infty} \Lambda(u) = (\frac{1}{2})^{2N} e^{\pm 2(N+1)(u-3\eta)-\varepsilon-\varepsilon'}[1 + 2\cosh(\varsigma' - \varsigma + 2\eta)] + \cdots,$$

$$\Lambda(0) = \Lambda(6\eta + i\pi) = (1 + 2e^{-\varepsilon}\sinh\eta) tr\{K^+(0)\} \prod_{l=1}^{N} \varphi_1(-\theta_l),$$

$$\Lambda(i\pi) = \Lambda(6\eta) = (1 - 2e^{-\varepsilon}\sinh\eta) tr\{K^+(i\pi)\} \prod_{l=1}^{N} \varphi_1(i\pi - \theta_l).$$

This facts also allows us to give the decomposition of $\Lambda(u)$ for an eigenvalue of the transfer matrix

$$\Lambda(u) = \Lambda_0 \prod_{j=1}^{2N+2} \sinh(\frac{u}{2} - \frac{z_j}{2} - \frac{3\eta}{2}) \sinh(\frac{u}{2} + \frac{z_j}{2} - \frac{3\eta}{2} + \frac{i\pi}{2}). \tag{33}$$

The energy spectrum of the Hamiltonian is determined by the zero roots $\{z_j\}$ as

$$E = -\sum_{j=1}^{2N+2} [\frac{\sinh(3\eta)}{\cosh(3\eta) - \cos(iz_j)} + \frac{\sinh(3\eta)}{\cosh(3\eta) - \cos(iz_j + i\pi)}]. \tag{34}$$

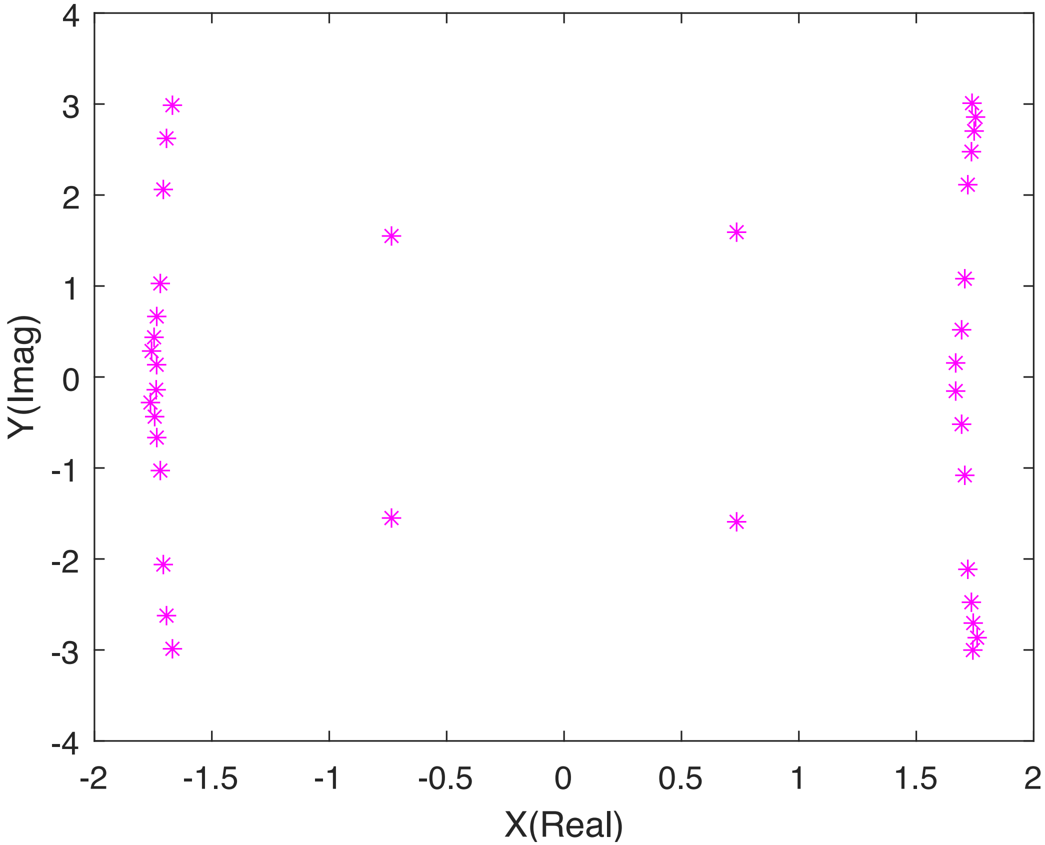

Figure 4: Exact numerical diagonalization results of the zero roots distributions at the ground state with $N = 8, \xi = 1, \xi', \varsigma = 0.6$ and $\varsigma' = 0.7$.

The corresponding ground state energy $E_0(N; p, q)$ is expressed in terms of roots as

$$\lim_{N\to\infty} E_0(N;\alpha,\alpha') = 2N \sum_{k=-\infty}^{\infty} [\tilde{a}_2(k) - \tilde{a}_8(k)e^{-i\pi k}]\tilde{\rho}(k),$$

where $\tilde{a}_n(k) = e^{-\eta|nk|}$ is the Fourier transformation of $a_n(u) = \frac{1}{2\pi}\frac{\sinh(n\eta)}{\cosh(n\eta)-\cos u}$ and $\tilde{\rho}(k)$ is the Fourier transformation of the density $\rho(z)$.

$$\tilde{\rho}(k) = [2N(\tilde{b}_4 + \tilde{b}_6 e^{-i\pi k})\tilde{\sigma}(k) + \tilde{b}_{|1+\frac{\alpha}{\eta}|} + \tilde{b}_{|1+\frac{\alpha'}{\eta}|} + \tilde{b}_6 - \tilde{b}_2 + 2(\tilde{b}_1 - \tilde{b}_3)\cos(\frac{\pi k}{2})$$
$$+(\tilde{b}_{|1-\frac{\alpha}{\eta}|} + \tilde{b}_{|1-\frac{\alpha'}{\eta}|} + \tilde{b}_6 - \tilde{b}_2)e^{-i\pi k}]/[2N(\tilde{b}_2 + \tilde{b}_8 e^{-i\pi k})],$$

(In $\frac{\alpha}{\eta} = \mathrm{arcsinh}(\frac{e^{\varepsilon}}{2}) \geq 3$ and $\frac{\alpha}{\eta}' = \mathrm{arcsinh}(\frac{e^{\varepsilon'}}{2}) \geq 3$ parameter region)

- Boundary energy (In $\frac{\alpha}{\eta} = \mathrm{arcsinh}(\frac{e^{\varepsilon}}{2}) \geq 3$ and $\frac{\alpha}{\eta}' = \mathrm{arcsinh}(\frac{e^{\varepsilon'}}{2}) \geq 3$ parameter region)

$$E_b(\alpha, \alpha') = e_b(\alpha) + e_b(\alpha') + e_{b0},$$

$$e_b(\alpha) = \sum_{k=-\infty}^{\infty} [1 - e^{-\eta|6k|-i\pi k}] \frac{e^{-\eta|(1+\frac{\alpha}{\eta})k|} + e^{-\eta|(1-\frac{\alpha}{\eta})k|-i\pi k}}{1 + e^{-\eta|6k|-i\pi k}},$$

$$e_b(\alpha') = \sum_{k=-\infty}^{\infty} [1 - e^{-\eta|6k|-i\pi k}] \frac{e^{-\eta|(1+\frac{\alpha'}{\eta})k|} + e^{-\eta|(1-\frac{\alpha'}{\eta})k|-i\pi k}}{1 + e^{-\eta|6k|-i\pi k}},$$

$$e_{b0} = \sum_{k=-\infty}^{\infty} [1 - e^{-\eta|6k|-i\pi k}] \frac{(e^{-\eta|6k|} - e^{-\eta|2k|})(1 + e^{-i\pi k}) + 2(e^{-\eta|k|} - e^{-\eta|3k|}) \cos\frac{\pi k}{2}}{1 + e^{-\eta|6k|-i\pi k}}.$$

V. Conclusion and comments

So far, many typical $U(1)$-symmetry-broken models have been solved by the method:

- The spin-$\frac{1}{2}$ Heisenberg chain with arbitrary boundary fields.
- The open spin chains with general boundary condition associated with the $A_n^{(1)}$ algebra.
- The t-J model with unparallel boundary fields.
- The Hubbard model with unparallel boundary fields.
- The open spin chains associated with the $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ algebras.
- The open spin chains associated with the $A_n^{(2)}$ and $D_n^{(2)}$ twisted algebras.

Thanks for your attentions