Yang-Baxter with bosons: from cold atoms to the failure of an exact solution

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Quotation

There are 'down-to-earth' physicists and chemists who reject lattice models as being unrealistic. In its most extreme form, their argument is that if a model can be solved exactly, then it must be pathological. I think this is defeatist nonsense: ....

Basically, I suppose the justification for studying these lattice models is very simple: they are relevant and they can be solved, so why not do so and see what they tell us?

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R.J. Baxter,
Exactly Solved Models in Statistical Mechanics,
Academic Press, London, 1982.

Bosons on a lattice

LETTER

doi:10.1038/nature17409

Quantum phases from competing short- and long-range interactions in an optical lattice

Renate Landig[1], Lorenz Hruby[1], Nishant Dogra[1], Manuele Landini[1], Rafael Mottl[1], Tobias Donner[1] & Tilman Esslinger[1]

Insights into complex phenomena in quantum matter can be gained from simulation experiments with ultracold atoms, especially in cases where theoretical characterization is challenging. However, these experiments are mostly limited to short-range collisional interactions; recently observed perturbative effects of long-range interactions were too weak to reach new quantum phases[1,2]. Here we experimentally realize a bosonic lattice model with competing short- and long-range interactions, and observe the appearance of four distinct quantum phases—a superfluid, a supersolid, a Mott insulator and a charge density wave. Our system is based on an atomic quantum gas trapped in an optical lattice inside a high-finesse optical cavity. The strength of the short-range on-site interactions is controlled by means of the optical lattice depth. The long (infinite)-range interaction potential is mediated by a vacuum mode of the cavity[3,4] and is independently controlled by tuning the cavity resonance. When probing the phase transition between the Mott insulator and the charge density wave in real time, we observed a behaviour characteristic of a first-order phase transition. Our measurements have accessed a regime for quantum simulation of many-body systems where the physics is determined by the intricate competition between two different types of interactions and the zero point motion of the particles.

Experiments with cold atoms have contributed in many ways to a stack of about 60 weakly coupled two-dimensional (2D) layers. These 2D layers are then exposed to a square lattice in the x–z plane formed by one free space lattice and one intracavity optical standing wave, both at a wavelength of $\lambda = 785.3$ nm. They create periodic optical potentials of equal depths V_{2D} along both directions, which we will specify in units of the recoil energy $E_R = h^2/2m\lambda^2$, where m denotes the mass of ^{87}Rb. In addition to the lattice potential, the atoms are exposed to an overall harmonic confinement, which results in a maximum density of 2.8 atoms per lattice site at the centre of the trap. The standing wave along the z axis fulfils a second role as it controls long-range interactions via off-resonant scattering into the optical resonator mode. The photons are scattered off the trapped atoms and are delocalized within the cavity mode, thereby mediating atom–atom interactions of infinite range (see Methods). These infinite-range interactions create λ-periodic atomic density–density correlations on the underlying $\lambda/2$-periodic square lattice[4]. The correlations can lead to the breaking of a $\mathbb{Z}_2$-symmetry between the two chequerboard sublattices[23], defined by either even or odd sites, resulting in the appearance of a self-consistent optical potential with alternating strength.

In a wide range of the parameter space, the system can be described by a lattice model with long-range interactions (see Methods and Extended Data Fig. 1), given by:

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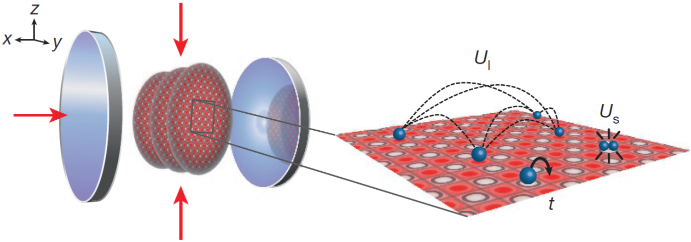

Figure 1 | Illustration of the experimental scheme that realizes a lattice model with on-site and infinite-range interactions. Left, a stack of 2D systems along the *y* axis is loaded into a 2D optical lattice (red arrows) between two mirrors (shown grey). The cavity induces atom–atom interactions of infinite range. Right, illustration of the competing energy scales: tunnelling t, on-site interactions U_s and long-range interactions U_l.

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ttice model with long-range interactions (see Me
ed Data Fig. 1), given by:

$$\hat{H} = -t \sum_{\langle e,o \rangle} (\hat{b}_e^\dagger \hat{b}_o + \mathrm{h.c.}) + \frac{U_\mathrm{s}}{2} \sum_{i \in e,o} \hat{n}_i(\hat{n}_i - 1) - \frac{U_\mathrm{l}}{K} \left(\sum_e \hat{n}_e - \sum_o \hat{n}_o \right)^2 - \sum_{i \in e,o} \mu_i \hat{n}_i$$

and o denote all even and odd lattice sites respectivel

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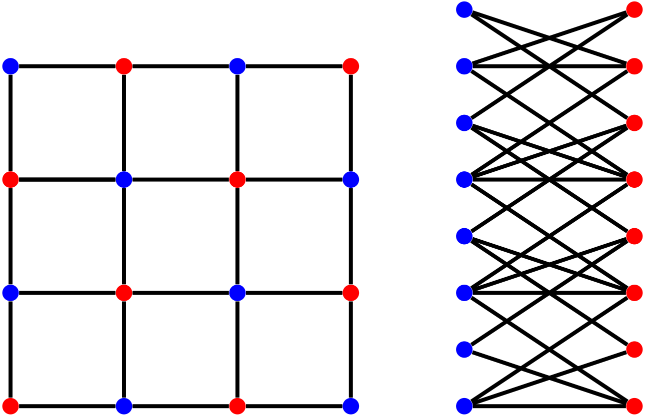
NOTE: $(N_e - N_o)^2 + N^2 = (N_e - N_o)^2 + (N_e + N_o)^2 = 2N_e^2 + 2N_o^2$.

Pictorial representation I



Adjacency matrix

$$\mathcal{A} = \left(\begin{array}{c|c} 0 & \mathcal{B} \\ - & - \\ \mathcal{B} & 0 \end{array} \right)$$

Hamiltonian

Let $\{a_j,\, a_j^\dagger : j = 1, \ldots, m\} \cup \{b_j,\, b_j^\dagger : j = 1, \ldots, m\}$ denote mutually commuting sets of canonical boson operators satisfying

$$
\begin{aligned}
[a_j,\, a_k^\dagger] &= [b_j,\, b_k^\dagger] = \delta_{jk} I, \\
[a_j,\, a_k] &= [a_j^\dagger,\, a_k^\dagger] = [b_j,\, b_k] = [b_j^\dagger,\, b_k^\dagger] = 0.
\end{aligned}
$$

For adjacency matrix $\mathcal{A} = \left(\begin{array}{c|c} 0 & \mathcal{B} \\ - & - \\ \mathcal{B} & 0 \end{array}\right)$ the Hamiltonian reads

$$
H = U(N_a^2 + N_b^2 - I) + \sum_{j,k=1}^{m} \mathcal{B}_{jk}(a_j^\dagger b_k + b_j^\dagger a_k)
$$

where $N_a = \sum_{j=1}^{m} a_j^\dagger a_j$, $N_b = \sum_{j=1}^{m} b_j^\dagger b_j$.

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$$H = U(N_a^2 + N_b^2 - I) + \sum_{j,k=1}^{m} \mathcal{B}_{jk}(a_j^\dagger b_k + b_j^\dagger a_k)$$

where $N_a = \sum_{j=1}^{m} a_j^\dagger a_j$, $N_b = \sum_{j=1}^{m} b_j^\dagger b_j$. The Hamiltonian admits a set of mutually-commuting conserved operators.

Quantum integrability

Explicitly, $[C(y),\ C(z)] = 0$ where

$$C(2p) = \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p}(a_j^\dagger a_k + b_j^\dagger b_k),$$

$$C(2p+1) = U\sum_{i=0}^{2p} D(2p,i) + \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p+1}(a_j^\dagger b_k + b_j^\dagger a_k)$$

with

$$D(2p,i) = \begin{cases} \displaystyle\sum_{j,k,r,q=1}^{m} \mathcal{B}_{jk}^{i}\mathcal{B}_{rq}^{2p-i}(a_j^\dagger a_q a_r^\dagger a_k + b_j^\dagger b_q b_r^\dagger b_k), & i \text{ even}, \\ \displaystyle\sum_{j,k,r,q=1}^{m} (\mathcal{B}_{jk}^{i}\mathcal{B}_{rq}^{2p-i} + \mathcal{B}_{jk}^{2p-i}\mathcal{B}_{rq}^{i})a_j^\dagger a_q b_r^\dagger b_k, & i \text{ odd}. \end{cases}$$

Note that $N = N_a + N_b = C(0)$ and $H = C(1) - U(I + C(0))$.

Canonical transformation

Let X denote a unitary operator that diagonalises $\mathcal{B}$, viz.

$$\sum_{p=1}^{m} X^{\dagger}_{jp} X_{pk} = \delta_{jk}, \qquad \sum_{p,q=1}^{m} X^{\dagger}_{jp} \mathcal{B}_{pq} X_{qk} = \varepsilon_j \delta_{jk},$$

with $\{\varepsilon_j : j = 1, ..., m\}$ the spectrum of $\mathcal{B}$. Introducing

$$a_k = \sum_{j=1}^{m} X_{kj} c_j, \quad b_k = \sum_{j=1}^{m} X_{kj} d_j, \quad a^{\dagger}_k = \sum_{j=1}^{m} X^{\dagger}_{jk} c^{\dagger}_j, \quad b^{\dagger}_k = \sum_{j=1}^{m} X^{\dagger}_{jk} d^{\dagger}_j,$$

leads to $N_a = \sum_{j=1}^{m} c^{\dagger}_j c_j = N_c$, $N_b = \sum_{j=1}^{m} d^{\dagger}_j d_j = N_d$ and

$$H = U(N_c^2 + N_d^2 - I) + \sum_{j=1}^{m} \varepsilon_j (c^{\dagger}_j d_j + d^{\dagger}_j c_j).$$

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leads to $N_a = \sum_{j=1}^{m} c_j^{\dagger} c_j = N_c$, $N_b = \sum_{j=1}^{m} d_j^{\dagger} d_j = N_d$ and $H = U(N_c^2 + N_d^2 - I) + \sum_{j=1}^{m} \varepsilon_j (c_j^{\dagger} d_j + d_j^{\dagger} c_j)$. Note that $\hat{N}_j = c_j^{\dagger} c_j + d_j^{\dagger} d_j$ are conserved operators; let N_j denote their eigenvalues. Then $\sum_{j=1}^{m} N_j = N$ is the total number of particles.

Bethe Ansatz solution

The energy eigenvalues are

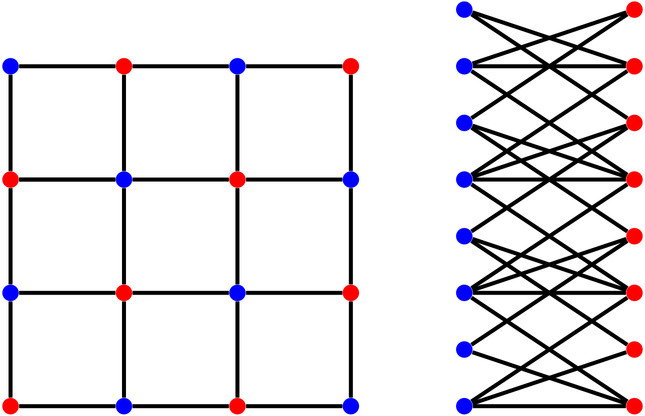
$$E = U(N^2 - 1) + 2U \sum_{j=1}^{m} \sum_{n=1}^{N} \frac{N_j \varepsilon_j^2}{v_n - \varepsilon_j^2}$$

subject to the Bethe Ansatz equations

$$\sum_{m \neq n}^{N} \frac{2v_n}{v_n - v_m} + \frac{\prod_{j=1}^{m} (v_n - \varepsilon_j^2)^{N_j}}{4U^2 \prod_{m \neq n}^{N} (v_n - v_m)} = N - 1 + \sum_{j=1}^{m} \frac{N_j \varepsilon_j^2}{v_n - \varepsilon_j^2}.$$

for $n = 1, \ldots, N$.

Pictorial representation I



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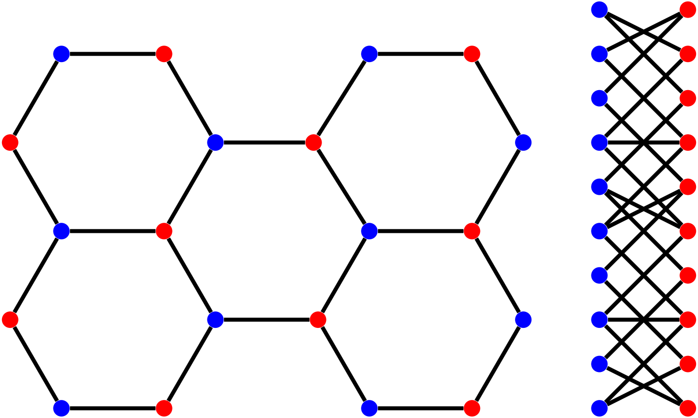

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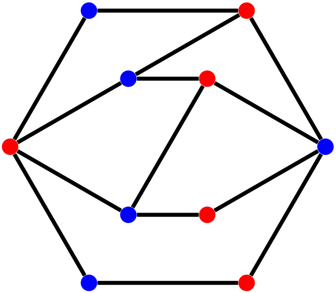

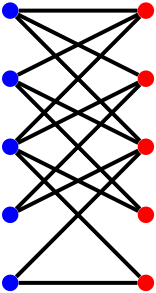

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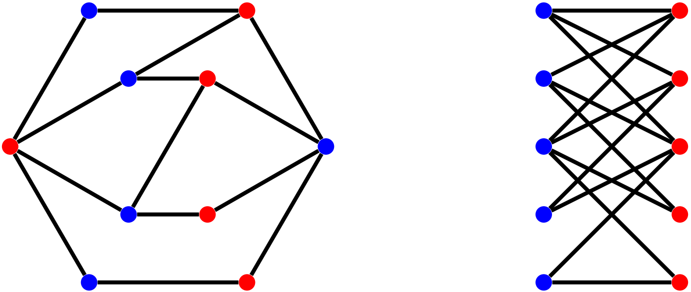

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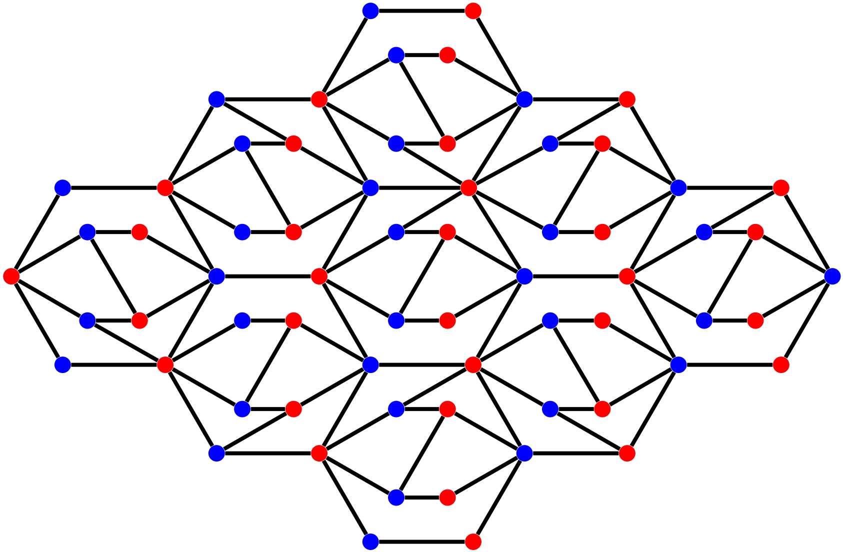
Pictorial representation IV



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Unsolvability

Acting on Fock space, the Hamiltonian

$$H = U(N_a^2 + N_b^2 - I) + \sum_{j,k=1}^{m} \mathcal{B}_{jk}(a_j^\dagger b_k + b_j^\dagger a_k)$$

commutes with the number operator $N = N_a + N_b$. It can be block-diagonalised as $H = \bigoplus_{\mathcal{N}=0}^{\infty} H(\mathcal{N})$.

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$$H(1) \cong \begin{pmatrix} 0 & | & \mathcal{B} \\ - & & - \\ \mathcal{B} & | & 0 \end{pmatrix}$$

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where the last expression is symmetry-adapted to $a_j \leftrightarrow b_j$.

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where the last expression is symmetry-adapted to $a_j \leftrightarrow b_j$. If the operator $\mathcal{B}$ is not exactly solvable, then H is not exactly solvable.

Some ancestors

- In 1891 Cantor published the "diagonal argument" to prove that the elements of an interval of $\mathbb{R}$, e.g. $[0, 1]$, cannot be put into one-to-one correspondence with $\mathbb{N}$. This gave rise to the concept that the set of elements in $[0, 1]$ is *uncountable*.
- In 1905 Richards applied the diagonal argument to formulate a paradox regarding elements in $\mathbb{R}$ that are *undefinable*.
- In 1927 Borel introduced the notion that there exist elements in $\mathbb{R}$ that are *uncomputable*.
- In 1929 Gödel expanded on these ideas to show that there are true statements in arithmetic that are *unprovable*.
- In 1937 Turing demonstrated that the halting problem is *undecidable*.

Classical Yang-Baxter equation and quantum integrability

For $r(u,v) = \sum_{j,k,p,q=1}^{n} r(u,v)^{jp}_{kq} e^k_j \otimes e^q_p$ the classical YBE reads

$$[r_{12}(u,v),\ r_{23}(v,w)] - [r_{21}(v,u),\ r_{13}(u,w)] + [r_{13}(u,w),\ r_{23}(v,w)] = 0.$$

Define the associated *generalised Gaudin algebra* with relations

$$[T^j_k(u),\ T^p_q(v)] = \sum_{\mu=1}^{n} \left(r^{jp}_{k\mu}(u,v)\, T^\mu_q(v) - r^{j\mu}_{kq}(u,v)\, T^p_\mu(v) \right)$$
$$- \sum_{\mu=1}^{n} \left(r^{pj}_{q\mu}(v,u)\, T^\mu_k(u) + r^{p\mu}_{qk}(v,u)\, T^j_\mu(u) \right).$$

This is an infinite-dimensional Lie algebra with defining representation π given by

$$\pi(T^j_k(u)) = \sum_{p,q=1}^{n} r^{jp}_{kq}(u,v) e^q_p.$$

Classical Yang-Baxter equation and quantum integrability

Assume also that

$$[r_{12}(u,v),\ r_{21}(v,u)] = [r_{12}^{t_2}(u,v),\ r_{21}^{t_2}(v,u)] = 0.$$

It is found that the *transfer matrix*

$$t(u) = \sum_{j,k=1}^{n} T_k^j(u) T_j^k(u)$$

satisfies, for all $u,\ v \in \mathbb{C}$, $[t(u),\ t(v)] = 0$. If $B(u)$ satisfies

$$[B_2(v),\ r_{12}(u,v)] = [B_1(u),\ r_{21}(v,u)]$$

we can extend the defining representation to the tensor product realisation

$$\bar{\pi}(T_k^j(u)) = B_k^j(u)I + \sum_{m=1}^{M}\sum_{p,q=1}^{n} r_{kq}^{jp}(u,v_m)(E_p^q)_m$$

where $[E_k^j,\ E_q^p] = \delta_k^p E_q^j - \delta_q^j E_k^p$.

Classical Yang-Baxter equation and classical integrability

We also define the associated *generalised Poisson algebra*

$$\{\mathcal{T}^j_k(u),\, \mathcal{T}^p_q(v)\} = \sum_{\mu=1}^{n}\Big(r^{jp}_{k\mu}(u,v)\mathcal{T}^{\mu}_q(v) - r^{j\mu}_{kq}(u,v)\mathcal{T}^{p}_{\mu}(v)\Big)$$
$$- \sum_{\mu=1}^{n}\Big(r^{pj}_{q\mu}(v,u)\mathcal{T}^{\mu}_k(u) + r^{p\mu}_{qk}(v,u)\mathcal{T}^{j}_{\mu}(u)\Big)\,.$$

Furthermore, set

$$(\mathcal{T}^{(2)})^j_k(u) = \sum_{l=1}^{n} \mathcal{T}^j_l(u)\mathcal{T}^l_k(u),$$

$$(\mathcal{T}^{(r+1)})^j_k(u) = \sum_{l=1}^{n}(\mathcal{T}^{(r)})^j_l(u)\mathcal{T}^l_k(u) = \sum_{l=1}^{n}\mathcal{T}^j_l(u)(\mathcal{T}^{(r)})^l_k(u),$$

$$\mathfrak{t}^{(r)}(u) = \sum_{j=1}^{n}(\mathcal{T}^{(r)})^j_j(u) \quad \Rightarrow \quad \{\mathfrak{t}^{(r)}(u),\, \mathfrak{t}^{(s)}(v)\} = 0.$$

Classical Yang-Baxter equation and integrability

Extend the defining representation to the tensor product realisation

$$\tilde{\pi}(\mathcal{T}^j_k(u)) = B^j_k(u)I + \sum_{m=1}^{M}\sum_{p,q=1}^{n} r^{jp}_{kq}(u, v_m)(\mathcal{E}^q_p)_m,$$

where $\{\mathcal{E}^j_k, \mathcal{E}^p_q\} = \delta^p_k \mathcal{E}^j_q - \delta^j_q \mathcal{E}^p_k$. Expanding $\mathfrak{t}^{(s)}(u) = \sum_j \mathfrak{t}^{(s)}_j u^j$

leads to "Poisson-commuting" functions $\{\mathfrak{t}^{(r)}_j, \mathfrak{t}^{(s)}_k\} = 0$.

The problem to "lift" or "quantise" Poisson invariants $\mathfrak{t}^{(s)}_j$ to central elements of $U(gl(n))$ is well-studied with a long history, e.g. Mishchenko and Fomenko (1978), Vinberg (1991), Feigin, Frenkel, and Reshetikhin (1994), Tarasov (2002), Chervov, Rybnikov, and Talalaev (2004), Skrypnyk (2007), Panyushev and Yakimova (2021), . . .

The specific example

Let P denote the permutation operator such that

$$P(\mathbf{x} \otimes \mathbf{y}) = \mathbf{y} \otimes \mathbf{x}, \qquad \mathbf{x},\, \mathbf{y} \in \mathbb{C}^n.$$

Set $r(u,v) = \left(\frac{1}{u-v} I \otimes I + \frac{1}{u+v} A \otimes A \right) P$. It may be checked that the classical YBE

$$[r_{12}(u,v),\, r_{23}(v,w)] - [r_{21}(v,u),\, r_{13}(u,w)] + [r_{13}(u,w),\, r_{23}(v,w)] = 0$$

holds provided $A^2 = I$. Moreover, setting $B(u) = uB$ then

$$[B_2(v),\, r_{12}(u,v)] = [B_1(u),\, r_{21}(v,u)]$$

holds provided $AB = -BA$. These conditions are satisfied by choosing $n = 2m$ and $A = \sigma^z \otimes I$, $B = \sigma^x \otimes \mathcal{B}$ for *arbitrary* $\mathcal{B} \in \mathrm{End}(\mathbb{C}^m)$. This solutions leads to the conserved operators for the integrable system described earlier.

The specific example - in summary

- Using $r(u, v)$ from the previous slide, construct the realisation of the generalised Poisson algebra with $M = 1$:

$$\tilde{\pi}(\mathcal{T}_k^j(u)) = B_k^j(u)I + \sum_{p,q=1}^{n} r_{kq}^{jp}(u, v)\mathcal{E}_p^q$$

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- Generalising the results of Vinberg shows that these operators quantise to commuting elements of $U(gl(2m))$.
- Map the elements E_k^j of $gl(2m)$ to operators on Fock space through the Jordan-Schwinger map

$$E_k^j \mapsto a_j^\dagger a_k, \quad j, k \text{ odd}, \qquad E_k^j \mapsto a_j^\dagger b_k, \quad j \text{ odd}, k \text{ even},$$
$$E_k^j \mapsto b_j^\dagger b_k, \quad j, k \text{ even}, \qquad E_k^j \mapsto b_j^\dagger a_k, \quad j \text{ even}, k \text{ odd}.$$

The specific example - conserved operators

Explicitly, $[C(y),\ C(z)] = 0$ where

$$C(2p) = \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p}(a_j^\dagger a_k + b_j^\dagger b_k),$$

$$C(2p+1) = U\sum_{i=0}^{2p} D(2p,i) + \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p+1}(a_j^\dagger b_k + b_j^\dagger a_k)$$

with

$$D(2p,i) = \begin{cases} \sum_{j,k,r,q=1}^{m} \mathcal{B}_{jk}^{i}\mathcal{B}_{rq}^{2p-i}(a_j^\dagger a_q a_r^\dagger a_k + b_j^\dagger b_q b_r^\dagger b_k), & i \text{ even}, \\ \sum_{j,k,r,q=1}^{m} (\mathcal{B}_{jk}^{i}\mathcal{B}_{rq}^{2p-i} + \mathcal{B}_{jk}^{2p-i}\mathcal{B}_{rq}^{i}) a_j^\dagger a_q b_r^\dagger b_k, & i \text{ odd}. \end{cases}$$

Note that $N = N_a + N_b = C(0)$ and $H = C(1) - U(I + C(0))$.

Summary

- Motivated by an optical lattice set-up in a cavity, a model was introduced for bosons on a *two-dimensional* square lattice with long-range interactions.
- The Hamiltonian, conserved operators, Bethe Ansatz solution follow from the formulation of system through a solution of the classical Yang-Baxter equation.
- The system generalises to models on generic bipartite graphs.
- The Bethe Ansatz equations are functions of the eigenvalues of the (weighted) adjacency submatrix. If these eigenvalues cannot be obtained exactly, then the Bethe Ansatz solution is also not exact.
- The result demonstrates that Yang-Baxter integrability does not imply *exact* solvability.

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