Quantum integrable models and $t-W$ scheme

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Introduction

• 1920, Wilhelm Lenz, to explain the physical phenomenon that the magnetism of magnet disappears when the temperature is larger than a critical value

$$H = J \sum_{j=1}^{N} \sigma_j \sigma_{j+1}$$

1. two kinds of states in nature: ordered and disordered.
paramagnetic and ferromagnetic states
normal and superfluid states
normal and superconductive states
2. the transition between two states.

1920, Ising, 1D
1944, Onsager, 2D

Open a new field: Ising model has many applications in many fields such as the folding of DNA in organisms, the spread of viruses, artificial intelligence, activation and deactivation of brain nerve cells, weather forecast, forest fire, social sciences, finance.

• 1928, Heisenberg model

$$H = J\sum_{j=1}^{N}\vec{\sigma}_j \cdot \vec{\sigma}_{j+1}$$

spin exchanging interaction
quantum magnetism: AFM, FM
coordinate Bethe ansatz
quantum inverse scattering method
spinon, fractional excitations
anisotropic couplings, magnetic ordered states, quantum phase transitions

$$H = -\sum_{j=1}^{N}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z)$$

$|\Delta| < 1$, gapless phase; $|\Delta| > 1$, gapped phase;
$|\Delta| = 1$, critical points. $\Delta = -1$, first order phase transition; $\Delta = 1$, KT phase transition

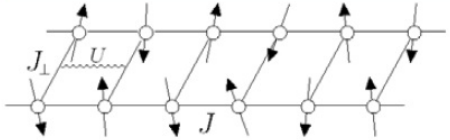
Exactly solvable models:

1. interacting particles with δ -function

$$H = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2c\sum_{i<j}\delta(x_i - x_j)$$

2. spin chain and spin ladder

$$H = \frac{1}{2}J\sum_{j=1}^{N}\vec{\sigma}_j\cdot\vec{\sigma}_{j+1} - \frac{1}{2}h\sum_{j=1}^{N}\sigma_j^z$$



3. Hubbard, supersymmetry t-J, Kondo

$$H = \sum_{i,j=1}^{N}\sum_{\sigma=\uparrow,\downarrow} t_{i,j}(C_{i,\sigma}^\dagger C_{j,\sigma} + h.c.) + U\sum_{i=1}^{N} n_{i\uparrow}n_{i\downarrow}$$

$$H = \sum_{j=1}^{N}\left\{-t\mathcal{P}\sum_{\sigma=\pm1}(c_{j,\sigma}^\dagger c_{j+1,\sigma} + H.c.)\mathcal{P} + J(\mathbf{S}_j\mathbf{S}_{j+1} - \frac{1}{4}n_n n_{j+1})\right\}$$

4. τ_2, Chiral Potts, vertex model

$$H_{P_9} = -\sum_{l=1}^{N/2}\sum_{k=1}^{8}\Omega_l^k - \sum_{l=1}^{N/2-1}\sum_{k=1}^{8}R_l^k R_{l+1}^{9-k}.$$

5. long range interaction

Gaudin model (1/r)

$$\hat{H}_n = 2gS_n^z + \sum_{m\neq n}^{N}\sum_{\alpha}\frac{2S_n^\alpha S_n^\alpha}{\delta_n - \delta_m}$$

Calogero-Sutherland model ($1/r^2$,continue case)

$$H_{CM} = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \sum_{j,k=1,j\neq k}^{N}\frac{\lambda^2 - \lambda P_{jk}}{(x_i - x_j)^2}$$

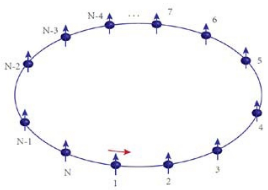
Haldane-Shastry model ($1/r^2$, lattice case)

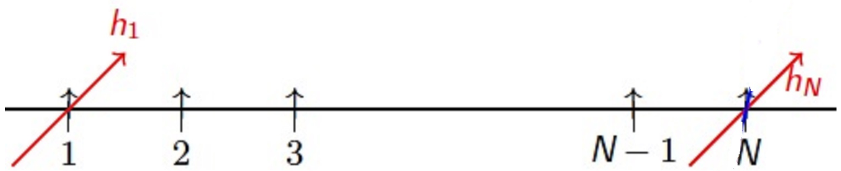
$$H_{HS} = \sum_{j<l}^{N}\frac{1}{\sin^2\frac{\pi}{N}(j-l)}\mathbf{S}_j\cdot\mathbf{S}_l$$

• Integrability: number of degree of freedom = number of conserved quantities.

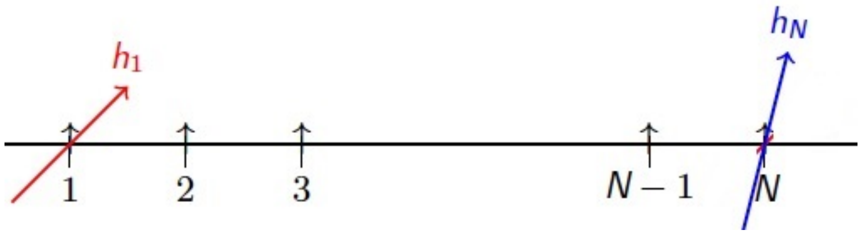
Quantum integrable systems can provide the benchmark for many new phenomena and physical concepts, and check the correction of numerical methods and numerical results. An important branch of modern physics.

1. $U(1)$ symmetry





2. $U(1)$ symmetry is broken



XYZ spin chain

The quantum spin chain with competing interactions

- Nearest neighbor interaction

$$H = J_1 \sum_{j=1}^{N} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}$$

- Next nearest neighbor interaction

$$H = \sum_{j=1}^{N} (J_1 \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + J_2 \vec{\sigma}_j \cdot \vec{\sigma}_{j+2})$$

$\diamond$ Phase diagram: Define $J = J_2/J_1$. When $0 < J < J_{cr}$, the ground state remains gapless, as is the case when $J = 0$. When $J > J_{cr}$, the spectrum becomes gapped. The critical point is $J_{cr} = 0.2411$, where there is a KT phase transition.
$\diamond$ When $J = 1/2$, the Hamiltonian is exactly solvable, Majumdar-Ghosh model

- Integrable quantum spin chain with competing interactions

$$H_{bulk} = \sum_{j=1}^{2N-1} \{J_1 \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + J_2 \vec{\sigma}_j \cdot \vec{\sigma}_{j+2} + J_3 (-1)^j \vec{\sigma}_{j+1} \cdot (\vec{\sigma}_j \times \vec{\sigma}_{j+2})\}$$

chiral three-spin interactions, spin liquid

✓ periodic or diagonal boundary reflection; Bethe asnatz

✓ anti-periodic and general open boundary conditions; off-diagonal Bethe asnatz

✓ References: Nucl. Phys. B 954, 115007 (2020); J. Phys. A 53, 075205 (2020); J. Phys. A 54, 315202 (2021); Commun. Theor. Phys. 73, 075001 (2021); Chin. Phys. B 30, 117501 (2021); Nucl. Phys. B 975, 115663 (2022); Phys. Rev. D 107, 056005 (2023).

- Heisenberg model with open boundary condition

$$H = \sum_{j=1}^{N-1} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + h_1\sigma_1^z + h_N^x\sigma_N^x + h_N^z\sigma_N^z$$

- The present model with open boundary condition

$$H = H_{bulk} + H_L + H_R$$

$$H_L = \frac{1-4a^2}{p^2-a^2}[p\sigma_1^z - a^2\sigma_1^z\sigma_2^z - iapD_1^z \cdot (\vec{\sigma}_1 \times \vec{\sigma}_2)]$$

$$H_R = \frac{4a^2-1}{a^2\xi^2+a^2-q^2}\Big[q(\xi\sigma_{2N}^x + \sigma_{2N}^z) - a^2(\xi\sigma_{2N-1}^x + \sigma_{2N-1}^z)(\xi\sigma_{2N}^x + \sigma_{2N}^z)$$

$$-iaq(\xi D_{2N}^x + D_{2N}^z) \cdot (\vec{\sigma}_{2N} \times \vec{\sigma}_{2N-1})\Big]$$

✠ Dzyloshinsky-Moriya interaction: $\vec{D} \cdot (\vec{\sigma}_j \times \vec{\sigma}_{j+1})$

• Integrability

Generating functional

$$t(u) = tr_0\{K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\}$$

$$T_0(u) = R_{0,2N}(u+a+\theta_{2N})R_{0,2N-1}(u-a-\theta_{2N-1})\cdots R_{0,2}(u+a+\theta_2)R_{0,1}(u-a-\theta_1)$$

$$\hat{T}_0(u) = R_{0,1}(u+a+\theta_1)R_{0,2}(u-a-\theta_2)\cdots R_{0,2N-1}(u+a+\theta_{2N-1})R_{0,2N}(u-a-\theta_{2N})$$

$$R_{0,j}(u) = u + P_{0,j} = u + \frac{1}{2}(1+\vec{\sigma}_0\cdot\vec{\sigma}_j)$$

$$K_0^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix}, \qquad K_0^+(u) = \begin{pmatrix} q+u+1 & \xi(u+1) \\ \xi(u+1) & q-u-1 \end{pmatrix}$$

$$H = c_2^{-1}\{t(a)t(-a)\left(\frac{\partial \ln t(u)}{\partial u}\Big|_{u=a} + \frac{\partial \ln t(u)}{\partial u}\right)\Big|_{u=-a}\}\Big|_{\{\theta_j\}=0} - c_0$$

The reflection matrix has the non-diagonal elements. The spin of quasi-particle could be changed after the boundary reflections. Thus the particle number of fixed spin is not conserved and the traditional Bethe ansatz does not work.

- Off-diagonal Bethe ansatz

$t(u)$ is a polynomial operator of u with the degree $4N+2$ and satisfies the crossing symmetry

$$t(u) = t(-u-1)$$

Thus the degree of polynomial is reduced ti $2N+1$, and its eigenvalue can be completely determined by $2N+2$ conditions.

fusion and operator product identities

$$t(\theta_j + a)t(\theta_j + a - 1) = a(\theta_j + a)d(\theta_j + a - 1), \quad j = 1, \cdots, 2N$$

In the homogeneous limit $\{\theta_j = 0\}$

$$[t(u+a)t(u+a-1)]^{(n)}|_{u=0} = [a(u+a)d(u+a-1)]^{(n)}|_{u=0}, \quad n = 1, \cdots, 2N$$

$$asymptotic\ behavior : t(u)|_{u\to\pm\infty} = 2u^{4N+2} + \cdots$$

$$t(0) = 2\,p\,q\prod_{j=1}^{2N}(1-\theta_j - a)(1+\theta_j + a)$$

- Inhomogeneous $T-Q$ relation with $\{\theta_j=0\}$

$$
\begin{aligned}
\Lambda(u) \quad = \quad & a(u)\frac{Q(u-1)}{Q(u)}+d(u)\frac{Q(u+1)}{Q(u)}+2[1-(1+\xi^2)^{\frac{1}{2}}]u(u+1) \\
& \times\frac{(u^2-a^2)^{2N}[(u+1)^2-a^2]^{2N}}{Q(u)}
\end{aligned}
$$

Bethe roots

$$
Q(u)=\prod_{j=1}^{2N}(u-\lambda_j)(u+\lambda_j+1)
$$

Bethe ansatz equations (BAEs)

$$
\begin{aligned}
& a(\lambda_j)Q(\lambda_j-1)+d(\lambda_j)Q(\lambda_j+1)=-2[1-(1+\xi^2)^{\frac{1}{2}}]\lambda_j(\lambda_j+1) \\
& \times(\lambda_j^2-a^2)^{2N}[(\lambda_j+1)^2-a^2]^{2N}, \quad j=1,\cdots,2N
\end{aligned}
$$

Thermodynamic limit and thermodynamic Bethe ansatz

The main idea of TBA

$$\prod = \prod \quad \xrightarrow{\ln \prod = \sum} \quad \sum = \sum$$

If $N \to \infty$, then $\sum \to \int$ and we obtain the integration equation for the Bethe roots. By solving it, we can calculate the density of Bethe roots, ground state energy density, elementary excitations, and thermodynamic quantities at finite temperature such as free energy, specific heat and magnetic susceptibility.

For the present case, the thermodynamic Bethe ansatz does not work. Reason:

$$\ln\left(\prod + \prod\right) \neq \sum .$$

Our methods:
1. Degenerate points.
2. Reduced $T - Q$ relation (numerical and analytical calculations).
3. The universal $t - W$ approach.

- Exact physical quantities in the thermodynamic limit

zero-roots parametrization

$$\Lambda(u) = 2\prod_{j=1}^{2N+1}(u - z_j + \frac{1}{2})(u + z_j + \frac{1}{2})$$

Substituting it into the fusion relation, we obtain

$$\Lambda(\theta_j + a)\Lambda(\theta_j + a - 1) = a(\theta_j + a)d(\theta_j + a - 1), \quad j = 1, \cdots, 2N$$

Bethe ansatz equations

$$4\prod_{l=1}^{2N+1}(\theta_j + a - z_l + \frac{1}{2})(\theta_j + a + z_l + \frac{1}{2})(\theta_j + a - z_l - \frac{1}{2})(\theta_j + a + z_l - \frac{1}{2})$$

$$= a(\theta_j + a)d(\theta_j + a - 1), \qquad j = 1, \cdots, 2N$$

energy spectrum

$$E = -\pi(4a^2 - 1)\sum_{j=1}^{2N+1}[a_1(iz_j - ia) + a_1(iz_j + ia)] - c_0$$

- Patterns of zero roots

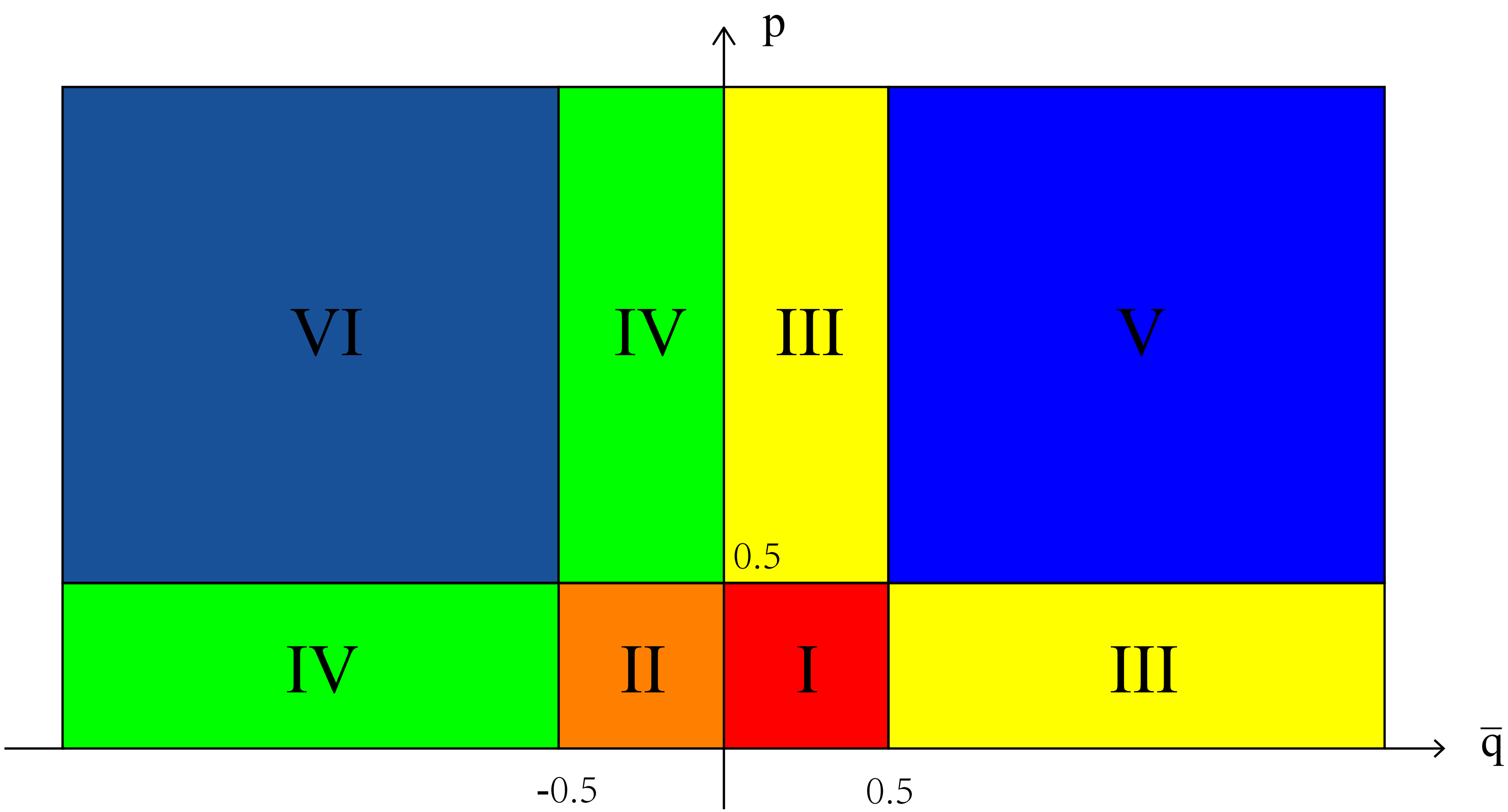


Figure 1: The distribution of $\bar{z}$-roots at the ground state in the upper $p-\bar{q}$ plane.

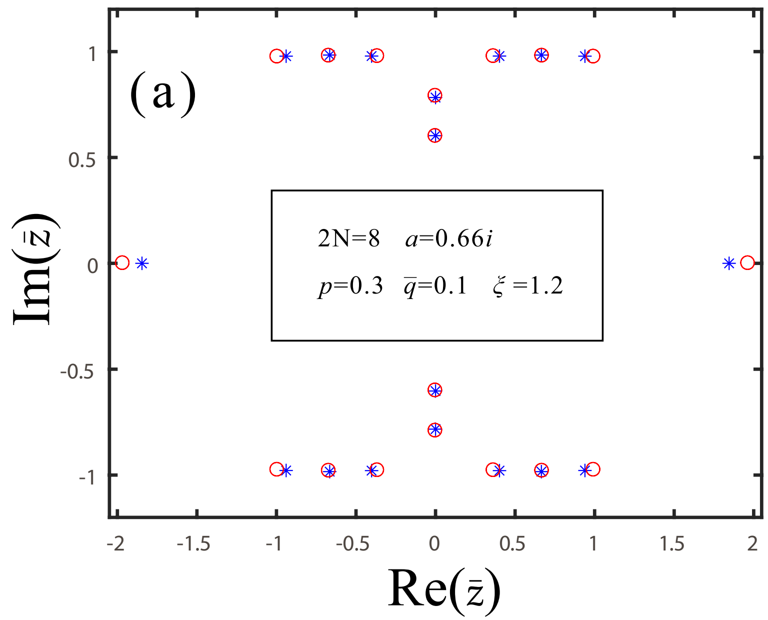


Figure 2: Pattern of $\bar{z}$-roots at the ground state in regime I with $2N = 8$. The blue stars indicate $\bar{z}$-roots for $\{\bar{\theta}_j = 0\}$ and the red circles specify $\bar{z}$-roots with the inhomogeneity parameters $\{\bar{\theta}_j = 0.1(j - N - 0.5)\}$.

In the regime I, where $0 \le p < \frac{1}{2}, 0 \le \bar{q} < \frac{1}{2}$, all the $\bar{z}$-roots form $2N - 2$ conjugate pairs as $\{\bar{z}_j \sim \tilde{z}_j \pm i | j = 1, \cdots, 2N - 2\}$ with real $\{\tilde{z}_j\}$, two boundary conjugate pairs $\{\pm i(|p| + \frac{1}{2}), \pm i(|\bar{q}| + \frac{1}{2})\}$ and two symmetrical real roots $\bar{z}_\pm = \pm\alpha$.

Taking derivative

$$
\begin{aligned}
&\ln|4|+\sum_{l=1}^{2N-1}\left[\ln|\bar\theta_j+\bar a-\tilde z_l+\frac{3i}{2}|+\ln|\bar\theta_j+\bar a-\tilde z_l+\frac{i}{2}|+\ln|\bar\theta_j+\bar a-\tilde z_l-\frac{i}{2}|+\ln|\bar\theta_j+\bar a-\tilde z_l-\frac{3i}{2}|\right]\\
&\quad+\ln|(\bar\theta_j+\bar a-\alpha+\frac{i}{2})(\bar\theta_j+\bar a-\alpha-\frac{i}{2})|+\ln|(\bar\theta_j+\bar a+\alpha+\frac{i}{2})(\bar\theta_j+\bar a+\alpha-\frac{i}{2})|\\
&\quad+\ln|(\bar\theta_j+\bar a-i|p|)(\bar\theta_j+\bar a+i|p|)|+\ln|(\bar\theta_j+\bar a-i|p|-i)(\bar\theta_j+\bar a+i|p|+i)|\\
&\quad+\ln|(\bar\theta_j+\bar a-i|\bar q|)(\bar\theta_j+\bar a+i|\bar q|)|+\ln|(\bar\theta_j+\bar a-i|\bar q|-i)(\bar\theta_j+\bar a+i|\bar q|+i)|\\
&=\ln|(\bar\theta_j+\bar a+i)(\bar\theta_j+\bar a-i)|-\ln|((\bar\theta_j+\bar a)+\frac{i}{2})((\bar\theta_j+\bar a)-\frac{i}{2})|\\
&\quad+\ln|(\bar\theta_j+\bar a+ip)(\bar\theta_j+\bar a-ip)|+\ln|((1+\xi^2)^{\frac{1}{2}}(\bar\theta_j+\bar a)+iq)((1+\xi^2)^{\frac{1}{2}}(\bar\theta_j+\bar a)-iq)|\\
&\quad+\sum_{k=1}^{2N}[(\ln|(\bar\theta_j-\bar\theta_k+i)(\bar\theta_j-\bar\theta_k-i)|+\ln|(\bar\theta_j-\bar\theta_k+2\bar a+i)(\bar\theta_j-\bar\theta_k+2\bar a-i)|]
\end{aligned}
$$

where $\bar a = -ia$

In the thermodynamic limit,

$$2N\int_{-\infty}^{\infty}[b_1(u+\bar{a}-\tilde{z})+b_3(u+\bar{a}-\tilde{z})]\rho(\tilde{z})d\tilde{z}+b_1(u+\bar{a}+\alpha)+b_1(u+\bar{a}-\alpha)$$

$$=2N\int_{-\infty}^{\infty}[b_2(u-\bar{\theta})+b_2(u+\bar{\theta}+2\bar{a})]\sigma(\bar{\theta})d\bar{\theta}+b_2(u+\bar{a})-b_1(u+\bar{a})$$

$$-b_{2|p|+2}(u+\bar{a})-b_{2|\bar{q}|+2}(u+\bar{a})$$

By using the Fourier transformation, the solution of $\tilde{z}$-roots density is

$$\tilde{\rho}(k)=[4N\tilde{b}_2(k)\cos(\bar{a}k)\tilde{\sigma}(k)+\tilde{b}_2(k)-\tilde{b}_1(k)-\tilde{b}_{2|p|+2}(k)$$

$$-\tilde{b}_{2|\bar{q}|+2}(k)-2\tilde{b}_1(k)\cos(\alpha k)]/[2N(\tilde{b}_1(k)+\tilde{b}_3(k))]$$

From now on, we use $\sigma(\theta)=\delta(\theta)$.

- The ground state energy

$$E_{g1} = N(4a^2 - 1)\int_{-\infty}^{\infty} [\tilde{a}_1(k) - \tilde{a}_3(k)] \cos(\bar{a}k)\tilde{\rho}(k)dk - c_0$$

$$-(4a^2 - 1)\Big[\frac{|p|}{a^2 - p^2} - \frac{|p| + 1}{a^2 - (|p| + 1)^2} + \frac{|\bar{q}|}{a^2 - \bar{q}^2} - \frac{|\bar{q}| + 1}{a^2 - (|\bar{q}| + 1)^2}\Big]$$

- Boundary energy

$$E_{b1} = E_{g1} - E_p$$

$$E_{b1} = e_b(p) + e_b(q) + e_{b0},$$

$$e_b(p) = \frac{(4a^2-1)}{4}\int_{-\infty}^{\infty}(1-e^{-|k|})\cosh(ak)\frac{e^{-|pk|}}{e^{-|k|/2}\cosh(k/2)}dk,$$

$$e_b(q) = \frac{(4a^2-1)}{4}\int_{-\infty}^{\infty}(1-e^{-|k|})\cosh(ak)\frac{e^{-|(q/\sqrt{1+\xi^2})k|}}{e^{-|k|/2}\cosh(k/2)}dk,$$

$$e_{b0} = \frac{(4a^2-1)}{4}\int_{-\infty}^{\infty}(1-e^{-|k|})\cosh(ak)\frac{e^{-|k|}-e^{-|k|/2}}{e^{-|k|/2}\cosh(k/2)}dk.$$

$e_b(p)$ and $e_b(q)$ are the contributions of left and right boundaries, respectively. e_{b0} exactly equals to the surface energy induced by the free boundaries.

We also find that the expression of surface energies in the rest regimes are the same. The reason is that the bare contributions of the boundary conjugate pairs to the ground state energy are exactly canceled by those of the back flow of continuum root density.

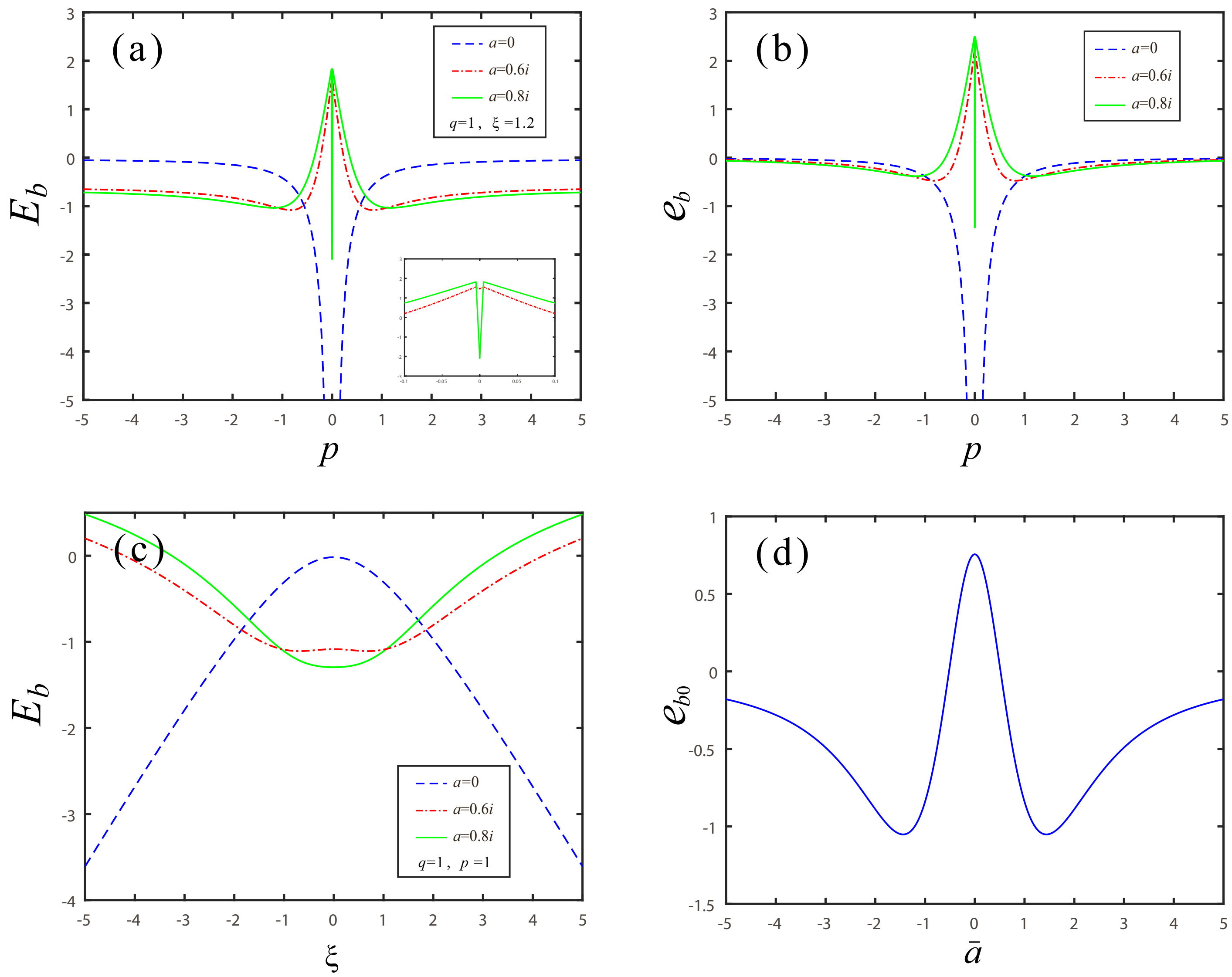


Figure 3: (a) The surface energy E_b versus the boundary parameter p. (b) The surface energy $e_b(p)$ versus the boundary parameter p. (c) The surface energy E_b versus the boundary parameter ξ. (d) The surface energy e_{b0} versus a.

• Elementary excitations

Type I: breaking one conjugate pair and putting the corresponding zero roots into the real axis.

Type II: the zero roots forming the conjugate pairs on the imaginary axis with more larger imaginary parts $\pm\frac{ni}{2}(n > 2)$.

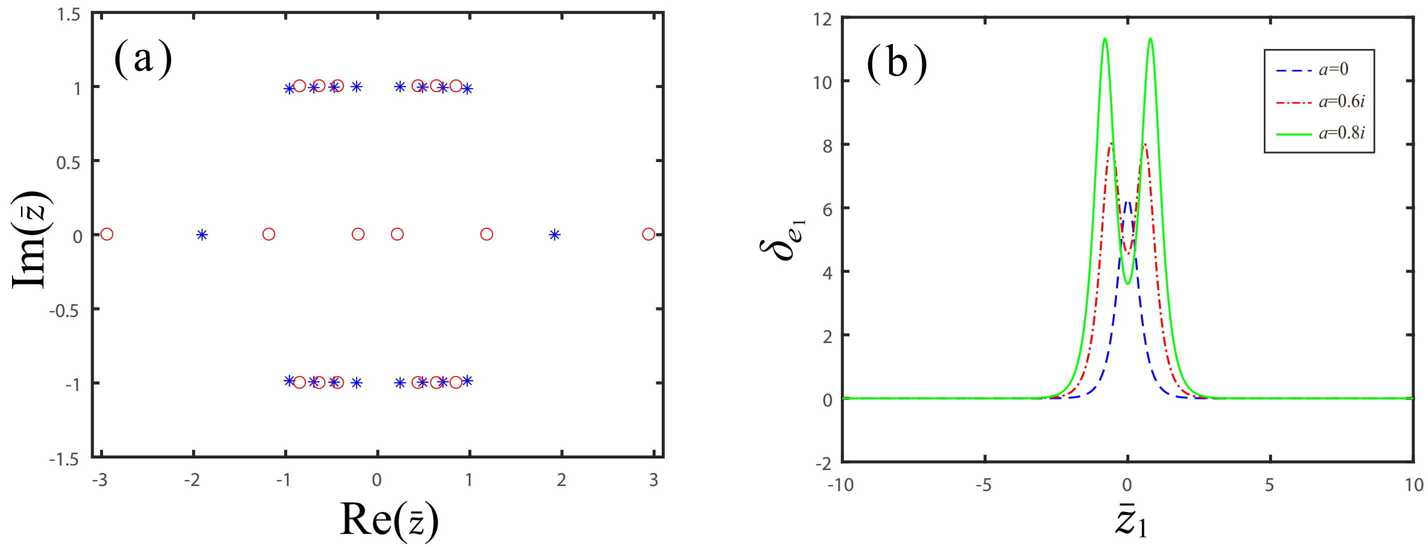


Figure 4: (a) The distribution of zero roots for $\{\bar{\theta}_j = 0\}$ at the ground state (blue asterisks) and at the first kind of excited state (red circles) with $2N = 8$, $a = 0.66i$, $p = 1.2$, $\bar{q} = 0.7$ and $\xi = 1.2$. (b) The excited energies δ_{e_1} versus $\bar{z}_1$ in the thermodynamic limit.

The non-Hermitian Bose-Hubbard model

- Fermi-Hubbard Model: strongly correlated electronic systems

$$H = -t \sum_{j=1}^{N} \sum_{\sigma=\uparrow,\downarrow} (c_{j,\sigma}^{\dagger} c_{j+1,\sigma} + h.c.) + U \sum_{j=1}^{N} n_{j,\uparrow} n_{j,\downarrow}$$

1968, Lieb and Wu, Phys. Rev. Lett. 20, 1445
Critical Metal-Insulator transition: $U_c = 0$

- Bose-Hubbard model

$$H = -t \sum_{j=1}^{N} (c_j^{\dagger} c_{j+1} + h.c.) + U \sum_{j=1}^{N} n_j(n_j - 1)$$

Lieb-Liniger model

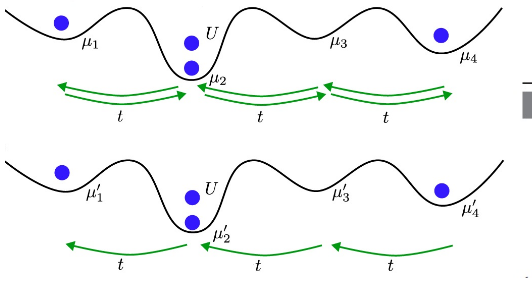
$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + c \sum_{i<j}^{N} \delta(x_i - x_j)$$

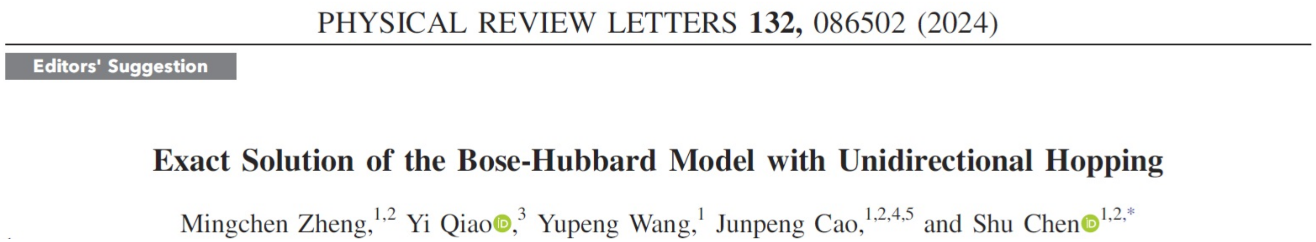
many applications in the cold atomic systems

- Integrable non-Hermitian Bose-Hubbard model with unidirectional hopping

$$H = -t\sum_{j=1}^{N-1}(b_j^\dagger b_{j+1} + \epsilon b_N^\dagger b_1) + \frac{U}{2}\sum_{j=1}^{N} n_j(n_j - 1)$$

$b_j^\dagger$ and b_j are bosonic creation and annihilation operators. $n_j = b_j^\dagger b_j$ is the number of bosons on site j and $M = \sum_{j=1}^{N} n_j$ is number of total particles. N is the number of sites. ϵ is the boundary term parameter.




PHYSICAL REVIEW LETTERS **132,** 086502 (2024)

Editors' Suggestion

Exact Solution of the Bose-Hubbard Model with Unidirectional Hopping

Mingchen Zheng,[1,2] Yi Qiao,[3] Yupeng Wang,[1] Junpeng Cao,[1,2,4,5] and Shu Chen[1,2,*]

This model is an unstable non-Hermitian model ($N > 2$) which all particles only hop to one side. Significantly, when $N = 2$ and $\epsilon = 1$, the model is Hermitian and reduces to the well-know integrable Bose-Hubbard dimer.
Reference: M. Zheng, Y. Qiao, Y. Wang, J. Cao, and S. Chen, Phys. Rev. Lett. 132, 086502 (2024).

• Integrability

The model is constructed by the spin-boson Lax operator

$$L_j(u) = \begin{pmatrix} u - n_j & gb_j \\ gb_j^\dagger & -g^2 \end{pmatrix}, \quad R_{12}(u) = \begin{pmatrix} u-1 & 0 & 0 & 0 \\ 0 & u & -1 & 0 \\ 0 & -1 & u & 0 \\ 0 & 0 & 0 & u-1 \end{pmatrix}$$

where g is a constant. The L operator satisfies the Yang-Baxter equation

$$R_{12}(u-v)\hat{L}_1(u)\hat{L}_2(v) = \hat{L}_2(v)\hat{L}_1(u)R_{12}(u-v)$$

Monodromy matrix,

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = L_1(u)L_2(u)\cdots L_N(u)$$

In order to characterize the twisted boundary coupling, we define a diagonal matrix K

$$K = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$$

the transfer matrix is

$$t(u) = tr[T(u)K] = A(u) + \epsilon D(u)$$

which satisfies the relation $[t(u), t(v)] = 0$, for any choice of u and v.

From the directly calculation, we get

$$t(u) = A(u) + \epsilon D(u) = u^N + C_1 u^{N-1} + C_2 u^{N-2} + \cdots,$$

$$C_1 = -\sum_{j=1}^{N} n_j,$$

$$C_2 = \frac{1}{2}\sum_{i\neq j}^{N} n_i n_j + g^2(\sum_{j=1}^{N-1} b_j^\dagger b_{j+1} + \epsilon b_N^\dagger b_1).$$

the non-Hermitian Hamiltonian

$$\begin{aligned} H = & -\frac{t}{g^2}(C_2 - \frac{1}{2}(C_1^2 + C_1)) \\ = & -t\sum_{j=1}^{N-1}(b_j^\dagger b_{j+1} + \epsilon b_N^\dagger b_1) + \frac{t}{2g^2}\sum_{j}^{N} n_j(n_j - 1) \end{aligned}$$

- Exact solution

By using the algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix

$$\Lambda(u) = u^N \prod_{j=1}^{M} \frac{u-\mu_j-1}{u-\mu_j} + \epsilon(-g^2)^N \prod_{j=1}^{M} \frac{u-\mu_j+1}{u-\mu_j}$$

and the Bethe ansatz equations

$$\frac{\mu_j^N}{\epsilon(-g^2)^N} = \prod_{j\neq l}^{M} \frac{\mu_j-\mu_l+1}{\mu_j-\mu_l-1} \qquad \epsilon \neq 0$$

Expanding $\Lambda(u)$ and comparing with the polynomial of $t(u)$, we get

$$C_1 = -M, \qquad C_2 = -\sum_{j=1} \mu_j - \frac{M(M-1)}{2}$$

So the eigenvalue of Hamiltonian

$$E = -\frac{t}{g^2}(C_2 - \frac{1}{2}(C_1^2 + C_1)) = \frac{t}{g^2}\sum_j \mu_j$$

Define $U = t/g^2$ and $\beta_j = U\mu_j$, the BAEs read

$$\beta_j^N = \epsilon(-t)^N \prod_{l \neq j}^{M} \frac{\beta_j - \beta_l + U}{\beta_j - \beta_l - U} \tag{1}$$

and the eigenvalue of Hamiltonian is

$$E = \sum_{j=1}^{M} \beta_j.$$

Taking the logarithm of (1), we have

$$N \ln(\frac{\beta_j}{-t}) = i \sum_{l \neq j}^{M} \Theta(\frac{-iU}{\beta_j - \beta_l}) + i2\pi I_j.$$

Here $\Theta(z) \equiv 2\arctan(z)$, and I_j is the quantum number.

In the fermion Bethe ansatz equations, the quantum number I_j of each particle shall not be equal. Here the $\{I_j | j = 1, 2, \ldots, M\}$ could be equal.

• Numerical check

The Hamiltonian is non-Hermitian, thus the energies are complex.

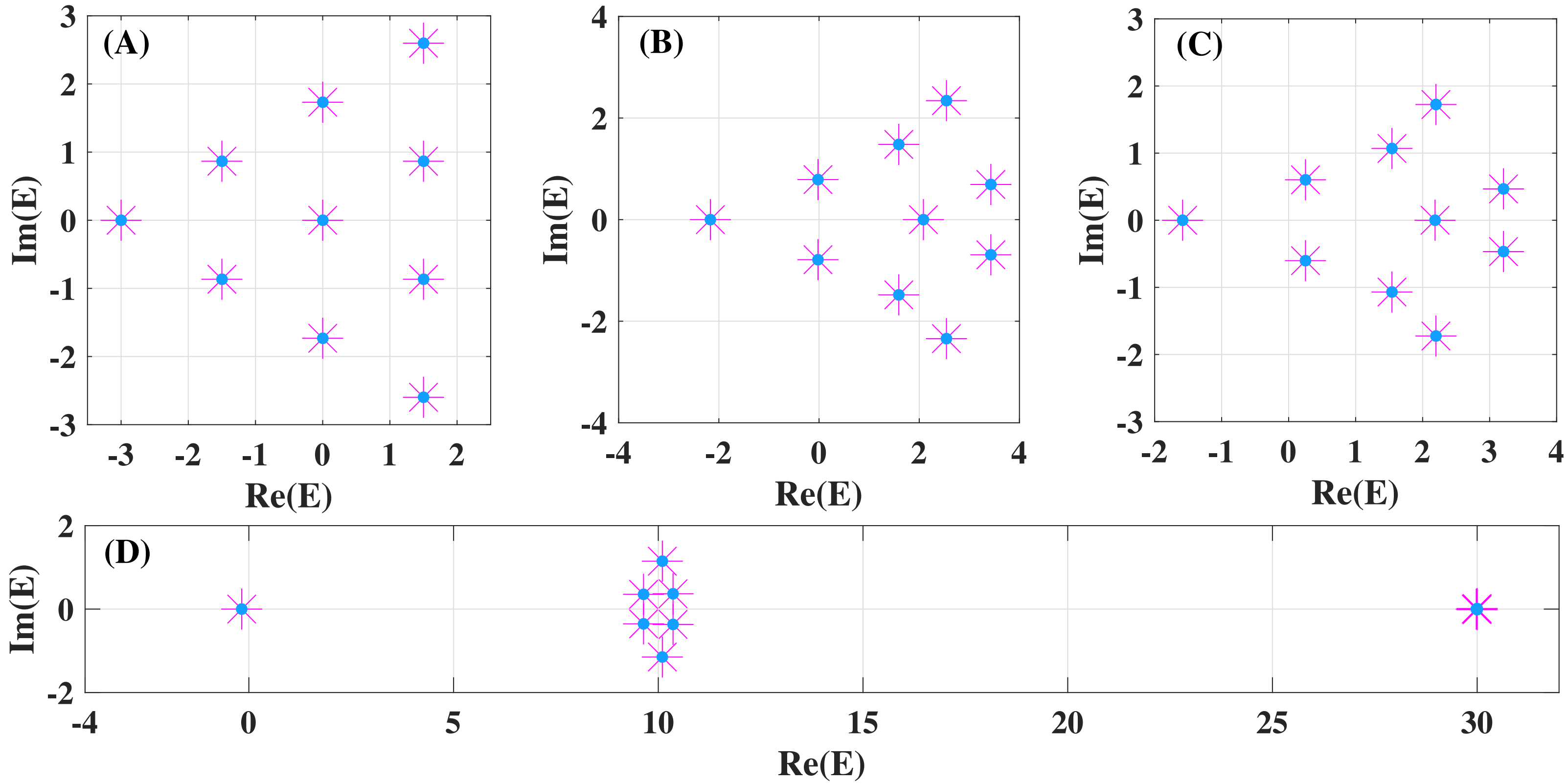


Figure 5: The numerical results of the eigenvalues of NHBH model. The blue points are obtained by directly solving the BAEs (1) and the data marked by purple asterisk are results from exact diagonalization with $U = 0.001$ $\epsilon = 1$ (A), $U = 1$ $\epsilon = 1$ (B), $U = 1$ $\epsilon = 0.5$ (C) and $U = 10$ $\epsilon = 1$ (D). Common parameters: $N = 3$, $M = 3$, $t = 1$.

• Superfluid-Mott like phase transition

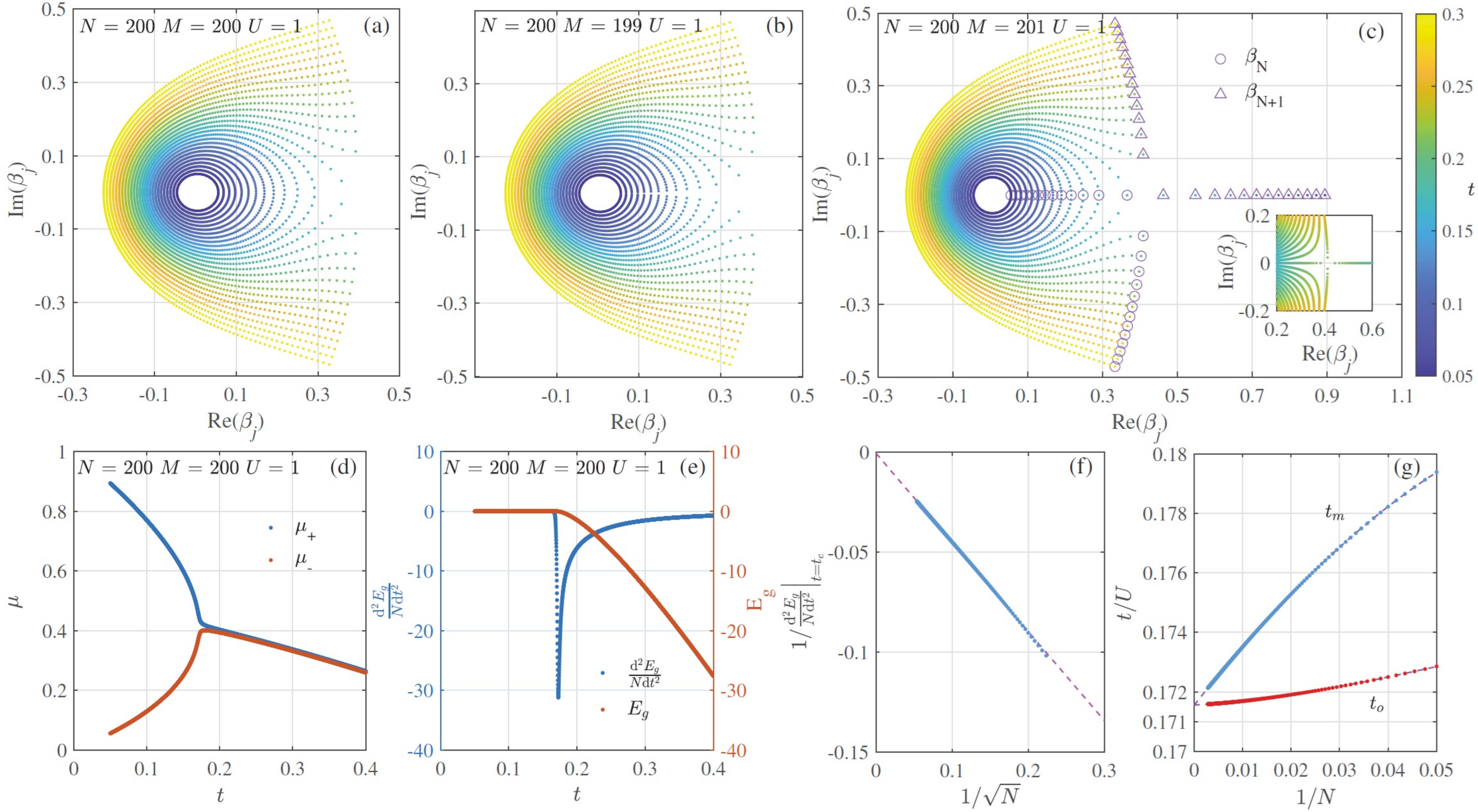
We define the ground state as the eigenstate whose eigenvalue has the lowest real part. To the present model, the ground state energy is real.

• filling: $N = M$

1. If U is small, the system is in the gapless superfluid phase.
   If U is large, the system is in the gapped Mott-insulator phase. That is every site is occupied by one boson due to the strong repulsive interaction.
   There is a Mott-superfluid transition for ceratin value of interaction.

2. Now, we determine the value of phase transition point. We define

$$\mu_+ = E_{M=N+1} - E_{M=N}$$
$$\mu_- = E_{M=N} - E_{M=N-1}$$



$N = 200\ M = 200\ U = 1$ (a)
$N = 200\ M = 199\ U = 1$ (b)
$N = 200\ M = 201\ U = 1$ (c)
β_N
β_{N+1}
Re(β_j)
Im(β_j)
t
$N = 200\ M = 200\ U = 1$ (d)
μ_+
μ_-
μ
$N = 200\ M = 200\ U = 1$ (e)
$\frac{d^2 E_g}{N dt^2}$
E_g
(f)
$1/\frac{d^2 E_g}{N dt^2}\Big|_{t=t_c}$
$1/\sqrt{N}$
(g)
t_m
t_o
t/U
$1/N$

- Superfluid-Mott like phase transition with $2N = M$.

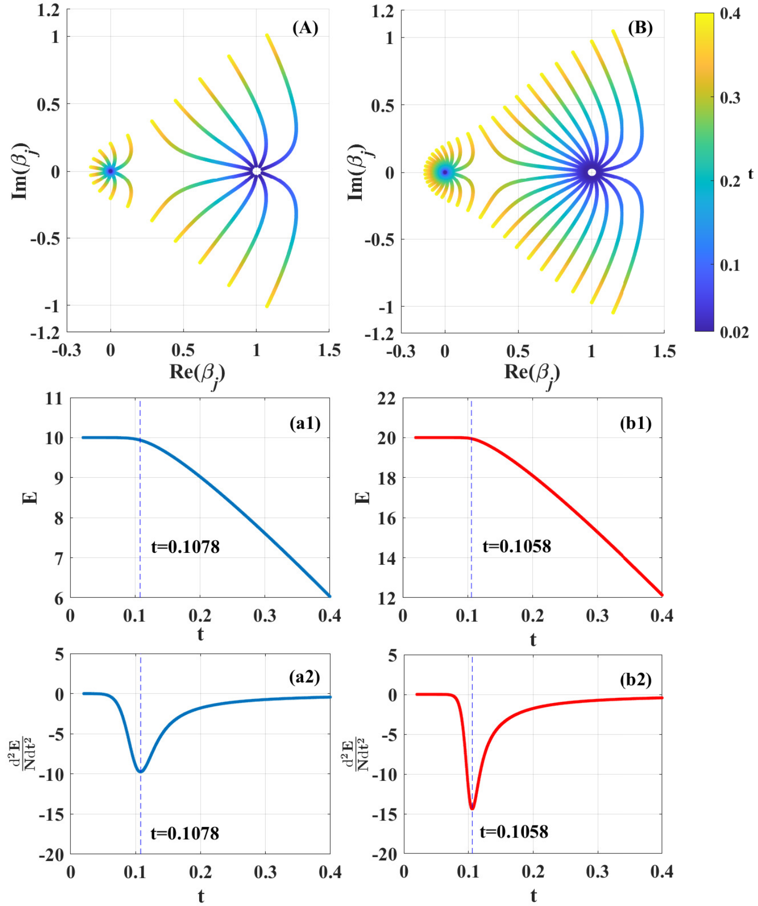


Figure 6: Numerical results of the trajectory of Bethe roots with kinetic energy term coefficient t varying from 0.4 to 0.02 for $N = 10,\ M =\ 20,\ U = 1$ (A), $N = 20,\ M =\ 40,\ U = 1$ (B). The numerical results of the eigenvalue E versus t/U and the second derivative of E/N with respect to t/U with $N = 10,\ M = 20$ (a1,a2), $N = 20,\ M = 40$ (b1,b2).

- The non-Hermitian skin effect (NHSE)

When there is no interaction, the NHBH model is the single particle system with bosons hopping only to one side. It is easy to imagine that in this case if the boundary parameter $\epsilon \neq 1$, the NHSE will appear.

Then if U is not zero, how will the NHSE of this system change?

The wave function is

$$\Psi = \sum_{x_1, x_2, \ldots, x_M = 1}^{N} \psi(x_1, \ldots, x_M) b^\dagger_{x_1} b^\dagger_{x_2} \ldots b^\dagger_{x_M} |0\rangle$$

$$\psi(x_1, \ldots, x_M) = \sum_{p,q} A_p(q) \exp[i \sum_{j=1}^{M} k_{p_j} x_{q_j}] \theta(x_{q_1} \le x_{q_2} \le \cdots \le x_{q_M})$$

$$= (-1)^M \sum_{p,q} A_p(q) \prod_{j=1}^{M} \beta_{p_j}^{x_{q_j}} \theta(x_{q_1} \le x_{q_2} \le \cdots \le x_{q_M})$$

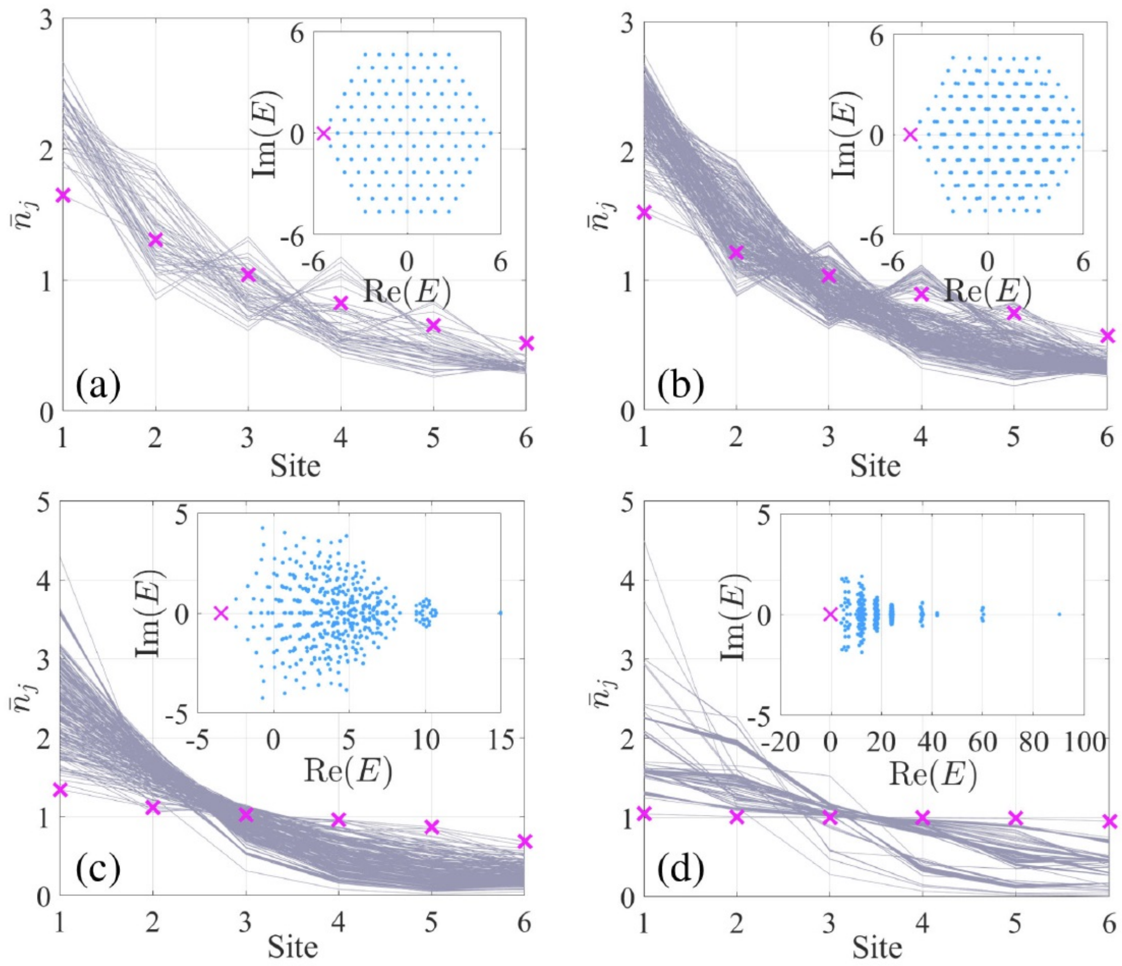
The relation between A_p is

$$\frac{A_{p_1 \ldots p_{i+1} p_i \ldots p_M}}{A_{p_1 \ldots p_i p_{i+1} \ldots p_M}} = S(k_{p_i}, k_{p_{i+1}})$$

and the scattering matrix of the system is

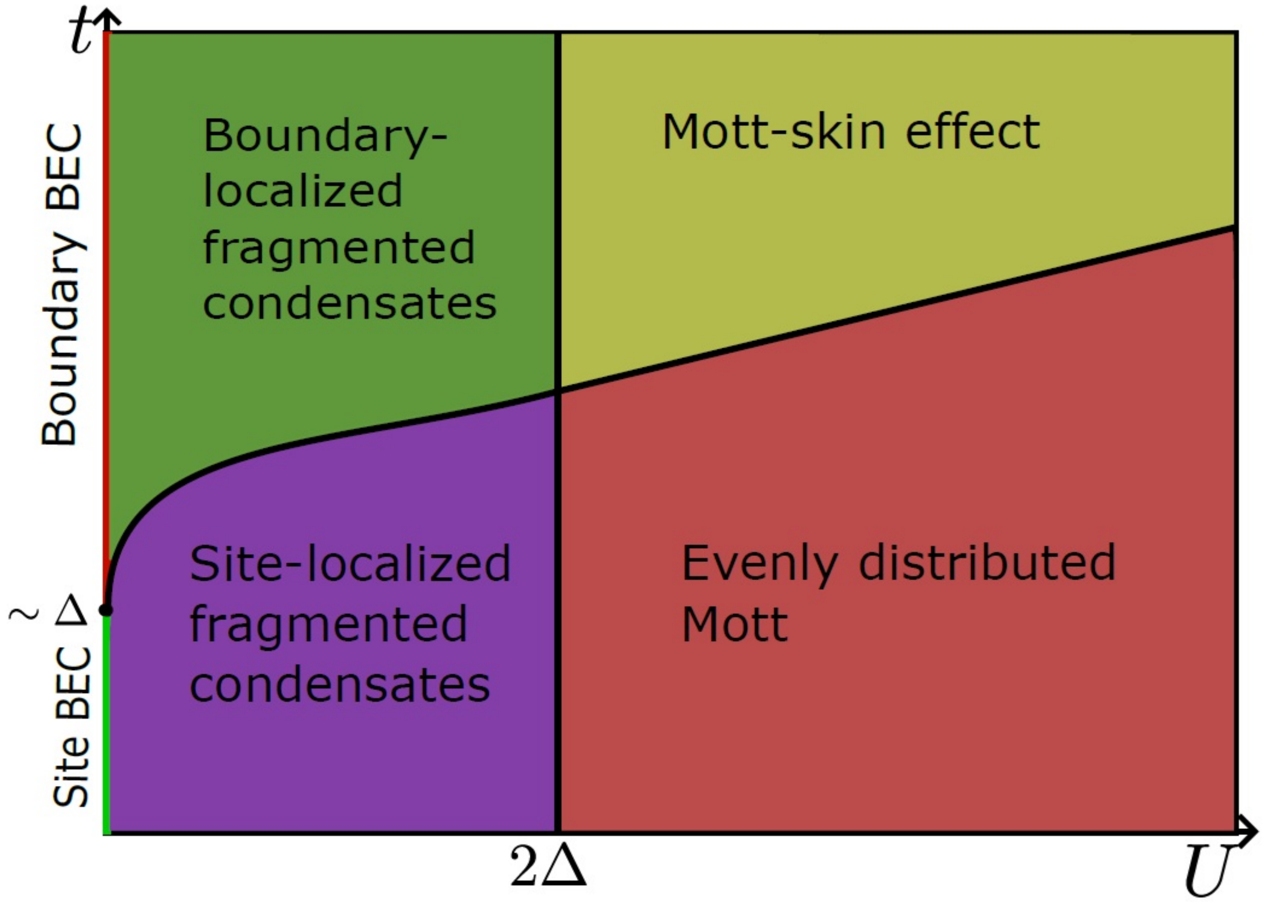
$$S(k_{p_i}, k_{p_{i+1}}) = \frac{\beta_{p_i+1} - \beta_{p_i} + U}{\beta_{p_i+1} - \beta_{p_i} - U}.$$

The solution of BAEs (1) can completely determine the wave function.



• For the case that $M = N$, there is only one state with eigenvalue close to 0 which is $|1, 1, \ldots, 1\rangle$, so that the NHSE will disappear. The particle number of the ground state is almost evenly distributed.

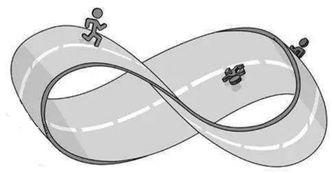
• When $N \neq M$, all state will have NHSE when $U \to \infty$.



Reference: C. Ekman and E. J. Bergholtz, Liouvillian skin effects and fragmented condensates in an integrable dissipative Bose-Hubbard model, arXiv:2402.10261.

The topological quantum spin ring

The XXZ spin chain with anti-periodic boundary condition is a typical quantum integrable models without $U(1)$ symmetry.



The Hamiltonian is

$$H = -\sum_{j=1}^{N} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh \eta \sigma_j^z \sigma_{j+1}^z \right].$$

• Anti-periodic boundary conditions: $\sigma_{N+1}^{\alpha} = \sigma_1^x \sigma_1^{\alpha} \sigma_1^x$ $(\alpha = x, y, z)$.
• The twisted bond could be shifted smoothly in the system with the spectrum of the Hamiltonian unchanged.
• Z_2-symmetry: $\tilde{H}_j = U_j^x H U_j^x, \quad U_j^x = \prod_{l=1}^{j} \sigma_l^x, \quad [H, U^x] = 0.$

The Hamiltonian is constructed as

$$H = -2\sinh\eta \left.\frac{\partial \ln t(u)}{\partial u}\right|_{u=0} + N\cosh\eta,$$

where

$$t(u) = tr_0\{\sigma_0^x R_{0,N}(u)\cdots R_{0,1}(u)\} = tr_0 \begin{pmatrix} C(u) & D(u) \\ A(u) & B(u) \end{pmatrix}.$$

The R-matrix

$$R_{0,j}(u) = \frac{\sinh(u+\eta)+\sinh u}{2\sinh\eta} + \frac{1}{2}(\sigma_j^x\sigma_0^x + \sigma_j^y\sigma_0^y) + \frac{\sinh(u+\eta)-\sinh u}{2\sinh\eta}\sigma_j^z\sigma_0^z,$$

From the Yang-Baxter relation, one can prove that $[t(u), t(v)] = 0$!

Conserved charges

Due to the topological boundary, the model possesses neither translational invariance nor $U(1)$ symmetry.

- Basic properties of the transfer matrix:

$$\mathbf{t}(0) = \sigma_1^x P_{1,N} P_{1,N-1} \cdots P_{1,2}, \qquad \mathbf{t}^{2N}(0) = 1,$$

is a conserved and represents the shift operator in the topological manifold.

Topological momentum: $\mathbf{P}_q = -i \ln \mathbf{t}(0)$. The eigenvalues of $\mathbf{P}_q$ are

$$k = \frac{\pi l}{N} \;\; mod \, \{\pi\}, \qquad l = \{-N, -N+1, \cdots, N-1\}.$$

- String charge:

$$\mathbf{M}_q = \frac{1}{2}(\mathbf{I}_q^+ + \mathbf{I}_q^-) = \frac{1}{4}e^{-\frac{(N-1)\eta}{2}} \lim_{u\to\infty}(2\sinh\eta e^{-u})^{N-1}\mathbf{t}(u),$$

where

$$\mathbf{I}_q^\pm = \frac{1}{2}\sum_{j=1}^{N} e^{\mp\frac{\eta}{2}\sum_{k=j+1}^{N}\sigma_k^z}\sigma_j^\pm e^{\pm\frac{\eta}{2}\sum_{k=1}^{j-1}\sigma_k^z},$$

are two generators of the quantum group associated with the model. The eigenvalues of the operator $\mathbf{M}_q$ is given by

$$M_q = \frac{1}{4}\sinh^{N-1}\eta\,\Lambda_0\,e^{-\sum_{k=1}^{N-1}z_k}.$$

♠ For generic η, the $\mathbf{M}_q$ is not an $U(1)$ charge.

♣ When $\eta\to 0$, the model tends to an isotropic spin chain and the $U(1)$ symmetry recovers with $\mathbf{M}_q = \sum_{j=1}^{N}\sigma_j^x/2$, which is just the $U(1)$ charge.

t-W relation

Using the fusion techniques, we have

$$\begin{aligned}\mathbf{t}(u)\mathbf{t}(u-\eta) &= tr_{1,2}\{P^{(-)}_{1,2}\sigma^x_1\sigma^x_2\mathbf{T}_2(u)\mathbf{T}_1(u-\eta)P^{(-)}_{1,2}\} \\ &\quad + tr_{1,2}\{P^{(+)}_{1,2}\sigma^x_1\sigma^x_2\mathbf{T}_2(u)\mathbf{T}_1(u-\eta)P^{(+)}_{1,2}\}.\end{aligned}$$

Thus the $t-W$ relation is

$$\mathbf{t}(u)\mathbf{t}(u-\eta) = -a(u)d(u-\eta)\times\mathbf{id} + d(u)\mathbf{W}(u),$$

where $\mathbf{W}(u)$ is an operator-valued degree N trigonometric polynomial of u. Meanwhile, $\mathbf{W}(u)$ and $\mathbf{t}(u)$ commute with each other

$$[\mathbf{W}(u), \mathbf{t}(u)] = 0,$$

which indicates they have common eigenstates.

Functional relations

$$\Lambda(u)\Lambda(u-\eta) = -a(u)d(u-\eta) + d(u)W(u). \tag{2}$$

Parameterization:

$$\Lambda(u) = \Lambda_0 \prod_{j=1}^{N-1} \sinh(u - z_j + \frac{\eta}{2}), \quad W(u) = W_0 \sinh^{-N}\eta \prod_{l=1}^{N} \sinh(u - w_l).$$

An important fact is that (2) is a degree $2N$ polynomial equation and thus gives $2N+1$ independent equations for the coefficients, which determines the $N-1$ z_j roots, N w_l roots and the two constants Λ_0 and W_0 completely.

Putting $u = z_j - \eta/2$ in (2), we obtain new Bethe ansatz equations

$$\sinh^N(z_j - \frac{3\eta}{2}) \sinh^N(z_j + \frac{\eta}{2}) = W_0 \sinh^N(z_j - \frac{\eta}{2}) \prod_{l=1}^{N} \sinh(z_j - w_l - \frac{\eta}{2}). \tag{3}$$

Putting $u = w_l$ in (2), we obtain

$$\Lambda_0^2 \prod_{j=1}^{N-1} \sinh(w_l - z_j + \frac{\eta}{2}) \sinh(w_l - z_j - \frac{\eta}{2}) = -\sinh^{-2N}\eta \sinh^N(w_l + \eta) \sinh^N(w_l - \eta).$$

Since $\Lambda(u)$ is a degree $N-1$ trigonometric polynomial of u, the leading terms in the right hand side of (2) must be zero. Therefore, $W_0^2 = 1$.
Selection rule: The coefficient Λ_0 can be determined by putting $u = 0$ in (2) as

$$\Lambda_0^2 \prod_{j=1}^{N-1} \sinh(z_j + \frac{\eta}{2}) \sinh(z_j - \frac{\eta}{2}) = (-1)^{N-1}.$$

The eigenvalue of the Hamiltonian can be expressed as

$$E = 2\sinh\eta \sum_{j=1}^{N-1} \coth(z_j - \frac{\eta}{2}) + N\cosh\eta.$$

Distribution of roots

From the intrinsic properties of the R-matrix, for imaginary η we have

$$\mathbf{t}^\dagger(u) = (-1)^{N-1}\mathbf{t}(u^* - \eta), \qquad \Lambda(u) = (-1)^{N-1}\Lambda^*(u^* - \eta).$$

The above relation implies that if z_j is a root, z_j^* must also be a root!
Therefore, z_j can be classified into 3 sets:
(1) real z_j;
(2) $\mathrm{Im} z_l = -i\pi/2$ (this is because its conjugate shifted by $i\pi$ becomes itself);
(3) complex conjugate pairs.

$W^*(u^*) = (-1)^N W(u)$ indicates if w_l is a root of $W(u)$, w_l^* must also be a root.

Ground state

At the ground state, all roots z_j and w_l take real values around zero symmetrically. Taking the logarithms of (3) and its complex conjugate we have

$$2\theta_1(z_j) - \theta_3(z_j) = \frac{4\pi I_j}{N} - \frac{1}{N}\sum_{l=1}^{N}\theta_1(z_j - w_l), \tag{4}$$

and

$$\ln|\Lambda_0 \sinh(z_j - \frac{3\eta}{2})| = \frac{1}{N}\sum_{l=1}^{N}\ln|\sinh(z_j - w_l - \frac{\eta}{2})|, \tag{5}$$

where I_j denote the quantum numbers (integers or half odd integers depending on the parity of N) associated with the root z_j and $\theta_n(x) = 2\cot^{-1}(\coth x \tan\frac{n\gamma}{2})$.

$$I_j = \left\{-\frac{N-2}{2}, -\frac{N-4}{2}, \cdots, \frac{N-4}{2}, \frac{N-2}{2}\right\}.$$

In the thermodynamic limit $N \to \infty$, we define the density of z-roots and the density of z-holes per unit site as $\rho(z)$ and $\rho^h(z)$, the density of w-roots as $\sigma(w)$, respectively.

Taking the continuum limits of (4) and (5) we have

$$
\begin{aligned}
2a_1(z) - a_3(z) &= 2\rho(z) + 2\rho^h(z) - a_1 * \sigma(z), \\
b_3(z) &= b_1 * \sigma(z),
\end{aligned}
$$

where $a_n(z) = \theta_n'(z)/(2\pi)$, $b_n(z) = \ln' |\sinh(z - n\eta/2)|/\pi$ and $*$ indicates convolution.

With Fourier transformation we readily have

$$
\rho(z) + \rho^h(z) = \frac{2\cosh(\frac{\pi z}{\pi-\gamma})\sin(\frac{\pi\gamma}{2\pi-2\gamma})}{(\pi-\gamma)[\cosh(\frac{2\pi z}{\pi-\gamma}) + \cos(\frac{\pi(\pi-2\gamma)}{\pi-\gamma})]}.
$$

Here $\rho^h(z)$ is non-zero only in the range $|z| > D$ ($D \to \infty$ in the thermodynamic limit) with $N\int_D^\infty \rho^h(z)dz = 1/2$ and $N\int_{-\infty}^{-D} \rho^h(z)dz = 1/2$.

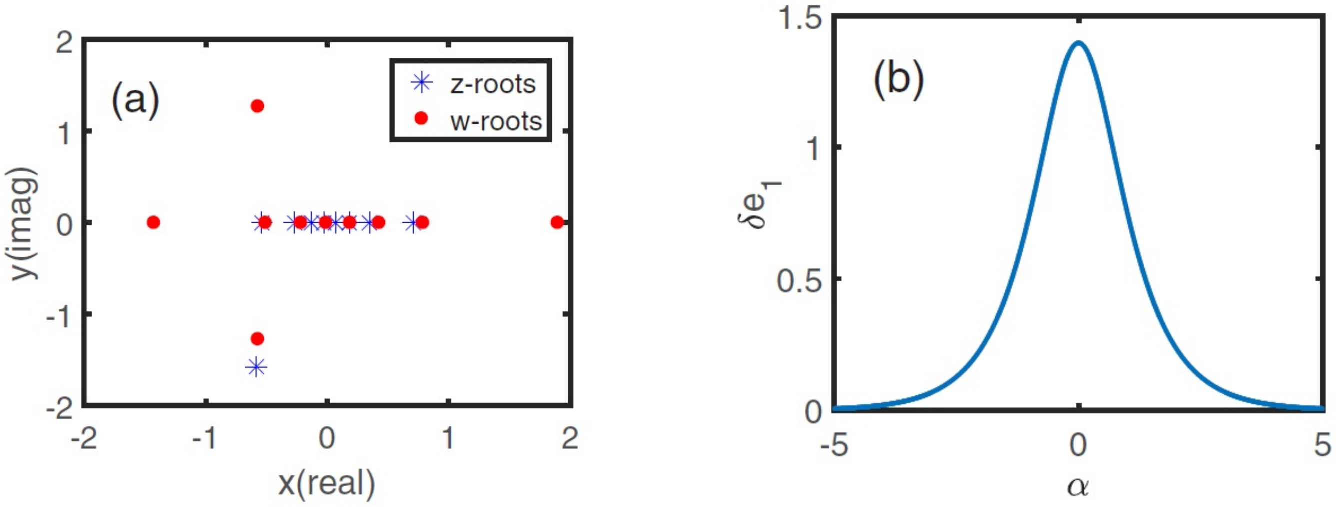
- The existence of the hole density is due to the fact that the total number of roots must be $N-1$ while the dimension of the Brillouin zone is N.
- However, the hole separates into two halves due to the topological restriction and each half hole locates at one edge of the spectral space.
- Clearly, the two half-holes contribute two half zero modes (carrying zero energy).

The ground state energy density reads

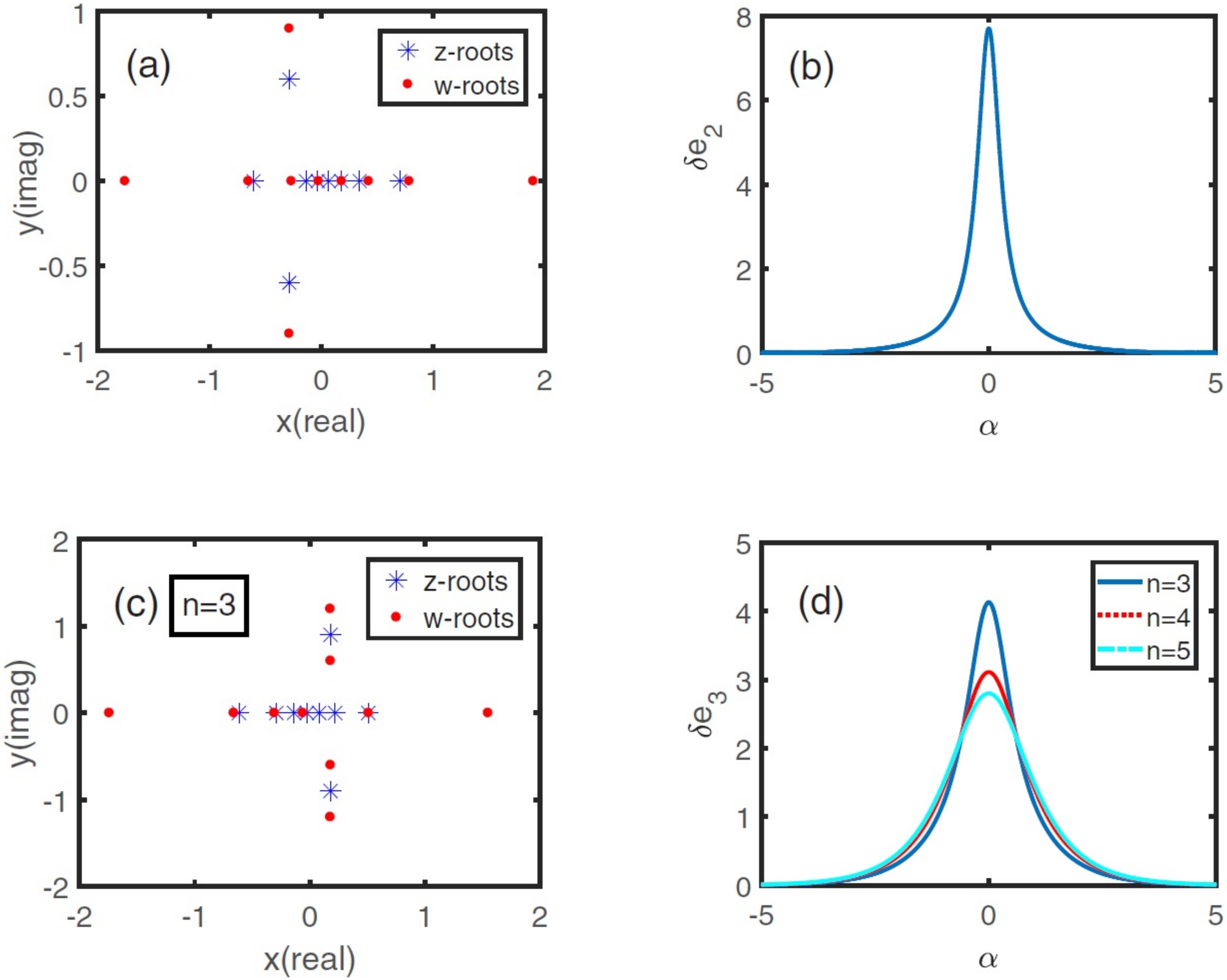
$$e_g = -\sin\gamma \int \frac{\cosh[\frac{(\pi-2\gamma)\tau}{2}]\tanh[\frac{(\pi-\gamma)\tau}{2}]}{\sinh(\frac{\pi\tau}{2})} d\tau + \cos\gamma.$$

Elementary excitations I

- The first kind of elementary excitations is described by a single root locating in the axis $\mathrm{Im}z = -i\pi/2$ and all the other roots remaining in the real axis.
- Accordingly, two w-roots form a conjugate pair $w_{\pm} = \beta \pm m\eta/2$ with β and m two real numbers, and all the other w-roots keep real.



Elementary excitations II and III



References of $t - W$ scheme:
Phys. Rev. B 102, 085115 (2020); Phys. Rev. B 103, L220401 (2021); Results Phys. 29, 104721 (2021); JHEP 11, 044 (2021); JHEP 07, 133 (2021).

Concluding remarks and perspective

• Non-Hermite physics is a hot topic recently. Most results are based on the single particle systems. By using the quantum inverse scattering method, we can construct some non-Hermite integrable models. Then we can study the correlation effects in the non-Hermite systems exactly.

• Besides the quantum inverse scattering, one can also use the fusion technique to construct new quantum integrable spin chain or the strongly correlated electronic model. The Mott insulator to superfluid transition at finite critical interaction is very interesting.

• Off-diagonal Bethe ansatz
1. spin chain, electronic model, statistical model, High ranks (A_n, B_n, C_n, D_n)
2. energy spectrum, eigenstates, thermodynamic limit, surface energy and elementary excitation, thermodynamic quantities at finite temperature
3. G_2 exceptional Lie algebra

Thank you for your attention!