New results in 1D repulsive Hubbard model: Quantum liquid, criticality and transport

Xi-Wen Guan

Innovation Academy for Precision Measurement Science and Technology
Chinese Academy of Science

Collaborators:

Jia-Jia Luo

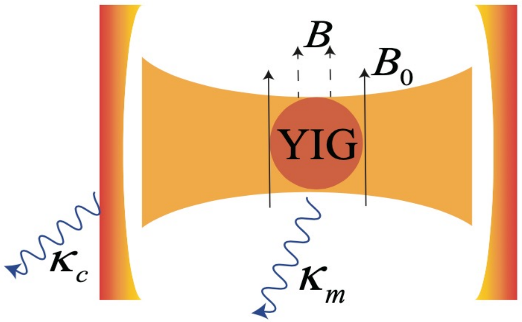
Han Pu

Mathematics and Physics of Integrability
Creswick, Melbourne, July 12, 2024

Recent and on-going research

Quantum metrology and supersonic flutter

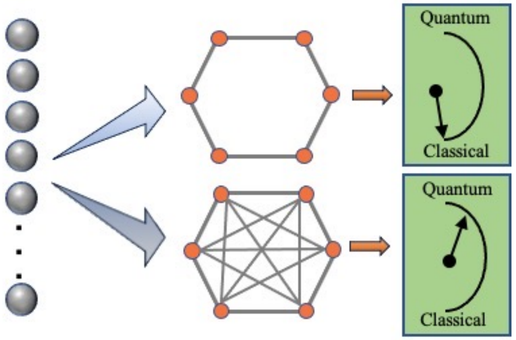
1. Wan, Shi, Guan, Magnonic sensor in Cavity, Phys. Rev. B 109, Letter 041301 (2024)
2. Shi, Guan and Yang, Growth limit of Fisher information, Phys. Rev. Lett. 132, 100803 (2024)
3. Zhang, Jiang, Lin, Guan, Quantum supersonic flutter, Phys. Rev. Lett. in refereeing



$$\hat{H} = \omega_c \hat{c}^\dagger \hat{c} + \omega_m \hat{b}^\dagger \hat{b} + g(\hat{c} + \hat{c}^\dagger)(\hat{b} + \hat{b}^\dagger)$$

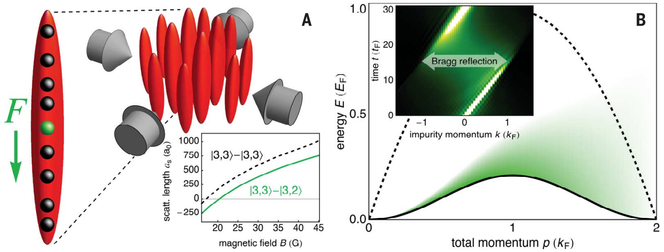
where $\omega_k = \mu(B_0 + B_z)$.

What is the role of bipartite entanglement in partially accessible metrological schemes?



Growth of quantum Fisher information

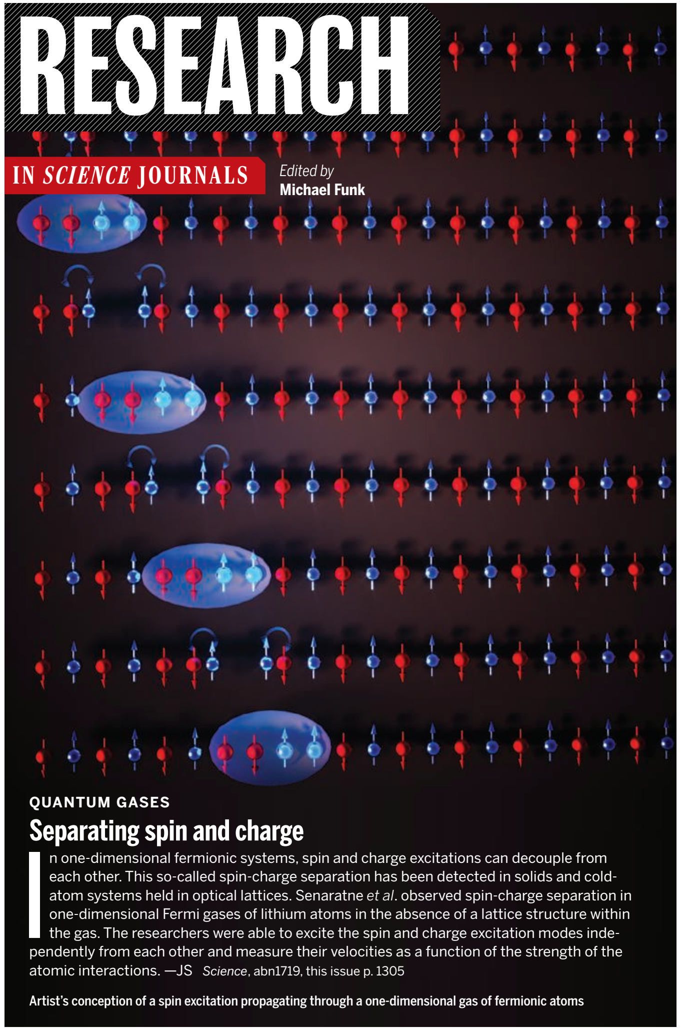
$$\frac{d\sqrt{I(t)}}{dt} \leq \Gamma(t) \equiv 2\sqrt{\mathrm{Var}\left([\partial_\lambda H_\lambda(t)]^{(\mathrm{H})}\right)_{|\psi_0\rangle}}$$



The injected particle never comes to a full stop, without a relaxation—Quantum flutter.

Quantum dynamical correlation functions: From coherent Luttinger liquid to Luther-Emery

1. Senaratne, et. al., Pu, Guan*, Hulet*, Determinant observation of S-C separation, Science 376, 1305 (2022)
2. Aashish, et. al., Giamarchi, Pu, Guan, Hulet, Measurement of the Luther-Emery liquid, 2024



RESEARCH

IN *SCIENCE* JOURNALS

Edited by
Michael Funk

QUANTUM GASES

Separating spin and charge

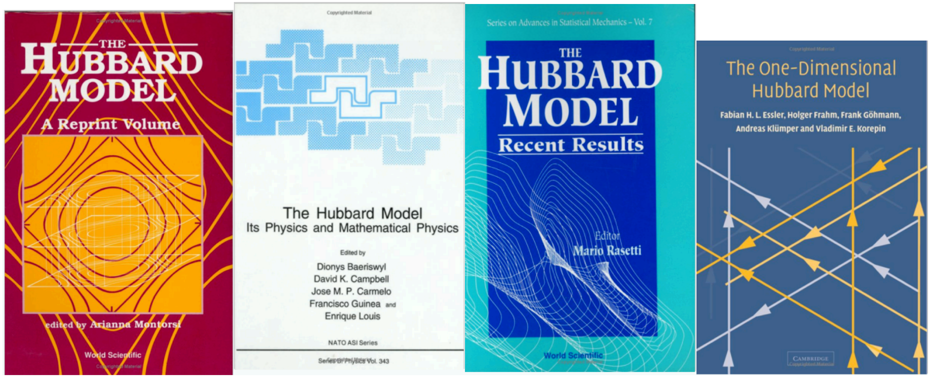
In one-dimensional fermionic systems, spin and charge excitations can decouple from each other. This so-called spin-charge separation has been detected in solids and cold-atom systems held in optical lattices. Senaratne *et al.* observed spin-charge separation in one-dimensional Fermi gases of lithium atoms in the absence of a lattice structure within the gas. The researchers were able to excite the spin and charge excitation modes independently from each other and measure their velocities as a function of the strength of the atomic interactions. —JS *Science*, abn1719, this issue p. 1305

Artist's conception of a spin excitation propagating through a one-dimensional gas of fermionic atoms

1D Hubbard model: from quantum liquids to quantum cooling and quantum transport

1. Luo, Pu, Guan, Phys. Rev. B 107, Letter 201103 (2023);
2. Luo, Pu, Guan, 51 pages, arXiv: 2307.00890; submitted to Report on Progress in Physics
3. Luo, Pu, Guan, Spin and charge Drude weights in 1D Hubbard model, in preparation

Hubbard model with cold atoms



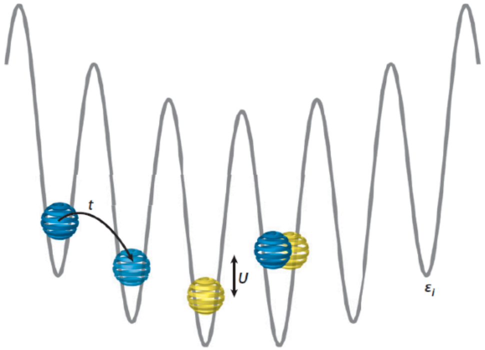
(a) (b) (c) (d)

A paradigm of physics in condensed matter:

- **Electronic properties of solids with narrow bands**
- **Band magnetism**
- **Metal-Mott insulator transition,**
- **Fractional excitations, FFLO pairing**
- **…**

The Hubbard model has also become increasingly important in

- cold atoms
- quantum metrology
- quantum information



Nichols et. al., Science 363, 383 (2019)
Brown, et. al. Science 363,379 (2019)
Shao, et. al. ArXiv:2402.14605 (2024)
Wei, et. al. Science 376, 716 (2024)

Hart, et al. Nature 519, 211 (2015)
Boll et al. Science 353, 1257 (2016)
Parsons et al. Science 353, 1253 (2016)
Cheuk, et al. Science 353, 1260 (2016)
Cheuk, et al. PRL 116, 235301 (2016)
Hilker, et al. Science 357, 484 (2017)
Cocchi, et al, Phys. Rev. X, 7, 031025 (2017)
Chiu, et al, Science 365, 251(2019)}
Hart, et al. Nature 565, 56 (2019)
Vijayan, et. al., Science, 367, 186 (2020)

Outline

I. 1D Hubbard model (Dynamical correlation)

Spin coherent and incoherent TLLs, critical scaling functions

II. Interaction driven criticality and Contact (Caloric effect)

Contact susceptibility and quantum cooling in a lattice

III. Quantum transport

Spin and charge Drude weights at zero and finite temperature

IV. Conclusion and discussion

I. 1D Hubbard model: A prototypical integrable model

$$H_0 = -\sum_{j=1}^{L}\sum_{a=\uparrow\downarrow}\left(c_{j,a}^{+}\,c_{j+1,a} + c_{j+1,a}^{+}c_{j,a}\right)$$
$$+u\sum_{j=1}^{L}(1-2n_{j\uparrow})(1-2n_{j\downarrow})$$
$$H_{GE} = H_0 - \mu\widehat{N} - 2B\hat{S}^z$$

- C_{ja} and C_{ja}^{+} : annihilation and creation operators of electrons with spin a at site j
- $n_{ja} = C_{ja}^{+}C_{ja}$
- $\widehat{N} = \sum_{j=1}^{L}(n_{j\uparrow}+n_{j\downarrow})$
- $u<0$ ($u>0$): on-site attractive (repulsive) interaction

Lieb, Wu PRL 20 , 1445 (1968)

Spin SU(2) symmetry

$$S^{\alpha} = \frac{1}{2}\sum_{j=1}^{L}\sum_{a,b=1}^{2} c_{j,a}^{\dagger}\,(\sigma^{\alpha})_b^a\, c_{j,b}, \alpha = x, y, z$$

Eta-pairing Symmetry

$$\eta^{+} = \sum_{j=1}^{L}(-1)^{j+1}\,c_{j,\uparrow}^{\dagger}\,c_{j,\downarrow}^{\dagger}, \qquad \eta^{z} = \frac{1}{2}(\widehat{N}-L)$$
$$\eta^{-} = \sum_{j=1}^{L}(-1)^{j+1}\,c_{j,\uparrow}c_{j,\downarrow},$$

Rich Symmetries: $SU(2)\otimes SU(2)/Z_2$; $U(1)\otimes U(1)$ ...

The model has been realized with ultracold atoms in lab

Essler, Frahm, Gohmann,Klumper, Korepin,
The One-Dimensional Hubbard Model
(Cambridge University Press,2005).

Exact Lieb-Wu Equations – Bethe ansatz:

$$e^{ik_jL} = \prod_{\ell=1}^{M} \frac{\lambda_\ell - \sin k_j - iu}{\lambda_\ell - \sin k_j + iu}, \qquad j = 1, \dots, N,$$

$$\prod_{j=1}^{N} \frac{\lambda_\ell - \sin k_j - iu}{\lambda_\ell - \sin k_j + iu} = \prod_{\substack{m=1\\ m\neq\ell}}^{M} \frac{\lambda_\ell - \lambda_m - 2iu}{\lambda_\ell - \lambda_m + 2iu}, \qquad \ell = 1, \dots, M.$$

String hypothesis for $u > 0$:

k: real quasimomentum root M_e

Λ: Spin Bound state $\Lambda - \Lambda$ String M_n

$$\lambda_\alpha^{m,j} = \Lambda'^m_\alpha + (m - 2j + 1)i\,u$$

$k - \Lambda$: charge bound state M'_n

$$k_\alpha^1 = \pi - \arcsin(\Lambda'^m_\alpha + mi\,u)\,;$$
$$k_\alpha^2 = \arcsin(\Lambda'^m_\alpha + (m-2)i\,u)\,;$$
$$k_\alpha^3 = \pi - k_\alpha^2;$$
$$\cdots$$
$$k_\alpha^{2m-1} = \pi - k_\alpha^{2m-1};$$
$$k_\alpha^{2m} = \pi - \arcsin(\Lambda'^m_\alpha - mi\,u)$$

Im sin k_α

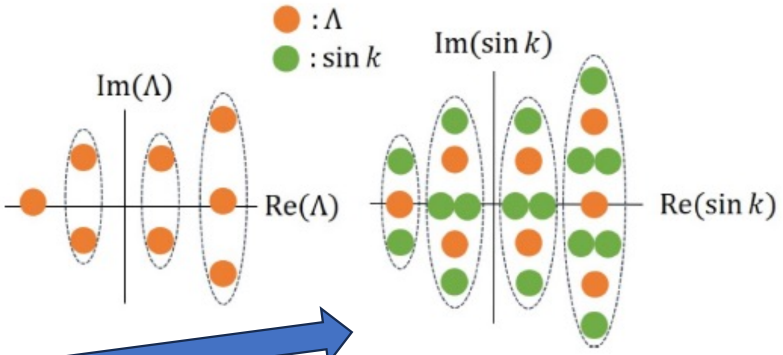
Energy and Momentum

$$E = -2\sum_{j=1}^{N} \cos k_j + u(L - 2N)$$

$$P = \left[\sum_{j=1}^{N} k_j\right] \bmod 2\pi\,.$$

$$N = \mathcal{M}_e + \sum_{n=1}^{\infty} 2nM'_n$$

$$M = \sum_{n=1}^{\infty} n\left(M_n + M'_n\right)$$



Length-n Λ strings(Orange dots):
n-magnons form a bound state

Length-n $k - \Lambda$ strings (Green dots):
2n electrons form a bound state

Lieb, Wu PRL 20 , 1445 (1968)

Bethe ansatz equations for ground and excited states:

$$
\begin{aligned}
&k_j L = 2\pi I_j - \sum_{n=1}^{\infty}\sum_{\alpha=1}^{M_n}\theta\left(\frac{\sin k_j - \Lambda_\alpha^n}{nu}\right) - \sum_{n=1}^{\infty}\sum_{\alpha=1}^{M'_n}\theta\left(\frac{\sin k_j - \Lambda'^n_\alpha}{nu}\right),\\
&\sum_{j=1}^{N-2M'}\theta\left(\frac{\Lambda^n_\alpha - \sin k_j}{nu}\right) = 2\pi J^n_\alpha + \sum_{m=1}^{\infty}\sum_{\beta=1}^{M_m}\Theta_{nm}\left(\frac{\Lambda^n_\alpha - \Lambda^m_\beta}{u}\right),\\
&2L\,\mathrm{Re}\left[\arcsin\left(\Lambda'^n_\alpha + niu\right)\right] = 2\pi J'^n_\alpha + \sum_{j=1}^{N-2M'}\theta\left(\frac{\Lambda'^n_\alpha - \sin k_j}{nu}\right) + \sum_{m=1}^{\infty}\sum_{\beta=1}^{M'_m}\Theta_{nm}\left(\frac{\Lambda^n_\alpha - \Lambda'^m_\beta}{u}\right),
\end{aligned}
\tag{5}
$$

where $\theta(x) = 2\arctan(x)$ and Θ_{nm} is defined as

$$
\Theta_{nm}(x) = \begin{cases} \theta\left(\frac{x}{|n-m|}\right) + 2\theta\left(\frac{x}{|n-m|+2}\right) + \cdots + 2\theta\left(\frac{x}{n+m-2}\right) + \theta\left(\frac{x}{n+m}\right), & \text{if } n \neq m \\ 2\theta\left(\frac{x}{2}\right) + 2\theta\left(\frac{x}{4}\right) + \cdots + 2\theta\left(\frac{x}{2n-2}\right) + \theta\left(\frac{x}{2n}\right), & \text{if } n = m \end{cases} .
\tag{6}
$$

The counting numbers $I_j, J^n_\alpha, J'^n_\alpha$ are integer or half-odd integers, which rely on the odevity of string number,

$$
I_j \text{ is } \begin{cases} \text{integer} & \text{if } \sum_m \left(M_m + M'_m\right) \text{ is even} \\ \text{half-odd integer} & \text{if } \sum_m \left(M_m + M'_m\right) \text{ is odd} \end{cases},
\tag{7}
$$

$$
J^n_\alpha \text{ is } \begin{cases} \text{integer} & \text{if } N - M_n \text{ is odd} \\ \text{half-odd integer} & \text{if } N - M_n \text{ is even,} \end{cases},
\tag{8}
$$

$$
J'^n_\alpha \text{ is } \begin{cases} \text{integer} & \text{if } L - N + M'_n \text{ is odd} \\ \text{half-odd integer} & \text{if } L - N + M'_n \text{ is even} \end{cases}.
\tag{9}
$$

Luo, Pu, Guan, 51 pages, arXiv: 2307.00890

Thermodynamics Bethe ansatz equations

- quantum many body systems
- microscopic state energy E_i
- partition function

  $Z = \sum_{i=1}^{\infty} W_i \mathrm{e}_i^{-E_i/(k_B T)}$
- free energy $F = -k_B T ln Z$
- challenge: finding new physics

Equation of state

$$f = -T\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\ln\left(1+\mathrm{e}^{-\frac{\kappa(k)}{T}}\right)+u$$
$$-T\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\Lambda}{\pi}\,\mathrm{Re}\,\frac{1}{\sqrt{1-(\Lambda-inu)^2}}\ln\left(1+\mathrm{e}^{-\frac{\varepsilon_n'(\Lambda)}{T}}\right)$$

$$\kappa(k) = -2\cos k-\mu-2u-B+\sum_{n=1}^{\infty}\int d\Lambda a_n(\sin k-\Lambda)T\ln\left(1+e^{-\frac{\varepsilon_n'(\Lambda)}{T}}\right)$$
$$-\sum_{n=1}^{\infty}\int \mathrm{d}\Lambda a_n(\sin k-\Lambda)\ln\left(1+e^{-\frac{\varepsilon_n(\Lambda)}{T}}\right)$$

Charge particle dispersion

Real k

$$\varepsilon_n(\Lambda) = 2nB-\int dk\cos k a_n(\sin k-\Lambda)T\ln\left(1+e^{-\frac{\kappa(k)}{T}}\right)+\sum_{m=1}^{\infty}A_{nm}*T\ln\left(1+e^{-\frac{\varepsilon_m(\Lambda)}{T}}\right)$$

Spin wave bound states

Length-n spin strings

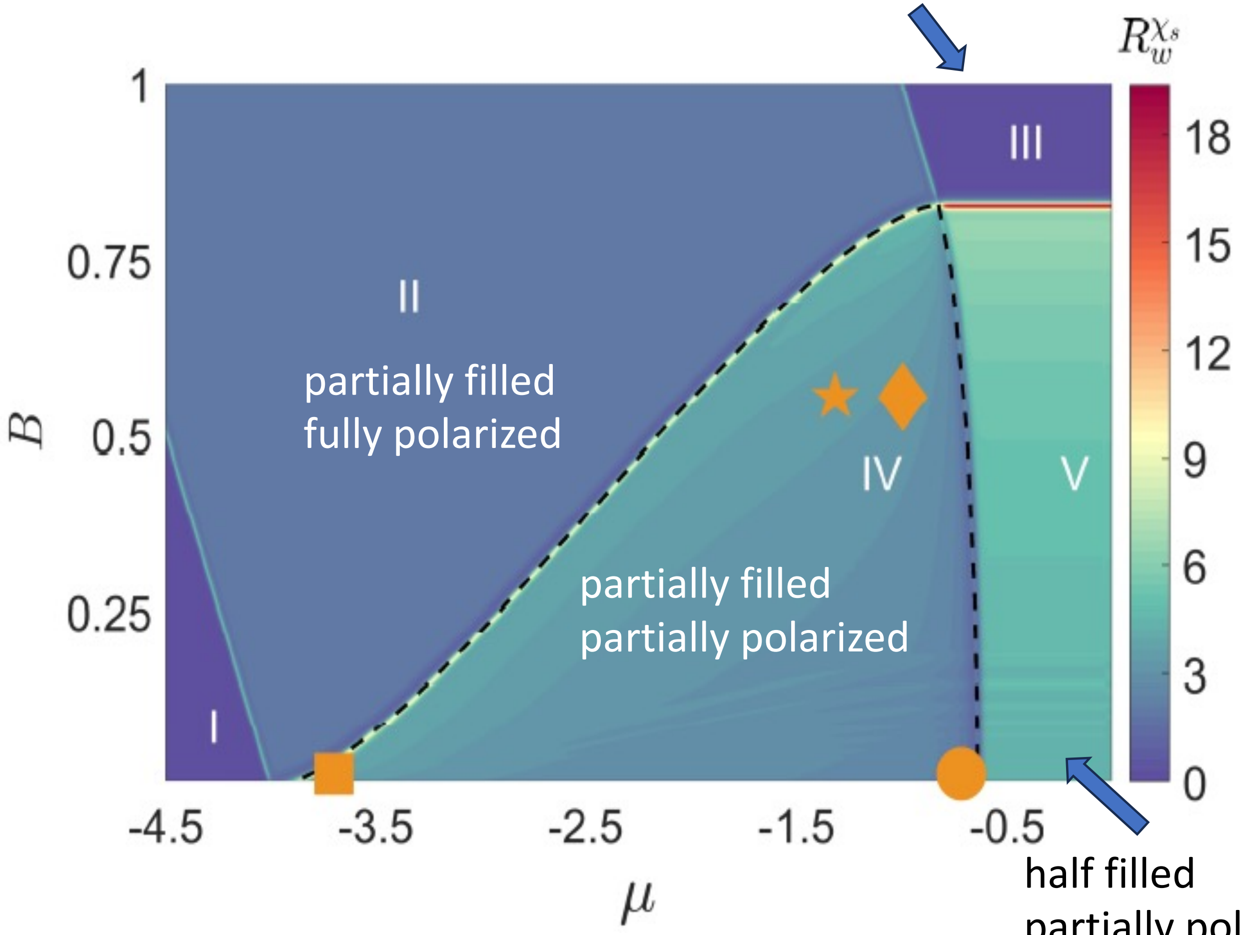
$$\varepsilon_n'(\Lambda) = 4Re\sqrt{1-(\Lambda-inu)^2}-2n\mu-4nu-\int dk\cos k a_n(\sin k-\Lambda)T\ln\left(1+e^{-\frac{\kappa(k)}{T}}\right)$$
$$+\sum_{m=1}^{\infty}A_{nm}*T\ln\left(1+e^{-\frac{\varepsilon_m'(\Lambda)}{T}}\right)$$

Charge particle bound states

Length-n electron BS

M. Takahashi One-dimensional Hubbard model at finite temperature, Progress of Theoretical Physics, 1972, 47(1): 69-82.

Wilson ratio maps out T=0 phase diagram



half filled
fully polarized

half filled
partially polarized

Wilson ratio: $R_w^{\chi_s} = \frac{4}{3}\left(\frac{\pi k_B}{\mu_B g}\right)^2 \frac{\chi_s}{C_v/T}$

χ -- susceptibility
c_v -- specific heat
T -- temperature

For Luttinger liquid phases at T=0

II: $R_w^{\chi_s} \approx 2$

IV: $R_w^{\chi_s} \approx 4(v_c K_s + v_s K_{sc})/(v_s + v_c)$

V: $R_w^{\chi_s} \approx 8k_s$

New result

I, III: $R_w^{\chi_s} \approx 0$

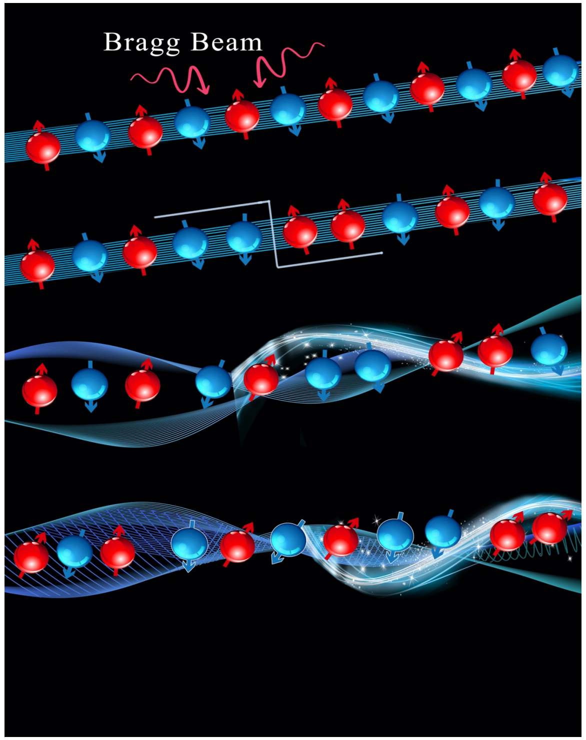
$K_{c,s}$ -- charge & spin Luttinger parameter

$v_{c,s}$ -- charge and spin velocities

Luo, Pu, Guan, PRB **107**, L201103 (2023)
Luo, Pu, Guan, arXiv: 2307.00890

Microscopic origin of the Spin-charge separation

$$H_0 = -\sum_{j=1}^{L} \sum_{a=\uparrow\downarrow} \left(c_{j,a}^{+} c_{j+1,a} + c_{j+1,a}^{+} c_{j,a} \right) + u \sum_{j=1}^{L} (1 - 2n_{j\uparrow})(1 - 2n_{j\downarrow})$$



The spin-charge separation involves the elementary excitations of 1D interacting fermions that dramatically decompose into the two collective motions of bosons: one solely carries charge, another solely carries spin.

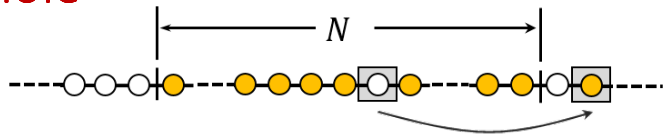
Recati et. al. PRL 90, 020401 (2003)
Hart, et al. Nature 565, 56 (2019)
Vijayan, et al. Science, 367, 186 (2020)
He, Jiang, Lin, Hulet, Pu, Guan, Phys. Rev. Lett. 125, 190401 (2020)

See: Giamarchi, 《Many-Body Physics in one dimension》

Elemental Fractional Excitations at ●

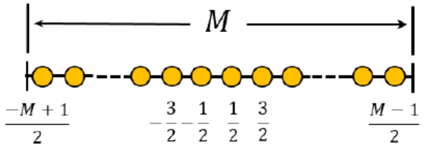
Particle-hole

Charge

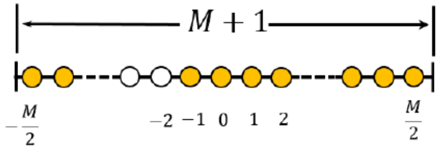


Fractional spinons

Length-1 Λ string
(Ground state)



Length-1 Λ string
(Two spinons)



Two fractional spinons:

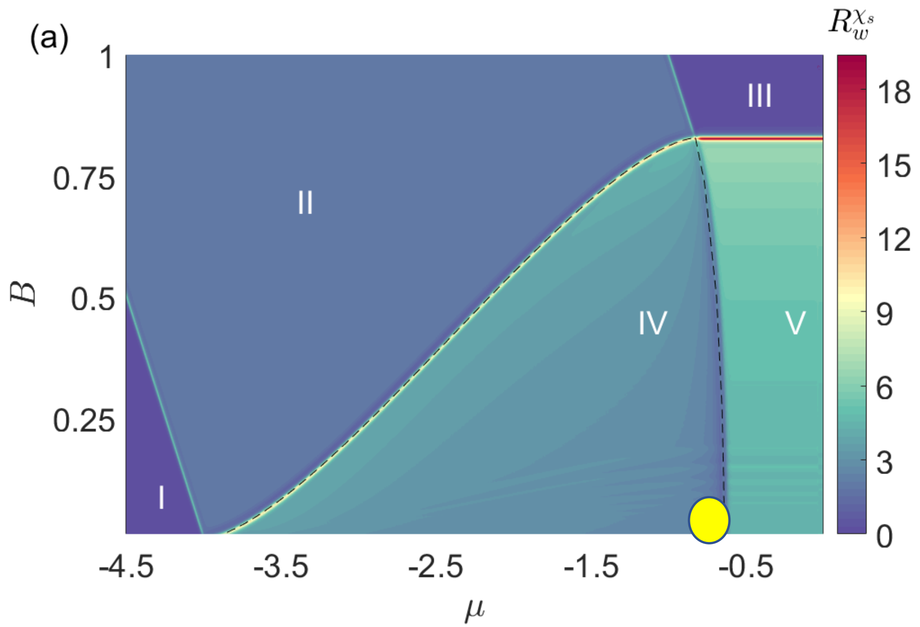
$\Delta S^{z}=(\mathrm{N}-2\mathrm{M})/2=1$

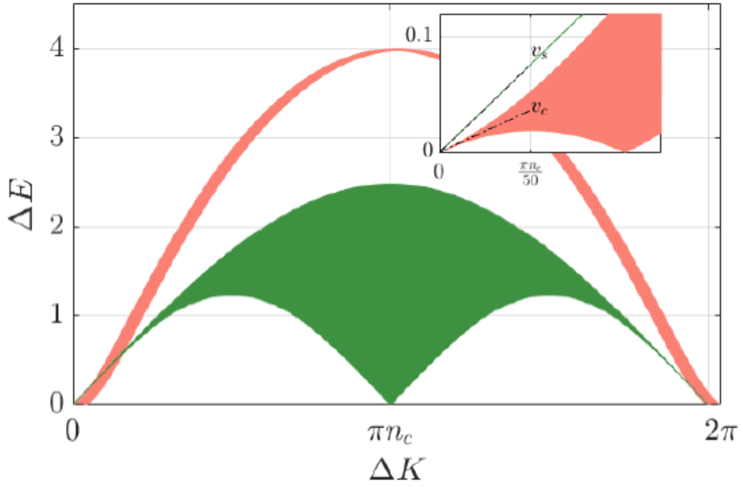
Fractional charge holons:

$\Delta \eta^{z}=(\mathrm{N}-\mathrm{L})/2=0$

Out of TLL

The system does not exist charge incoherent liquid!



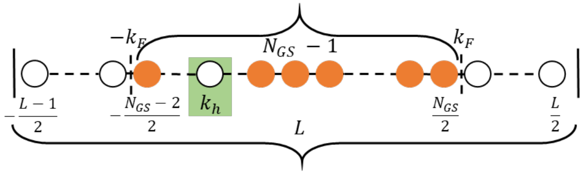


Fractional Excitations

Removing a spin-down fermions

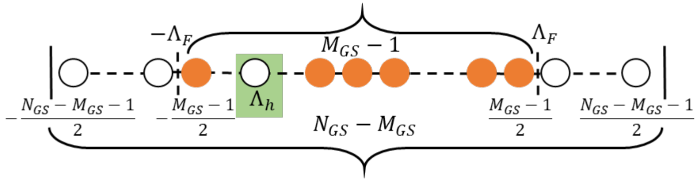
Fractional holon

Charge: excited
(one holon)

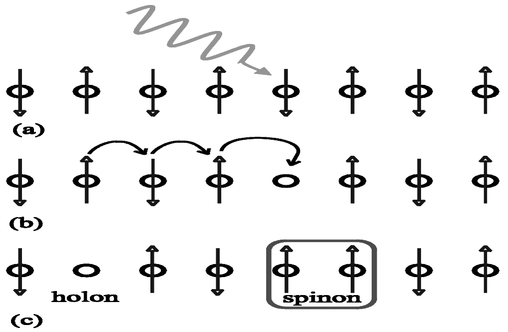


Fractional spinion

Spin: excited
(one spinon)

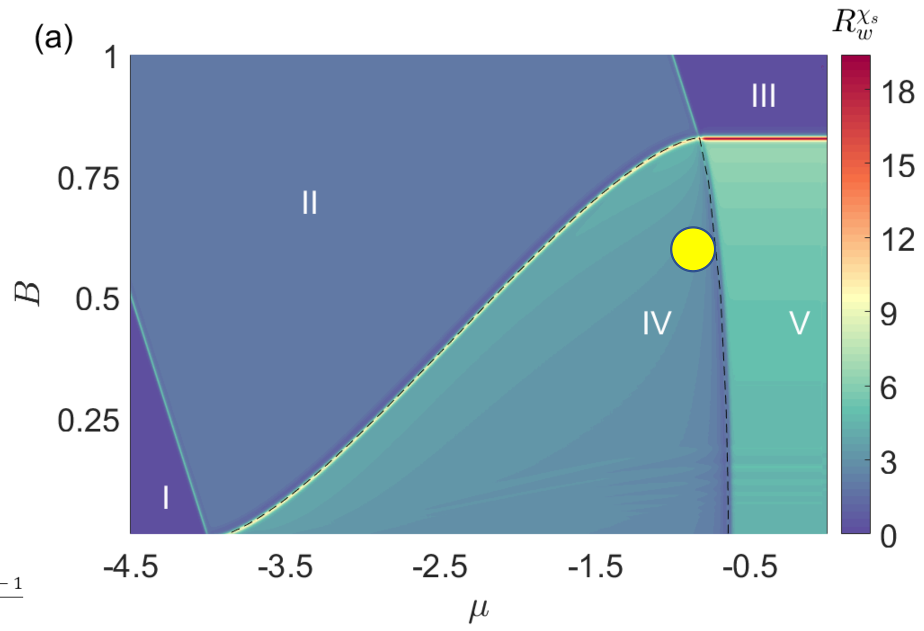


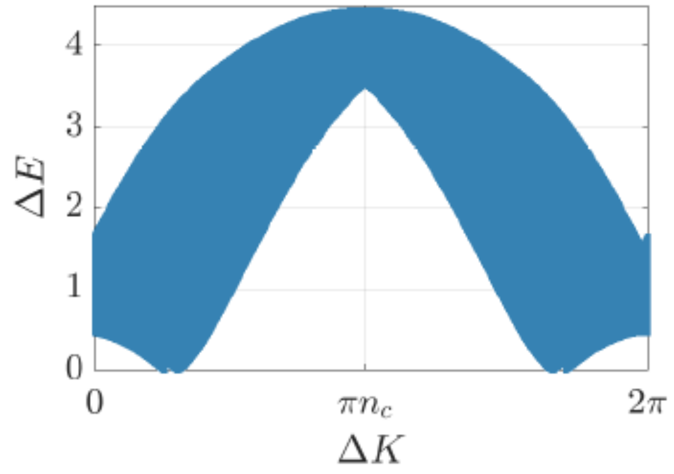
holon-spinon: $(\Delta\eta^z, \Delta S^z) = (-\frac{1}{2}, \frac{1}{2})$



Out of the TLL

Spin-charge scattering
Rather than Separation!

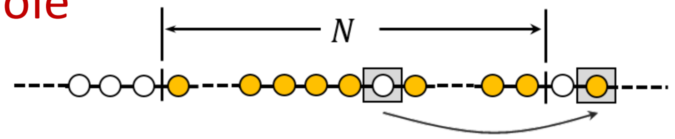




Elemental Fractional Excitations

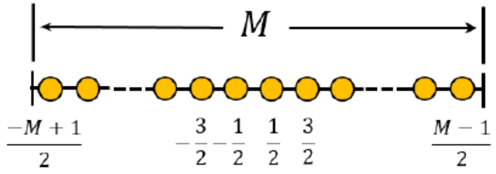
Particle-hole

Charge

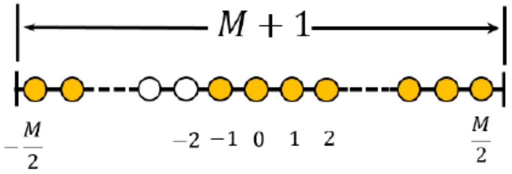


Fractional spinons

Length-1 Λ string
(Ground state)



Length-1 Λ string
(Two spinons)



New result

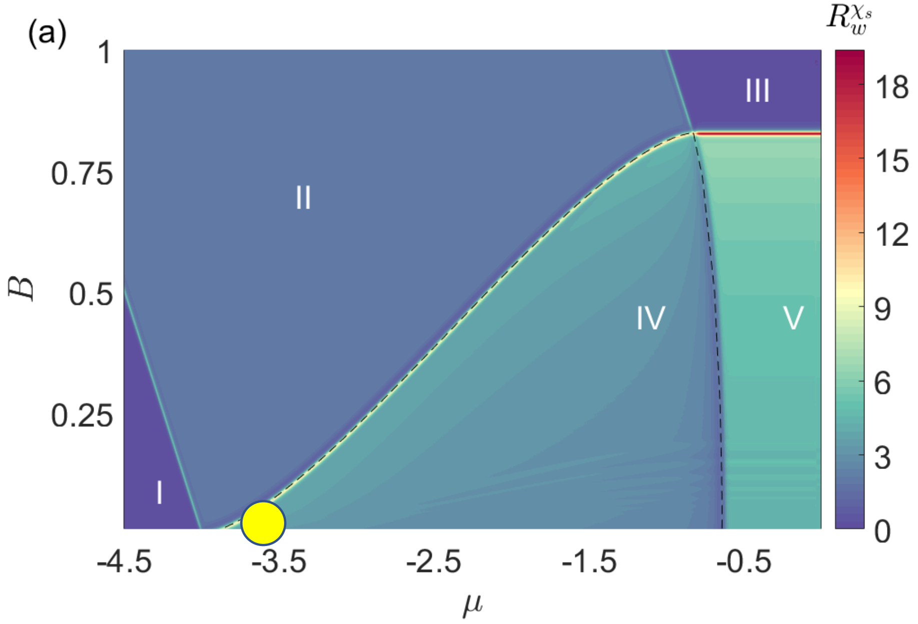
$K_BT \ll E_{spin} \ll E_{charge}$

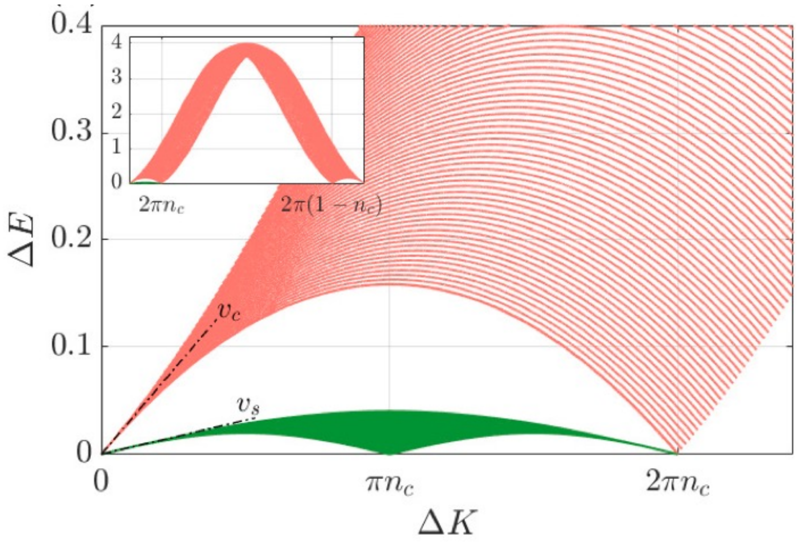
Two fractional spinons: ΔS^z=1

Paticle-hole excitations: $\Delta\eta^z = 0$

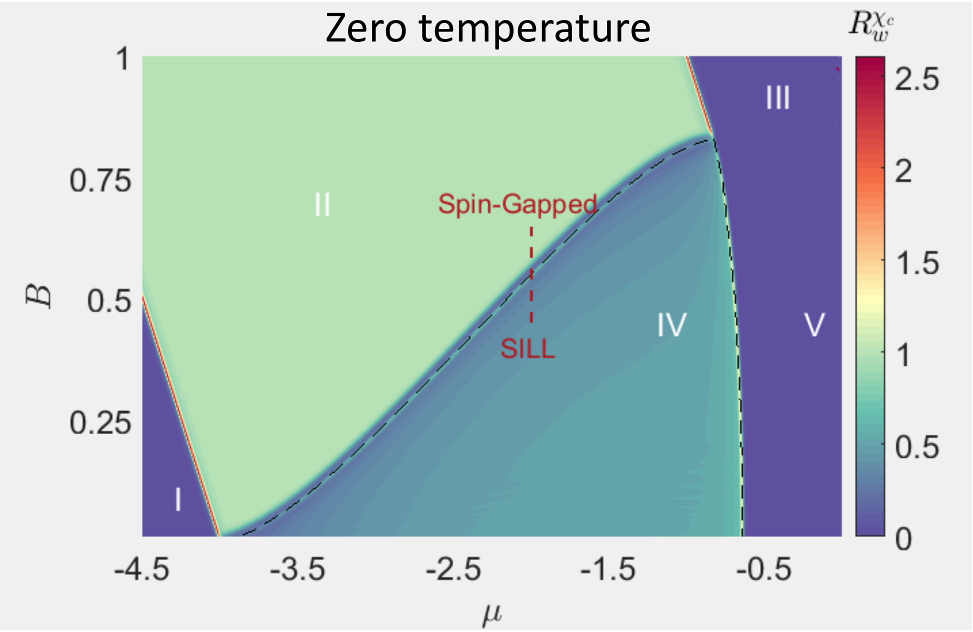
Spin incoherent liquid condition:

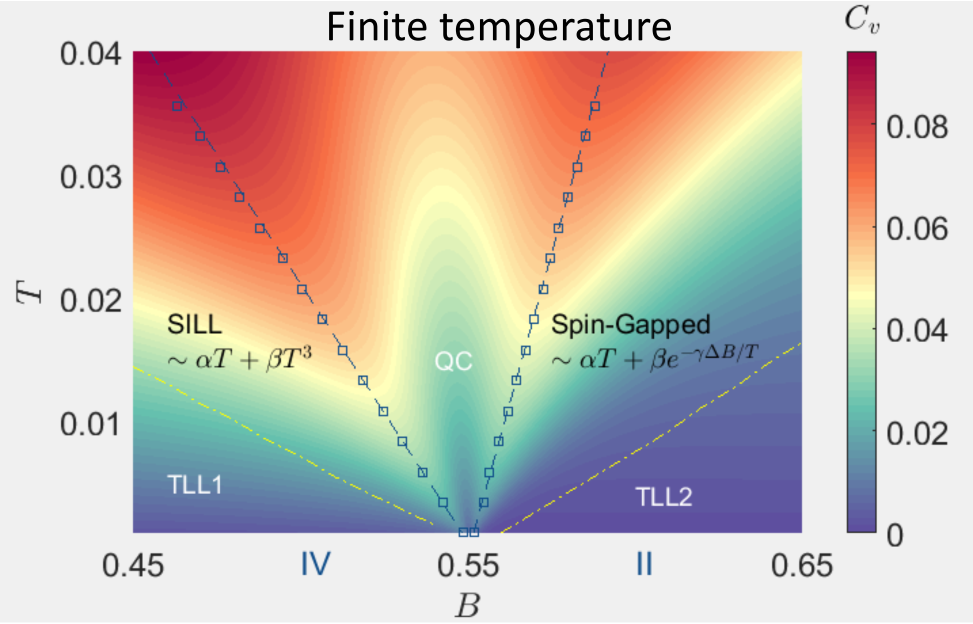
$E_{spin} \ll K_BT \ll E_{charge}$





Finite temperature: spin-coherent and spin–incoherent Luttinger liquids





QC — Quantum criticality $\boldsymbol{|B - B_c| \ll K_B T}$

$$\frac{C_v}{T} = C_v^0 + T^{\frac{d}{z}+1-\frac{2}{vz}} K\left(\frac{\mu - \mu_c}{T^{1/vz}}\right) \qquad z = 2, v = 1/2$$

TLL—Tomonaga-Luttinger liquid $\boldsymbol{K_B T \ll E_{spin} \ll E_{charge}}$

$$H_v = \int dx \left(\frac{\pi v_v K_v}{2} \Pi_v^2 + \frac{v_v}{2\pi K_v} (\partial_x \phi_v)^2\right), v = c, s$$

$$f = f_0 - \frac{\pi T^2}{6}\left(\frac{1}{v_c} + \frac{1}{v_s}\right) \quad \text{phase IV}$$

$$f = f_0 - \frac{\pi T^2}{6}\frac{1}{v_c} \quad \text{phase II}$$

$$f = f_0 - \frac{\pi T^2}{6}\frac{1}{v_s} \quad \text{phase V}$$

Spin incoherent liquid in 1D Hubbard model

Distinguishing TLL and SILL:

$$G^{\uparrow} = \left\langle \phi_{\uparrow}(x,t)\phi_{\uparrow}^{\dagger}(0,0) \right\rangle$$

$$G^{p} = \left\langle \phi_{\downarrow}(x,t)\phi_{\uparrow}(x,t)\phi_{\uparrow}^{\dagger}(0,0)\phi_{\downarrow}^{\dagger}(0,0) \right\rangle$$

SILL Conditions

$$|x \pm \mathrm{i}v_c t| \ll v_c/T$$

$$|x \pm \mathrm{i}v_s t| \gg v_s/T$$

$$T \sim E_s \sim J \sim \left(k_{F\uparrow} + k_{F\downarrow}\right)/2 \cdot v_s \equiv k_F v_s$$

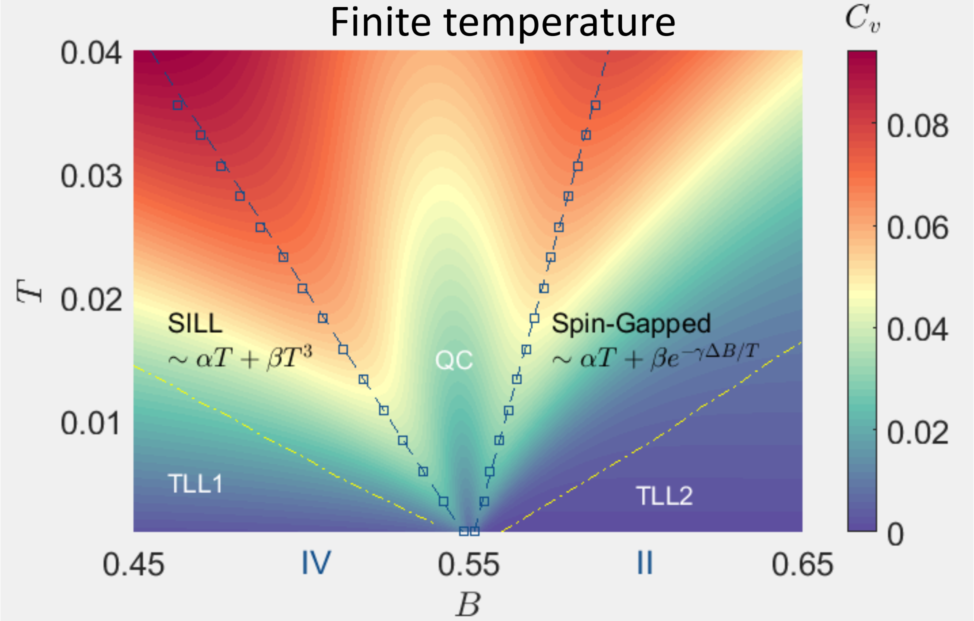
Near B_c from conformal field theory

$$\langle \phi(x,t)\phi(0,0)\rangle_0 = \sum A(D_c, D_s, N_c^{\pm}, N_s^{\pm}) \frac{\exp(-2iD_c k_{F,\uparrow}x)\exp(-2i(D_c+D_s)k_{F,\downarrow}x)}{(x-iv_ct)^{2\Delta_c^+}(x+iv_ct)^{2\Delta_c^-}(x-iv_st)^{2\Delta_s^+}(x+iv_st)^{2\Delta_s^-}}$$

$$\begin{aligned}\langle \phi(x,t)\phi(0,0)\rangle_T = &\sum A(D_c, D_s, N_c^{\pm}, N_s^{\pm})\exp(-2iD_c k_{F,\uparrow}x)\exp(-2i(D_c+D_s)k_{F,\downarrow}x) \\ &\times \left(\frac{\pi T}{v_c \sinh(\pi T(x-iv_ct)/v_c)}\right)^{2\Delta_c^+} \left(\frac{\pi T}{v_c \sinh(\pi T(x+iv_ct)/v_c)}\right)^{2\Delta_c^-} \\ &\times \left(\frac{\pi T}{v_s \sinh(\pi T(x-iv_st)/v_s)}\right)^{2\Delta_s^+} \left(\frac{\pi T}{v_s \sinh(\pi T(x+iv_st)/v_s)}\right)^{2\Delta_s^-}\end{aligned}$$

Essler, Frahm, Göhman, Klümper and Korepin, the one-dimensional Hubbard model, Cambridge University Press, 2010

Finite temperature: spin-coherent and spin–incoherent Luttinger liquids



SILL — Spin incoherent TLL

$$\boldsymbol{E_{spin} \ll K_B T \ll E_{charge}}$$

$$C_v = \frac{\pi T}{3}\left(\frac{1}{v_c}+\frac{1}{v_s}\right)+\frac{7\pi^3 T^3}{40 v_s\left(-\varepsilon_1(0)\right)^2}+O(T^4)$$

$$G_\sigma(x,t) = \langle \psi_\sigma(x,t)\psi_\sigma^\dagger(0,0)\rangle$$

$$G^{\uparrow}_{B\to B_c} \approx e^{-ik_{F,\uparrow}x} C_\uparrow^-(x - iv_c t)\langle S_R^+(x,t)S_R(0,0)\rangle + h.c.$$

$$\langle S_R^+(x,t)S_R(0,0)\rangle \sim (2\pi\alpha k_F)^{\frac{1}{2}-\frac{1}{\pi}\sqrt{1-\frac{B}{B_c}}}\, e^{-\pi\alpha\left(\frac{1}{2}-\frac{1}{\pi}\sqrt{1-\frac{B}{B_c}}\right)k_F x}$$

$$C_\uparrow^-(Z) = \frac{const}{Z^{2\Delta_c^-}}$$

$$2\Delta_c^- = 1 - \frac{2}{\pi}\sqrt{1-\frac{B}{B_c}}$$

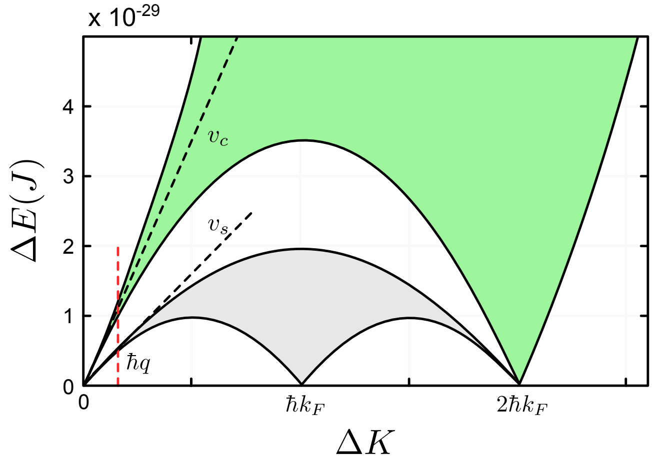
Spin-Charge Separation: Spin coherent liquid

Particle-hole spectrum (green) $q = \hbar\Delta K$

$$\omega_{\pm} = v_c q \pm \frac{1}{2m^*} q^2 \qquad \frac{m}{m^*} = \frac{\varepsilon_c''(k_0)}{2(2\pi\rho_c(k_0)^2} - \frac{\pi\rho_c'(k_0)\varepsilon_c'(k_0)}{(2\pi\rho_c(k_0))^3}$$

Two-spinons spectrum (grey):

$$\omega_{s+}(q) = v_s|q| - \frac{v_s q^3}{2k_s^2} + \cdots \qquad \omega_{s-}(q) = v_s|q| - \frac{2v_s q^3}{k_s^2} + \cdots$$



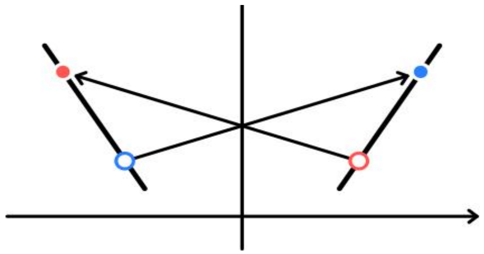
Yang-Gaudin model: pin and change excitations

Effective Field Theory: separated spin and charge TLLs

Charge: $$H_v = \frac{1}{2\pi}\int dx \left[u_v K_v \left(\pi\Pi_v(x)\right)^2 + \frac{u_v}{K_v}(\nabla\varphi_v(x)){}^\wedge 2\right], v = c$$

Spin: $$H_\sigma = \frac{1}{2\pi}\int dx \left[u_\sigma K_\sigma \left(\pi\Pi_\sigma(x)\right)^2 + \frac{u_\sigma}{K_\sigma}(\nabla\varphi_\sigma(x)){}^\wedge 2\right]$$

Backward scattering $$H_g = \frac{2g_1}{(2\pi\alpha)^2}\int dx\cos\left(\sqrt{8}\varphi_\sigma\right)$$



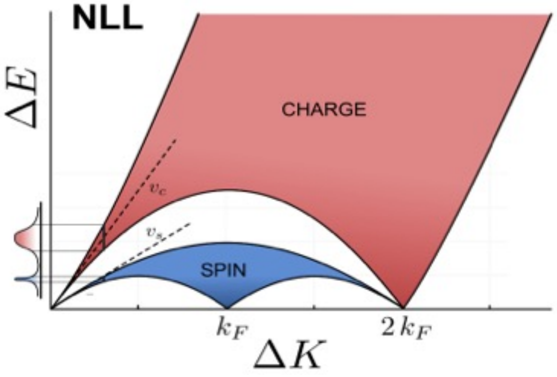
Spin backward scattering

He, Jiang, Lin, Hulet, Pu, Guan, PRL 125, 190401 (2020)

DSFs of Spin and Charge: TLL and nonlinear TLL

Charge and spin dynamic structure factor for Liner Luttinger liquid

$$S^{\rho\rho}(q,\omega) = \frac{qK_\rho}{2\pi^2}\delta(\omega - u_v q), \qquad S^{\sigma\sigma}(q,\omega) = \frac{qK_\sigma}{2\pi^2}\delta(\omega - v_\sigma q)$$



Charge DSF with band curvature correction

(Interaction effect: $\mathrm{k_F}$ *is replaced by* $k_c = m^* v_c/\hbar$)

$$S^{\rho\rho}(q,\omega) = \frac{Im\,\chi(q,\omega,k_F,T,N)}{\pi(1-e^{-\beta\hbar w})}, Im\,\chi(q,\omega,k_F,T,N) = \frac{Nm^*}{2\hbar^2 q k_F}\pi\left(n_{q-} - n_{q+}\right)$$

$$q_\pm = \frac{\omega\, m^*}{\hbar q} \pm \frac{q}{2}, n_q = \frac{1}{e^{\beta(\epsilon-\mu)}+1}, \epsilon = \frac{\hbar^2 q^2}{2m^*},$$

$$\frac{m}{m^*} = \frac{\varepsilon_c''(k_0)}{2(2\pi\rho_c(k_0)^2} - \frac{\pi\rho_c'(k_0)\varepsilon_c'(k_0)}{(2\pi\rho_c(k_0))^3}$$

Spin DSF with backward scattering: (perturbation from $-\pi v_s g(J_L^+ J_R^- + \mathrm{H.c.})$)

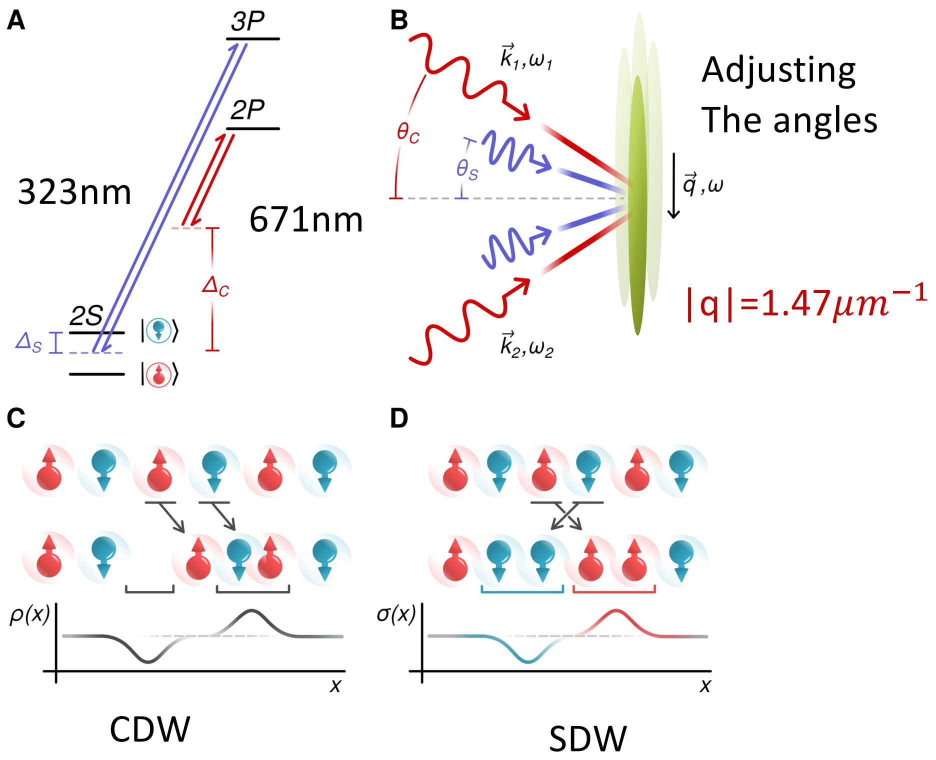
$$Im\,\chi = \frac{2K_s\tau_s(T)}{\pi[\tau_s(T)(\omega - v_s q)]^2 + \pi} \qquad Im\sum_q = -\frac{1}{\tau_s(T)} = -\frac{\pi}{2}[g(T)]^2 k_B T \qquad g(T) \approx \frac{g}{1 + g\,ln(T_F/T)}$$

Pereira, Sela , et. al., Phys. Rev. B 82, 115324 (2010)

He, et. al., Phys. Rev. Lett. 125, 190401 (2020)

Two-photon Bragg Spectroscopy:

^{6}Li Ultracold atoms



Momentum transfer (spin state detuning Δ_σ)

$$P(q,\omega) \propto \left(\frac{1}{\Delta_\uparrow^2}+\frac{1}{\Delta_\downarrow^2}\right) S_{\uparrow\uparrow} + \frac{2}{\Delta_\uparrow\Delta_\downarrow} S_{\uparrow\downarrow}$$

Charge excitation S_C: $\Delta_\uparrow \approx \Delta_\downarrow \gg \Delta_{\uparrow\downarrow}$

Spin excitation S_S : $\Delta_\uparrow = -\Delta_\downarrow = |\Delta_{\uparrow\downarrow}|/2$

$$S_{C,S}(q,\omega) \equiv 2\left[S_{\uparrow\uparrow}(q,\omega) \pm S_{\uparrow\downarrow}(q,\omega)\right]$$

$$S_{\sigma\sigma'}(q,\omega) = \frac{1}{2\pi}\int \mathrm{d}z \int \mathrm{d}t\, \mathrm{e}^{-i(q\cdot z-\omega t)} \langle\rho_\sigma(x,t)\rho_{\sigma'}(0,0)\rangle$$

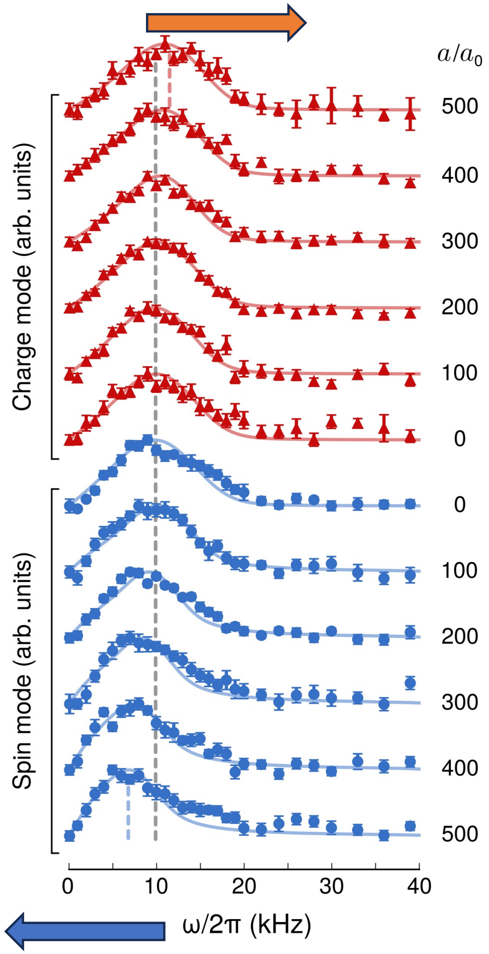
Using $|1> and\ |3>$ states and narrow $2S-3P_{3/2}$ (UV) transition to reduce the rate of spontaneous emission in spin excitation; $|1> and\ |2>$ energy levels are used for charge excitations.

Sign of the light shift potential:

- Symmetrical light shift for charge density wave
- Asymmetrical light shift for spin density wave

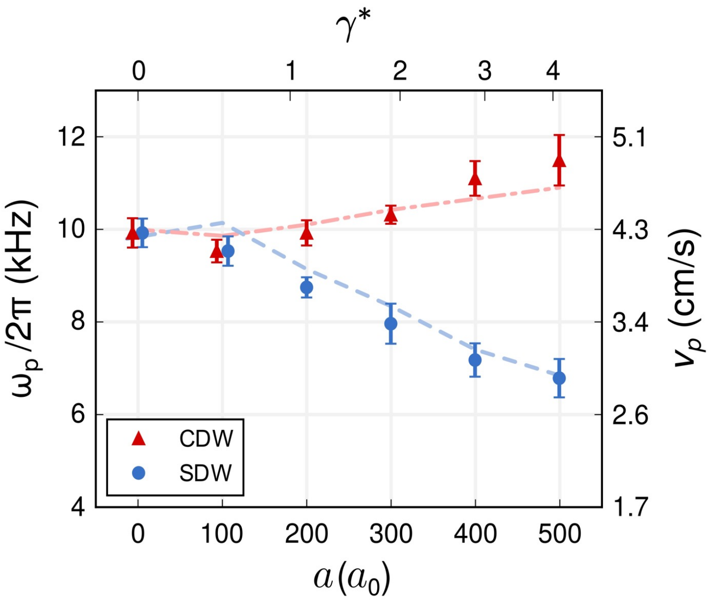
Senaratne, et. al., Pu, Guan*, Hulet*, Science 376, 1305 (2022) **He, Jiang Lin*, Pu, Guan*, Phys. Rev. Lett. 125, 190401 (2020)**

Observation of Spin-coherent liquid: Spin-charge separation



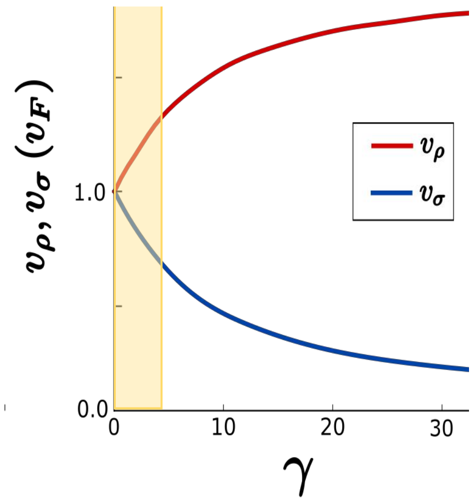
Charge(red) and spin(blue) dynamical structure factors

Encoding Nonlinear TLL Effect



Peak frequencies and velocities

$$v_p = \omega\ _p/q$$

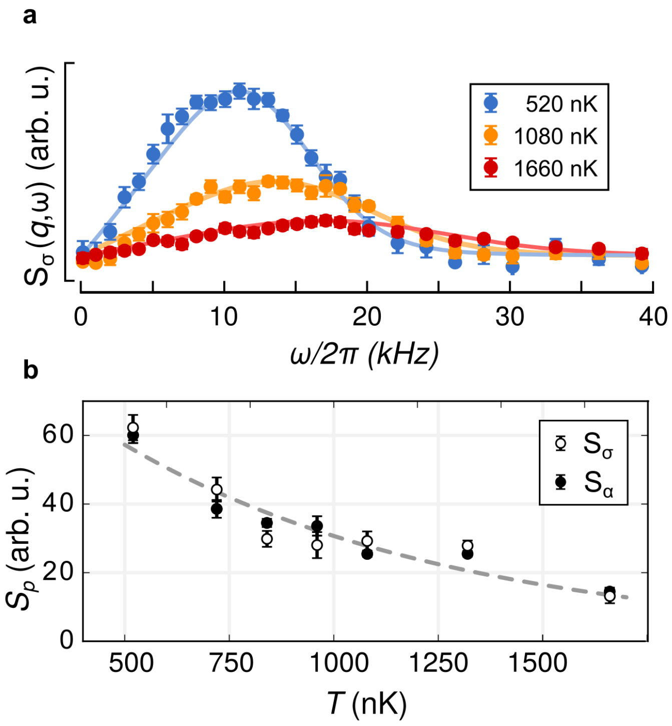


Velocities of spin and charge shift in opposite directions!

Senaratne, et. al., Pu, Guan*, Hulet*, Science 376, 1305 (2022)

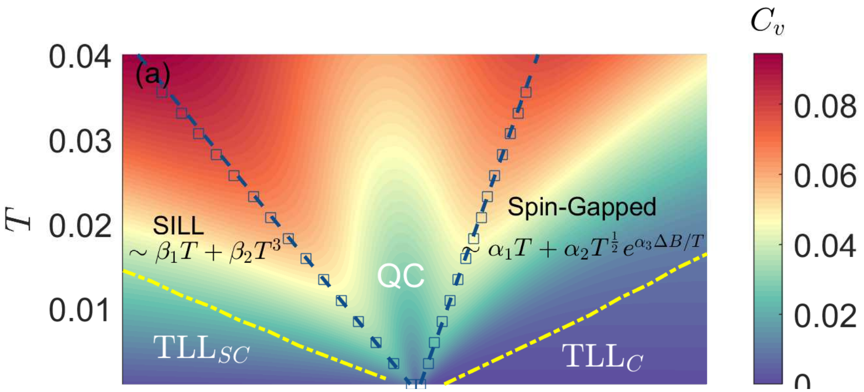
Guan, Batchelor, Lee, Rev. Mod. Phys. 85, 163 (2013)

Evidence for spin-Incoherent Liquid

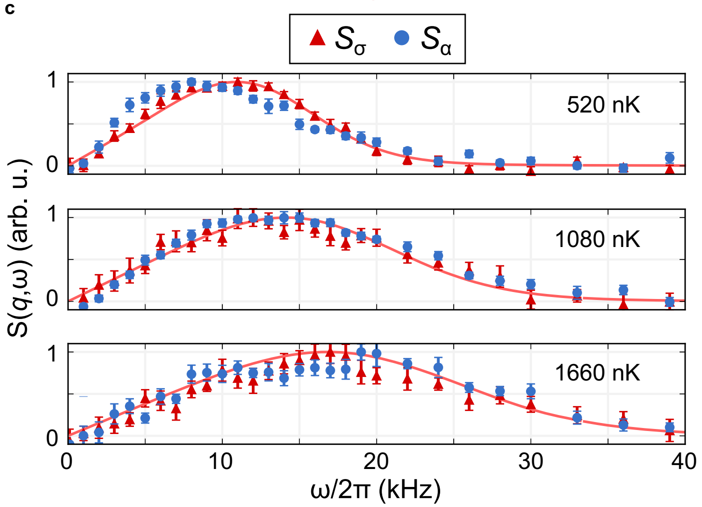


(a) Charge DSF; (b) charge and spin peak velocities
SILL shows a suppression of spin-change separation

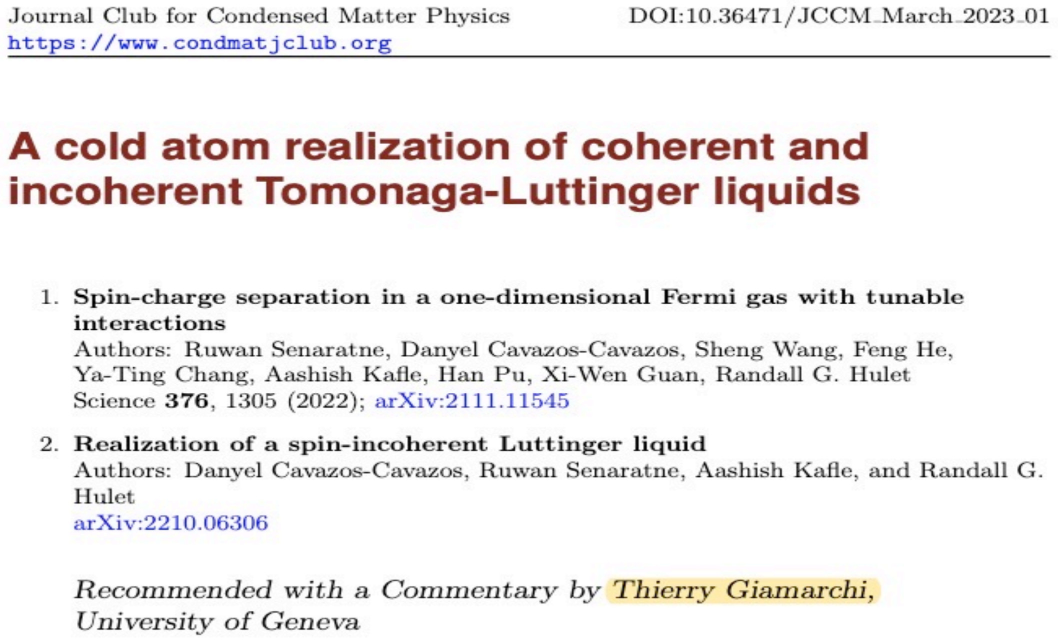
Cavazos-Cavazos et al. Nat. Comms. (2023)14:3154
He, et. al., Phys. Rev. Lett. 125, 190401 (2020)



Criticality of the Yang-Gaudin model



Peak frequencies of spin and charge



Journal Club for Condensed Matter Physics DOI:10.36471/JCCM_March_2023_01
https://www.condmatjclub.org

A cold atom realization of coherent and incoherent Tomonaga-Luttinger liquids

1. **Spin-charge separation in a one-dimensional Fermi gas with tunable interactions**
   Authors: Ruwan Senaratne, Danyel Cavazos-Cavazos, Sheng Wang, Feng He, Ya-Ting Chang, Aashish Kafle, Han Pu, Xi-Wen Guan, Randall G. Hulet
   Science **376**, 1305 (2022); arXiv:2111.11545
2. **Realization of a spin-incoherent Luttinger liquid**
   Authors: Danyel Cavazos-Cavazos, Ruwan Senaratne, Aashish Kafle, and Randall G. Hulet
   arXiv:2210.06306

Recommended with a Commentary by Thierry Giamarchi, University of Geneva

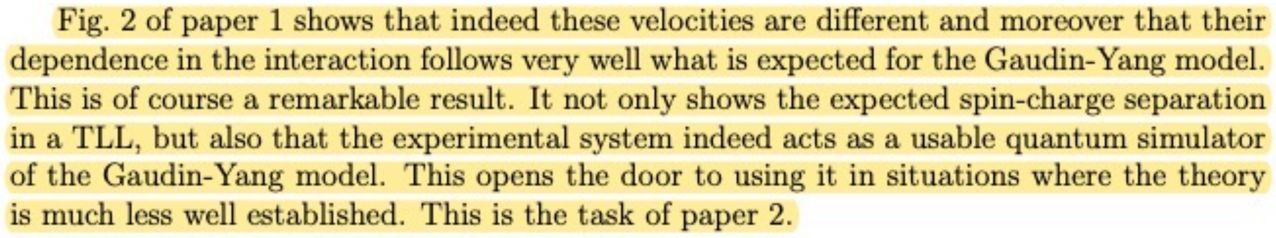
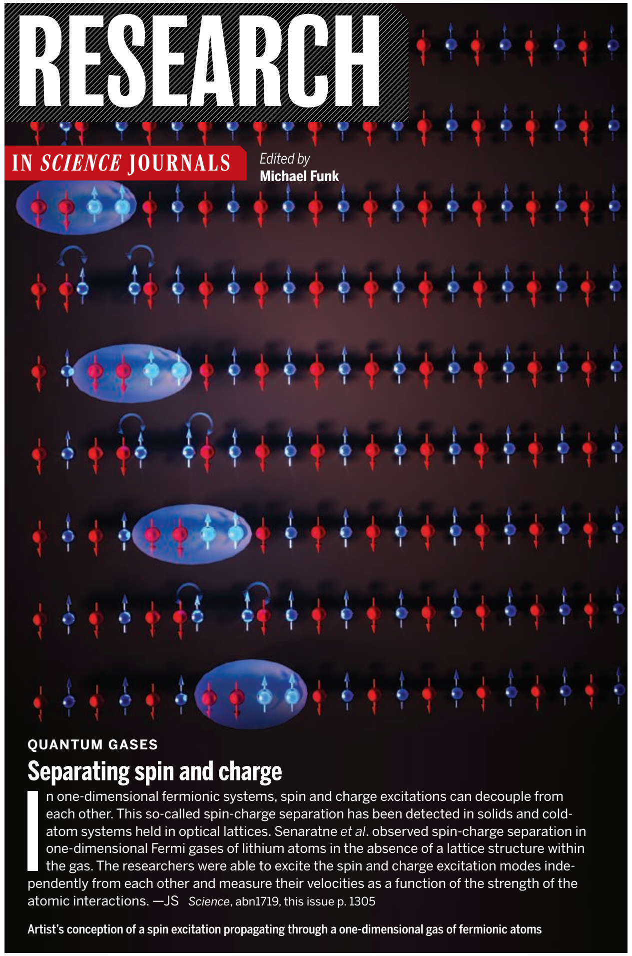


Fig. 2 of paper 1 shows that indeed these velocities are different and moreover that their dependence in the interaction follows very well what is expected for the Gaudin-Yang model. This is of course a remarkable result. It not only shows the expected spin-charge separation in a TLL, but also that the experimental system indeed acts as a usable quantum simulator of the Gaudin-Yang model. This opens the door to using it in situations where the theory is much less well established. This is the task of paper 2.

Separating Spin and Charge



QUANTUM GASES

Separating spin and charge

In one-dimensional fermionic systems, spin and charge excitations can decouple from each other. This so-called spin-charge separation has been detected in solids and cold-atom systems held in optical lattices. Senaratne *et al*. observed spin-charge separation in one-dimensional Fermi gases of lithium atoms in the absence of a lattice structure within the gas. The researchers were able to excite the spin and charge excitation modes independently from each other and measure their velocities as a function of the strength of the atomic interactions. —JS *Science*, abn1719, this issue p. 1305

Artist's conception of a spin excitation propagating through a one-dimensional gas of fermionic atoms

Notorious difficulties:

Spectral function
Dynamical structure factor
Quantum transport and nonequilibrium physics

Outline

I. 1D Hubbard model (Dynamical correlation)

Spin coherent and incoherent TLLs, critical scaling functions

II. Interaction driven criticality and Contact (Caloric effect)

Contact susceptibility and quantum cooling in a lattice

III. Quantum transport

Spin and charge Drude weights at zero and finite temperature

IV. Conclusion and discussion

To better capture the **interaction-driven** effects, we define:

New Result

Contact (interaction driven)

$$C = \frac{\partial f}{\partial u} = 4d - 2n_c + 1$$

$$d = \frac{1}{N}\sum_i \langle n_{i,\uparrow} n_{i,\downarrow} \rangle$$ double occupancy

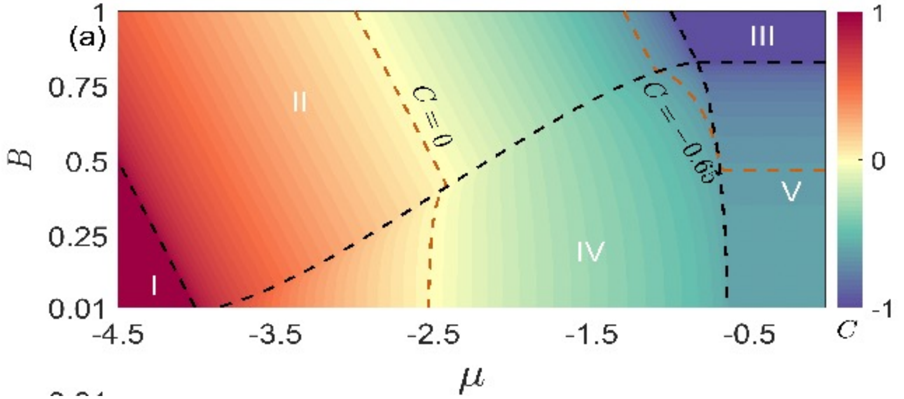
Contact Susceptibilities

$$f = e - \mu n_c - 2Bm - Ts - uC$$

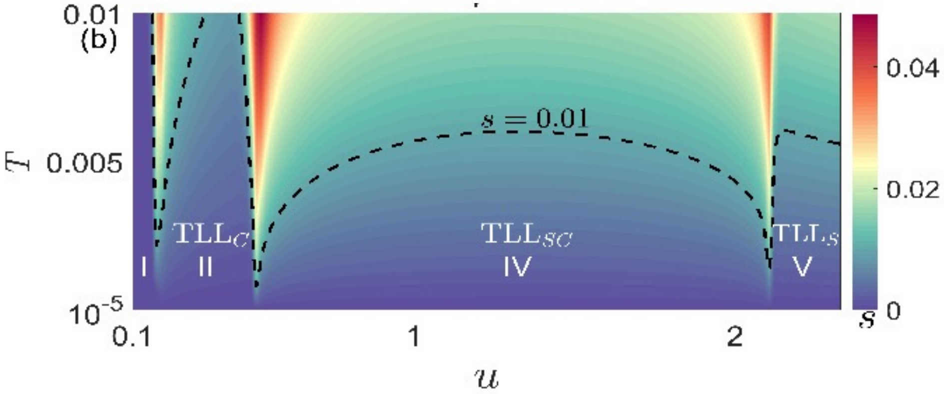
Maxwell relations

$$\frac{\partial n_c}{\partial u} = -\frac{\partial C}{\partial \mu} \qquad \frac{\partial m}{\partial u} = -\frac{\partial C}{\partial (2B)} \qquad \frac{\partial s}{\partial u} = -\frac{\partial C}{\partial T}$$

Contour plot of the Contact @ $T = 0.005$ and $u = 1$



Contour plot of the entropy @ $B = 0.15, \mu = -2.5$



- **Interaction-driven phase transitions (II-IV) and (V-IV)**

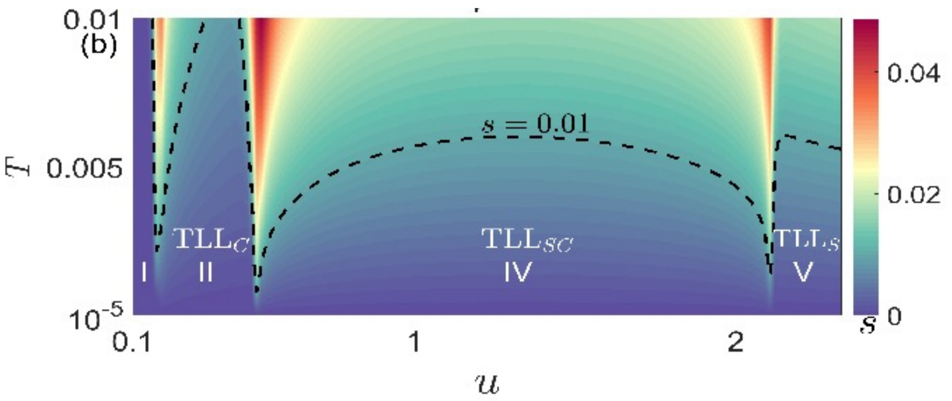
$$f = f_0 - \frac{\pi T^2}{6v_c} + T^{\frac{3}{2}}\pi^{\frac{1}{2}}\sigma_1(0)\left(\frac{\varepsilon_1''(0)}{2}\right)^{-\frac{1}{2}} \mathrm{Li}_{\frac{3}{2}}(-e^{-\frac{\varepsilon_1(0)}{T}})$$

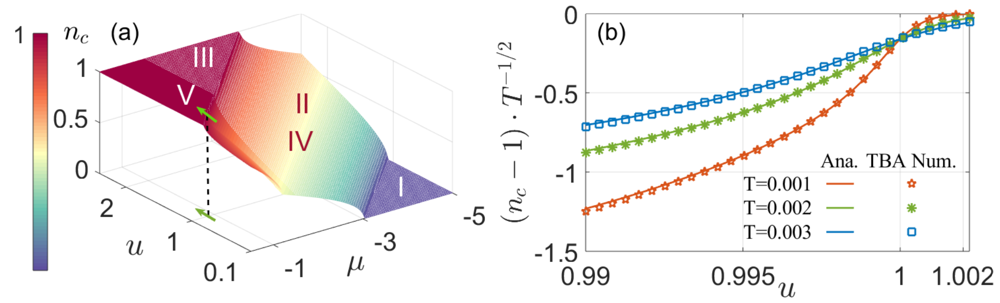
$$f = f_0 - \frac{\pi T^2}{6v_s} + T^{\frac{3}{2}}\pi^{\frac{1}{2}}\rho(\pi)\left(\frac{-\kappa''(\pi)}{2}\right)^{-\frac{1}{2}} \mathrm{Li}_{\frac{3}{2}}(-e^{\frac{\kappa(\pi)}{T}})$$

$$\boldsymbol{\varepsilon_1(0), \kappa(\pi) = \alpha_B \Delta B + \alpha_\mu \Delta\mu + \alpha_u \Delta u}$$

$$\frac{\alpha_u}{\alpha_B} = -\frac{\partial B}{\partial u}, \qquad \frac{\alpha_u}{\alpha_\mu} = -\frac{\partial \mu}{\partial u}, \qquad \frac{\alpha_B}{\alpha_\mu} = -\frac{\partial \mu}{\partial B}$$

Entropy accumulation at phase transitions!





Upper: Contour plot of the entropy in T-u plane for $B = 0.15, \mu = -2.5$, a maximum entropy at QC.
Lower: IV-V phase transition: density shows universal scaling behaviour driven by interaction.

• Contact susceptibilities and applications

$$\frac{\partial s}{\partial u} = -\frac{\partial C}{\partial T}; \quad \frac{C_v}{T}\frac{\partial T}{\partial u} = \frac{\partial C}{\partial T}$$

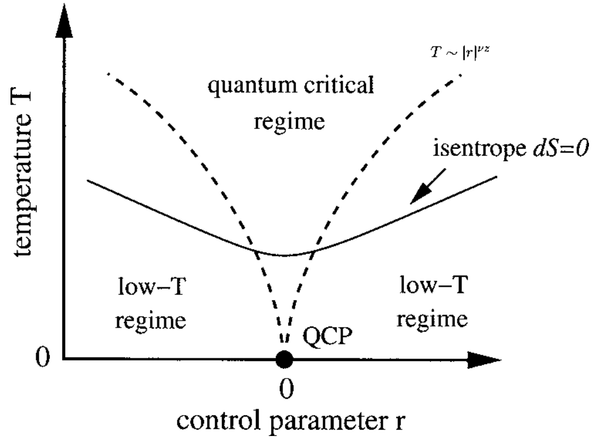
For quantum cooling

Grüneisen parameter

$$\frac{\partial C}{\partial u}\Big|_{s,N,V,H} = \frac{T}{u}\Gamma_{int}, \qquad \Gamma_{int} = \frac{\partial C}{\partial T}\frac{u}{c_v}$$

Also a large change of $\boldsymbol{\Gamma_{int}}$

$\frac{\partial s}{\partial u}, \frac{\partial C}{\partial T}$ change rapidly



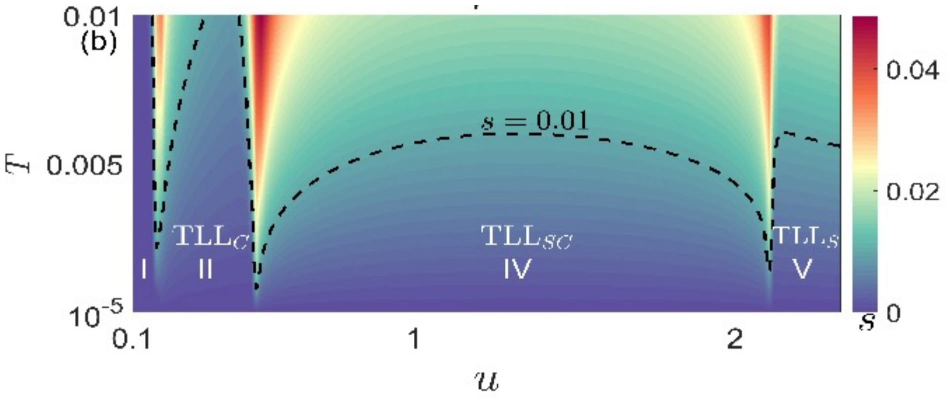
Isentrope:

$$ds = \frac{\partial s}{\partial u}\text{du} + \frac{\partial s}{\partial T}\text{dT} = 0$$

Also see Adiabatic demagnetization cooling:

Wolf et. al. PNAS, 108, 6862 (2011)

New result



Quantum Cooling

- Entropy peaks near phase boundaries.
- Isentropic process:
  maximum entropy → minimum T

A potentially novel way of cooling quantum gases in lattice!

Adiabatic interaction ramping cooling!

- **Contact susceptibilities and applications**

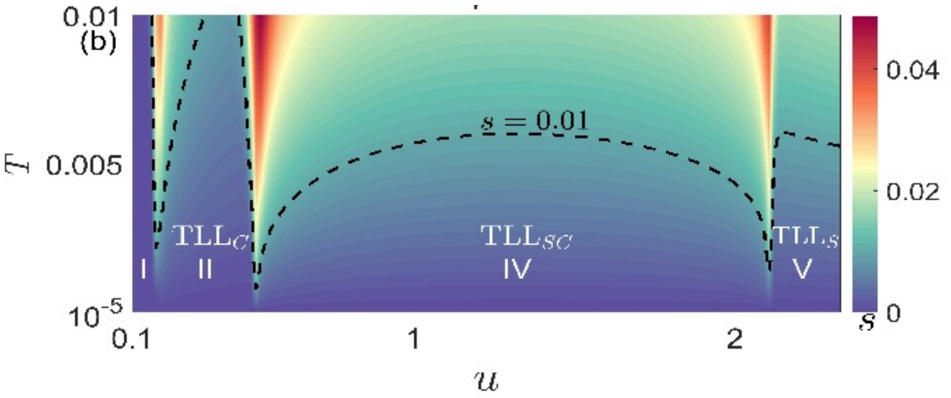
$$\frac{\partial s}{\partial u} = -\frac{\partial C}{\partial T};\quad \frac{C_v}{T}\frac{\partial T}{\partial u} = \frac{\partial C}{\partial T}$$

$$\frac{\partial C}{\partial u}|_{s,N,V,H} = \frac{T}{u}\Gamma_{int},\qquad \Gamma_{int} = \frac{\partial C}{\partial T}\frac{u}{c_v}$$

For quantum cooling

Grüneisen parameter

$\frac{\partial s}{\partial u}, \frac{\partial C}{\partial T}$ change rapidly !

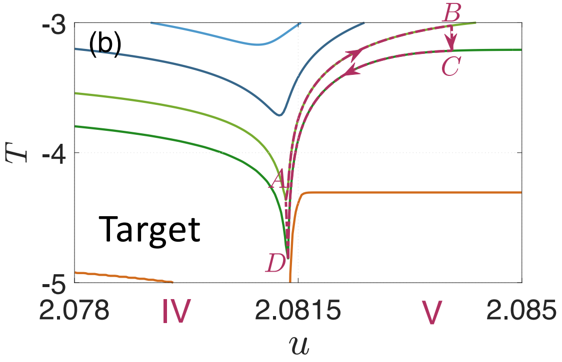


Quantum refrigeration

1) A->B: adiabatically ramp up;
2) B->C: hot isochore process;
3) C->D: adiabatically ramp down;
4) D->A: cold isochore process.

Target material T_{tar}
Substance: lattice model
Hot Ambient T_C

Ambient



Isentropic lines

Quantum Cooling

- Entropy peaks near phase boundaries.
- Isentropic process:
  maximum entropy → minimum T

A potentially novel way of cooling quantum gases in lattice!

Outline

I. 1D Hubbard model (Dynamical correlation)

Spin coherent and incoherent TLLs, critical scaling functions

II. Interaction driven criticality and Contact (Caloric effect)

Contact susceptibility and quantum cooling in a lattice

III. Quantum transport

Spin and charge Drude weights at zero and finite temperature

IV. Conclusion and discussion

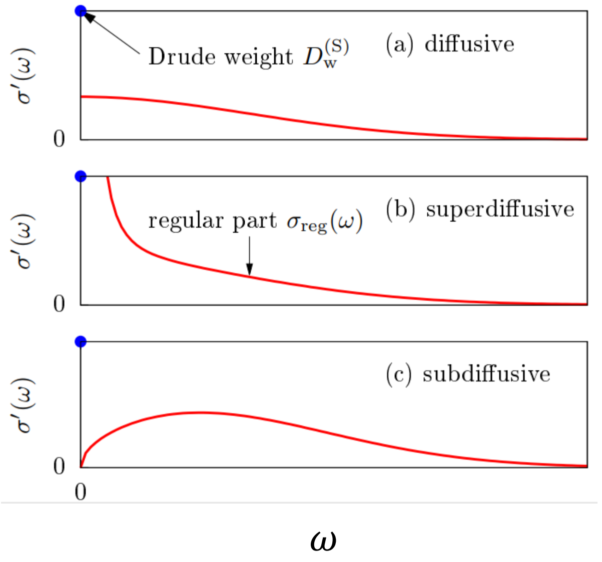
Quantum Transport

Kubo formulas for conductivities

Generalized hydrodynamics (GHD)

Bosonization , Bethe ansatz

$\boldsymbol{\sigma'(\omega \to 0) \sim |\omega|^{\alpha}}$ **Conductor:** $\boldsymbol{\alpha = -1}$



Finite $\sigma_{reg}(\omega)$

$\alpha \to 0$

Divergent $\sigma_{reg}(\omega)$

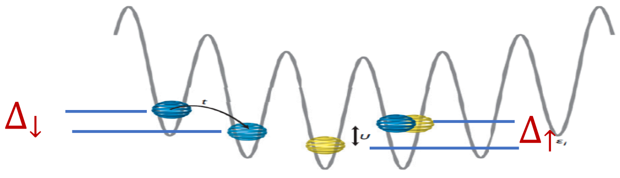
$-1 < \alpha < 0$

Vanishing $\sigma_{reg}(\omega)$

$\alpha > 0$

Drude weight can be obtained from real-time equilibrium current-current correlation function

Theoretical challenging!



$\boldsymbol{\Delta H}$

$\boldsymbol{\Delta T}$

Transport Coefficients in spin chain

$$\begin{pmatrix} \mathcal{J}^{\rm th} \\ \mathcal{J}^{s} \end{pmatrix} = \begin{pmatrix} \kappa_{\rm th} & C_s^{\rm th} \\ C_{\rm th}^{s} & \sigma_s \end{pmatrix} \begin{pmatrix} -\nabla T \\ \nabla h \end{pmatrix}$$

Thermal and Spin conductivities:

κ_{th} σ_s

$$\sigma_s'(k=0,\omega) = 2\pi D_s \delta(\omega) + \sigma_s^{\rm reg}(\omega)$$

Thermal current: $\mathcal{J}^{th} = \mathcal{J}^{E} - \mathcal{J}^{s}$

$$\boldsymbol{\lim_{t\to\infty} \frac{1}{t} \int_0^t dt' < J(t')J(0) >}$$

Bertini, et. al. Rev. Mod. Phys. **93**, 025003 (2021)
Sirker, SciPost Phys. Lect. Notes 17, 2020

Linear response theory

$$\sigma_s(\omega) = \frac{\mathrm{i}}{\omega}\left[\frac{\langle H_{\mathrm{kin}}\rangle}{N} - \frac{\mathrm{i}}{N}\int_0^\infty dt\,\mathrm{e}^{\mathrm{i}\omega t}\langle[\mathcal{J}^s(t),\mathcal{J}^s(0)]\rangle\right]$$

$$\sigma'_s(\omega) = -\frac{\pi}{N}\sum_{n,m}\frac{p_n-p_m}{E_n-E_m}|\langle n|\mathcal{J}^2|m\rangle|^2\delta(\omega-(E_m-E_n))$$

$$= \frac{\beta\pi}{N}\sum_{E_n=E_m} p_n|\langle n|\mathcal{J}^2|m\rangle|^2\delta(\omega) + \frac{\pi}{N}\sum_{E_n\neq E_m}\frac{p_n-p_m}{E_m-E_n}|\langle n|\mathcal{J}^2|m\rangle|^2\delta(\omega-(E_m-E_n))$$

Spectral representation $p_n = \exp(-\beta E_n)/Z$

Drude weight D

Regular part: Diffusion constant $\mathcal{D}$

finite at $t \to \infty$

vanish at $t \to \infty$

$$\sigma'_s(\omega) = \frac{1-\mathrm{e}^{-\beta\omega}}{2\omega}\int_{-\infty}^{\infty} dt\,\mathrm{e}^{\mathrm{i}\omega t}\,[(\mathcal{J}^s\mathcal{J}^s)_\infty + C_s^{\mathrm{reg}}(t)]$$

$$= 2\pi\frac{(\mathcal{J}\mathcal{J})_\infty}{2T}\delta(\omega) + \frac{1-\mathrm{e}^{-\beta\omega}}{2\omega}C_s^{\mathrm{reg}}(\omega)\,.$$

Drude weight & Onsager coefficients

$$D_s = \frac{(\mathcal{J}^s\mathcal{J}^s)_\infty}{2T} = \lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{2NT}\langle\mathcal{J}^s(t)\mathcal{J}^s(0)\rangle$$

$$\sigma_s^{\mathrm{reg}}(\omega\to 0) = \beta\int_0^\infty dt\,C_s^{\mathrm{reg}}(t) = \chi_s(\beta)\mathcal{D}_s$$

Static spin susceptibility: χ_s

Diffusion constant: $\mathfrak{D}_s$

$$\mathcal{D}_s = \frac{\beta}{\chi(\beta)}\int_0^\infty dt\;[C(t) - 2TD_s]$$

Mazur bound of the Drude weight

time average of current-current correlation function

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{2NT}\langle\mathcal{J}(t)\mathcal{J}(0)\rangle = \lim_{N\to\infty}\frac{1}{2NT}\sum_k\frac{\langle\mathcal{J}Q_k\rangle^2}{\langle Q_k^2\rangle}$$

Bertini, et. al. Rev. Mod. Phys. **93**, 025003 (2021)
Nardis, Bernard, Doyon, SciPost Phys. 6, 049 (2019)
Sirker, arXiv:191012155

Diffusion in GHD:

$$\partial_t \mathfrak{q}_i(x,t) + \partial_x \mathfrak{j}_i(x,t) = 0$$

$$\partial_t \langle \mathfrak{q}_i(x,t)\rangle + \partial_x \langle \mathfrak{j}_i(x,t)\rangle = 0$$

$$\langle \mathfrak{j}_i(x,t)\rangle =: \bar{\mathfrak{j}}_i[\bar{\mathfrak{q}}_\cdot(\cdot,t)](x,t)$$

Currents depend on charge densities nearby their locations

$$\bar{\mathfrak{j}}_i[\bar{\mathfrak{q}}_\cdot(\cdot,t)](x,t) = \mathcal{F}_i(\bar{\mathfrak{q}}_\cdot(x,t)) - \frac{1}{2}\sum_{j\in I} \mathfrak{D}_i{}^j(\bar{\mathfrak{q}}_\cdot(x,t))\partial_x \bar{\mathfrak{q}}_j(x,t) + O\big(\partial_x^2 \bar{\mathfrak{q}}_\cdot(x,t)\big)$$

Navier-Stokes Equation

$$\partial_t \bar{\mathfrak{q}}_i(x,t) + \partial_x \mathcal{F}_i(\bar{\mathfrak{q}}_\cdot(x,t)) - \frac{1}{2}\partial_x\big(\mathfrak{D}_i{}^j(\bar{\mathfrak{q}}_\cdot(x,t))\,\partial_x \bar{\mathfrak{q}}_j(x,t)\big) = 0$$

Two-point correlation & Static susceptibilities

$$S_{ij}(x,t) := \langle \mathfrak{q}_i(x,t)\mathfrak{q}_j(0,0)\rangle^c \qquad C_{ij} := \int \mathrm{d}x\, S_{ij}(x,t) = \int \mathrm{d}x\, S_{ij}(x,0)$$

$$\frac{1}{2}\int \mathrm{d}x\, x^2 \big(S_{ij}(x,t) + S_{ij}(x,-t) - 2S_{ij}(x,0)\big) = \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \int \mathrm{d}x\, \langle \mathfrak{j}_i(x,s)\mathfrak{j}_j(0,s')\rangle^c$$

Spreading of the correlation

Conservation law

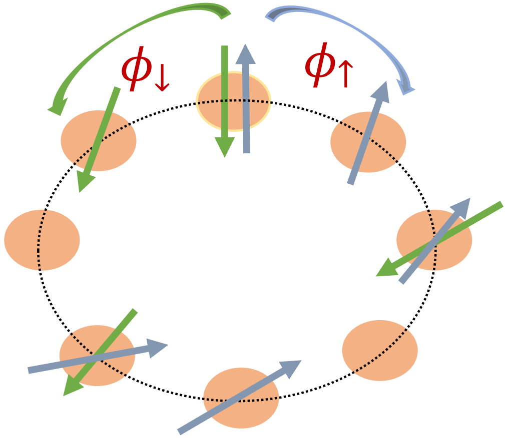
Space and time translation invariance

$$\frac{1}{2}\int \mathrm{d}x\, x^2 \left(S_{ij}(x,t) + S_{ij}(x,-t)\right) = D_{ij}t^2 + \mathfrak{L}_{ij}t + o(t)$$

Spreading coefficients: Drude weight & Onsager coefficients

$$D_{ij} := \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} \mathrm{d}s \int \mathrm{d}x\, \langle \mathfrak{j}_i(x,s)\mathfrak{j}_j(0,0)\rangle^c \qquad \mathfrak{L}_{ij} := \lim_{t\to\infty}\int_{-t}^{t} \mathrm{d}s \left(\int \mathrm{d}x\, \langle \mathfrak{j}_i(x,s)\mathfrak{j}_j(0,0)\rangle^c - D_{ij}\right)$$

Inducing flux for spin & charge: Two U(1) symmetries



$$H = -\sum_{j=1}^{L}\sum_{a=\uparrow\downarrow}\left(\mathrm{e}^{\mathrm{i}\phi_a/L}c^{+}_{j,a}\,c_{j+1,a} + \mathrm{H.c.}\right) + u\sum_{j=1}^{L}(1-2n_{j\uparrow})(1-2n_{j\downarrow})$$

Twisted boundary condition: $c^{\dagger}_{L+1,a} = \mathrm{e}^{\mathrm{i}\phi_a}c^{\dagger}_{1,a}$

$$\mathrm{e}^{ik_jL} = \boxed{\mathrm{e}^{i\phi_\uparrow}}\prod_{L=1}^{M}\frac{\lambda_l - \sin k_j - iu}{\lambda_l - \sin k_j + iu}$$

$$\boxed{\mathrm{e}^{i(\phi_\uparrow-\phi_\downarrow)}}\prod_{j=1}^{N}\frac{\lambda_l - \sin k_j - iu}{\lambda_l - \sin k_j + iu} = -\prod_{m=1}^{M}\frac{\lambda_l - \lambda_m - 2iu}{\lambda_l - \lambda_m + 2iu}$$

$$k_j = k_j^{\infty} + \frac{x_1}{L} + \frac{x_2}{L^2} + \frac{x_3}{L^3} + \frac{x_4}{L^4}\dots$$

$$\Lambda_\alpha^n = \Lambda_\alpha^{n\infty} + \frac{y_{1n}}{L} + \frac{y_{2n}}{L^2} + \frac{y_{3n}}{L^3} + \frac{y_{4n}}{L^4}\dots$$

$$\Lambda_\alpha'^n = \Lambda_\alpha'^{n\infty} + \frac{z_{1n}}{L} + \frac{z_{2n}}{L^2} + \frac{z_{3n}}{L^3} + \frac{z_{4n}}{L^4}\dots$$

$$\boldsymbol{\frac{E}{L} = E_0 + \frac{E_1}{L} + \frac{E_2}{L^2} + \frac{E_3}{L^3} + \dots}$$

$$\boldsymbol{D^{(l)} = \frac{L^l}{2Z}\sum\nolimits_l e^{-\beta E_i}\frac{\partial^{l+1}E_i}{\partial\phi^{l+1}}\bigg|_{\phi_{c,s}=0}}$$

$(n_\downarrow + n_\uparrow) \rightarrow D^c\!: \phi_\uparrow = \phi_\downarrow = \phi_c$ for charge

$(n_\uparrow - n_\downarrow) \rightarrow D^s\!: \phi_\uparrow = -\phi_\downarrow$ for spin

Linear $D^{(1)} \sim \langle J(t_1)J(t_2)\rangle$

Nonlinear $D^{(3)} \sim \langle J(t_1)J(t_2)J(t_3)J(t_4)\rangle$

$D^{(N)} \sim \langle J(t_1)\dots\dots J(t_{N+1})\rangle$

Luo, Pu, Guan, PRB **107**, L201103 (2023)
Luo, Pu, Guan, arXiv: 2307.00890
Guan, Yang, Nucl. Phys. B 512, 601 (1998)

Dressed charge: q_{α}^{dr}

$$D^{c,s} = \frac{1}{2T} \sum_{\alpha=k,\Lambda,k-\Lambda} \int d\theta_\alpha \rho_\alpha (1-n_\alpha) v_\alpha^2 \underline{\left(2\pi(\rho_\alpha + \rho_\alpha^h) \frac{dx_\alpha}{d\phi} \right)}^2$$

$$x_1 = g_1^x \phi;\ x_n = g_n^x \phi^n / n!$$

$$y_1 = g_1^y \phi;\ x_n = g_n^y \phi^n / n!$$

$$z_1 = g_1^z \phi;\ x_n = g_n^z \phi^n / n!$$

$$\theta_\gamma = \theta_\gamma^\infty + \frac{x_{\gamma\,1}}{L} + \frac{x_{\gamma\,2}}{L^2} + \frac{x_{\gamma\,3}}{L^3} + \cdots$$

$$D = \int d\theta \rho(\theta)(1-n(\theta)) v^{\mathrm{eff}}(\theta)^2 q^{dr}(\theta)^2$$

x_α **can be expressed in terms of** x_1

bare charge q $\xrightarrow[\text{dressing}]{\text{interaction}}$

$$q_a^{\mathrm{dr}} = \mathrm{sign}(p_a') 2\pi (\rho_a + \rho_a^h) \frac{dx_a}{d\phi}$$

New result

$$\mathrm{sign}(p_a'(\theta)) = 1, 1, -1 \ \text{for}\ k, \Lambda, k-\Lambda$$

Confirm the validity of the GHD!

numerically

$$q_a^{\mathrm{dr}} = (\mathrm{I} - \boldsymbol{B})_{ab}^{-1} * q_b$$

related to TBA kernels

$$\boldsymbol{B} = \begin{bmatrix} 0 & [a_n(\sin k - \Lambda) n_n]\,|_{1\times N} & -[a_n(\sin k - \Lambda) n_m']\,|_{1\times M} \\ \cos k\,[a_n(\sin k - \Lambda) n_k]\,|_{N\times 1} & \left[-\frac{1}{2\pi}\left(\frac{\partial}{\partial\Lambda}\Theta_{nm}\left(\frac{\Lambda-\Lambda'}{u}\right)\right) n_m\right]|_{N\times N} & 0|_{N\times M} \\ \cos k\,[a_n(\sin k - \Lambda) n_k]\,|_{M\times 1} & 0|_{M\times N} & \left[-\frac{1}{2\pi}\left(\frac{\partial}{\partial\Lambda}\Theta_{nm}\left(\frac{\Lambda-\Lambda'}{u}\right)\right) n_m'\right]|_{M\times M} \end{bmatrix}$$

Bare charges q

q_a^{bare}: particle number , magnetization number, energy

$$\begin{array}{lllrcl} k: & o_k = 1 & m_k = 1/2 & e_k & = & -2\cos k - \mu - 2u - B \\ \Lambda: & o_{n|\Lambda} = 0 & m_{n|\Lambda} = -n & e_{n|\Lambda} & = & 2nB \\ k-\Lambda: & o_{n|k-\Lambda} = 2n & m_{n|k-\Lambda} = 0 & e_{n|k\Lambda} & = & 4\mathrm{Re}\sqrt{1-(\Lambda - \mathrm{i}nu)^2} - 2n\mu - 4nu \end{array}$$

$$\boxed{q_a^{\text{dr}} = (\mathrm{I} - \boldsymbol{B})_{ab}^{-1} * q_a^{\text{bare}}}$$

Dressed charges q^{dr} at T=0 ($k - \Lambda$ strings are gapped)

$$\begin{aligned} q_k^{\text{dr}} &= 1 + \int_{-A}^{A} \mathrm{d}\Lambda a_1(\sin k - \Lambda) q_\Lambda^{\text{dr}} \\ q_\Lambda^{\text{dr}} &= \alpha + \int_{-Q}^{Q} \mathrm{d}k \cos k a_1(\Lambda - \sin k) q_k^{\text{dr}} - \int_{-A}^{A} \mathrm{d}\Lambda' a_2(\Lambda - \Lambda') q_\Lambda^{\text{dr}} \end{aligned}$$

$\alpha = 0, -2$ for charge and spin transport

Beyond the bosonization result: finite magnetic field at T=0

New Result

Bosonization at $H = 0$

$$D^c = \frac{K_c v_c}{\pi}, \chi^c = \frac{2K_c}{\pi v_c}$$

$$D^s = \frac{K_s v_s}{\pi}, \chi^s = \frac{K_s}{2\pi v_s}$$

spin rotation symmetry $K_s = 1$

Drude weight

at $H \neq 0, \mu \neq 0$ for Phase IV

$$D^c = \frac{1}{2\pi} q_k^{c,\mathrm{dr}\,2} v_k|_Q + \frac{1}{2\pi} q_\Lambda^{c,\mathrm{dr}\,2} v_\Lambda|_A$$

$$D^s = \frac{1}{2\pi} q_k^{s,\mathrm{dr}\,2} v_k|_Q + \frac{1}{2\pi} q_\Lambda^{s,\mathrm{dr}\,2} v_\Lambda|_A$$

Contributions from another degrees of states

$\{Z_{\alpha\beta}\}$ are the dressed charges

Susceptibility

at $H \neq 0$

$$\chi_c|_B = \frac{\mathrm{Z}_{cc}^2}{\pi v_c} + \frac{\mathrm{Z}_{cs}^2}{\pi v_s},$$

$$\chi_s|_\mu = \frac{(\mathrm{Z}_{cc} - 2\mathrm{Z}_{sc})^2}{4\pi v_c} + \frac{(\mathrm{Z}_{cs} - 2\mathrm{Z}_{ss})^2}{4\pi v_s}$$

$$\mathrm{Z} = \begin{pmatrix} \xi_{cc}(Q) & \xi_{cs}(A) \\ \xi_{sc}(Q) & \xi_{ss}(A) \end{pmatrix}$$

$$\xi_{ab}(x_b) = \delta_{ab} + \sum_d \int_{-X_d}^{X_d} \mathrm{d}x_d \xi_{ad}(x_d) K_{db}(x_d, x_b)$$

General result:

arbitrary H, μ

For all phases

$$D^c = \frac{K_c v_c}{\pi} + \frac{K_{cs} v_s}{\pi}, \chi^c = \frac{2K_c}{\pi v_c} + \frac{2K_{cs}}{\pi v_s}$$

$$D^s = \frac{K_s v_s}{\pi} + \frac{K_{sc} v_c}{\pi}, \chi^s = \frac{K_s}{2\pi v_s} + \frac{K_{sc}}{2\pi v_c}$$

$q_k^{c,dr}=\xi_{cc}$, $q_\Lambda^{c,dr}=\xi_{cs}$

$q_k^{s,dr}=\xi_{cc} - 2\xi_{sc}$, $q_\Lambda^{s,dr}=\xi_{cs} - 2\xi_{ss}$

Crossing Luttinger parameters: K_{cs}, K_{sc}

Luttinger parameters v.s. Dressed charges

Phase II

$$K_c = q_k^{c,dr\,2} = \mathrm{Z}_{cc}^2 = 1$$

free lattice

Phase V

$$K_s = \frac{q_\Lambda^{s,dr\,2}}{4} = \mathrm{Z}_{ss}^2 \xrightarrow{h=0} \frac{1}{2}$$

spin chain

Phase IV

$$K_c = \frac{q_k^{c,dr\,2}}{2} = \frac{\mathrm{Z}_{cc}^2}{2}$$

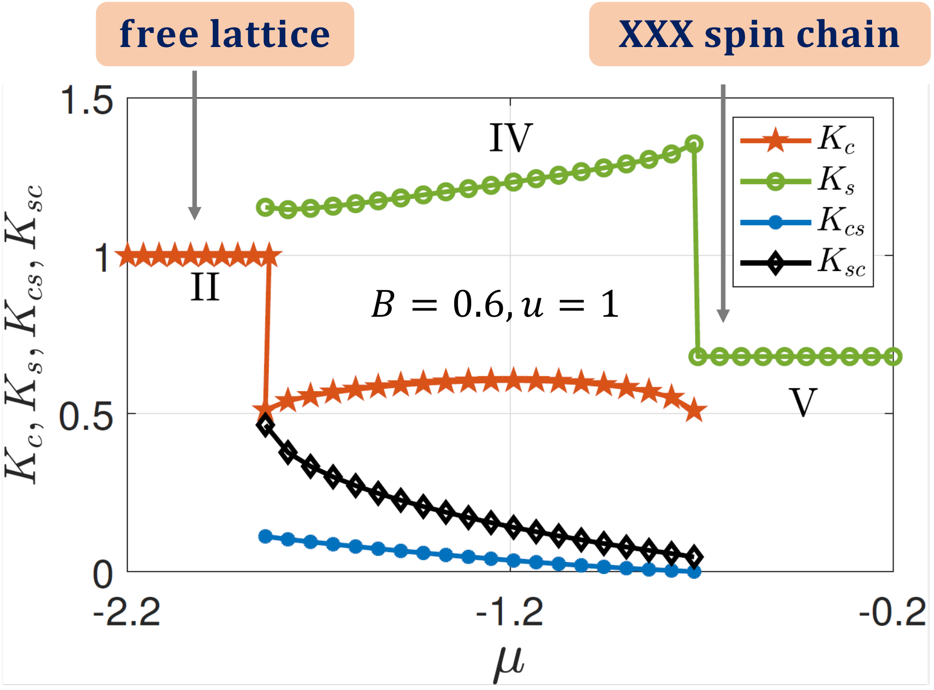
$$K_s = \frac{q_\Lambda^{s,dr\,2}}{2} = \frac{(\mathrm{Z}_{cs} - 2\mathrm{Z}_{ss})^2}{2} \xrightarrow{h=0} 1$$

$$K_{cs} = \frac{q_\Lambda^{c,dr\,2}}{2} = \frac{\mathrm{Z}_{cs}^2}{2} \xrightarrow{h=0} 0$$

$$K_{sc} = \frac{q_k^{s,dr\,2}}{2} = \frac{(\mathrm{Z}_{cc} - 2\mathrm{Z}_{sc})^2}{2} \xrightarrow{h=0} 0$$

Cover the bosonization result at vanishing magnetic field.

$\{Z_{\alpha\beta}\}$ **are the dressed charge**



For fixed B and u

Dressed charges at infinite interaction

$$D^c = \frac{K_c v_c}{\pi} + \cancel{\frac{K_{cs} v_s}{\pi}}$$

$$D^s = \cancel{\frac{K_s v_s}{\pi}} + \frac{K_{sc} v_c}{\pi}$$

$$v_s \xrightarrow{c=\infty} 0$$

$$v_c \xrightarrow{c=\infty} 2\sin(\pi n_c)$$

Drude Weights displays a feature of spin charge coupling!

Phase IV

$$K_c = \frac{q_k^{c,dr\,2}}{2} = \frac{Z_{cc}^2}{2} \xrightarrow{c=\infty} \frac{1}{2}$$

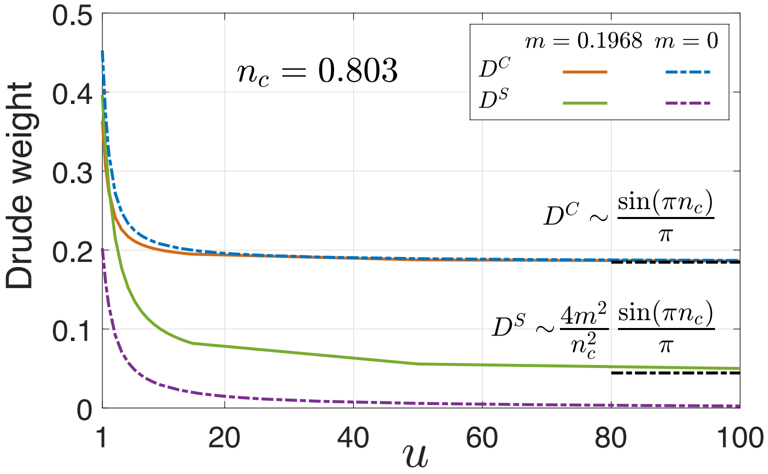
New result

$$K_s = \frac{q_\Lambda^{s,dr\,2}}{2} = \frac{(Z_{cs} - 2Z_{ss})^2}{2}$$

$$K_{cs} = \frac{q_\Lambda^{c,dr\,2}}{2} = \frac{Z_{cs}^2}{2} \xrightarrow{c=\infty} 0$$

Subtle spin polarization!

$$K_{sc} = \frac{q_k^{s,dr\,2}}{2} = \frac{(Z_{cc} - 2Z_{sc})^2}{2} \xrightarrow{c=\infty} \frac{2m^2}{n_c^2}$$



DWs v.s. interaction for fixed n and m

DWs essentially depends on polarization and filling factor!

Linear Drude weight at finite temperature

New Result

- **Dressed charges:**

$$q_a^{\mathrm{dr}} = \mathrm{sign}(p_a')2\pi(\rho_a + \rho_a^h)\frac{dx_a}{d\phi}$$

Density, magnetization, energy

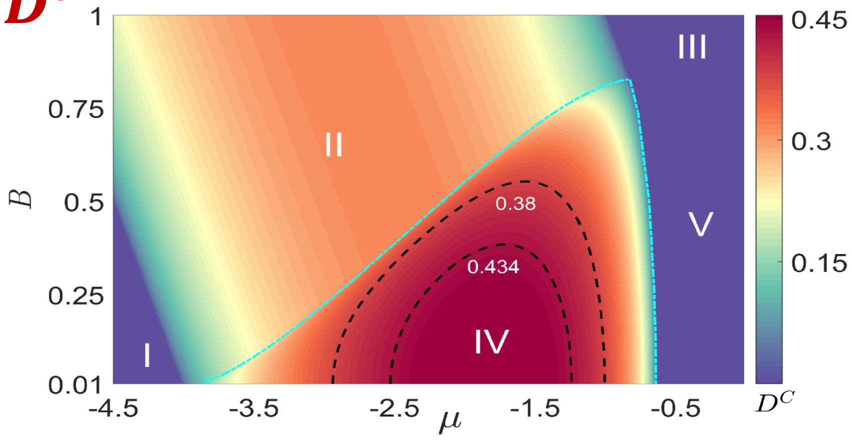
- **Universal laws in TLL**

$k-\Lambda$ strings has no contribution

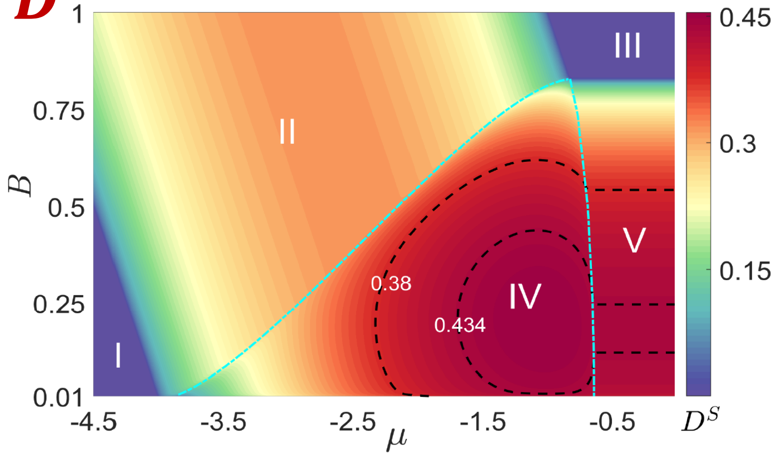
$$D^{c,s} = \sum\nolimits_{a=k,\Lambda,k-\Lambda}\left\{\frac{1}{2\pi}\left(q_a^{c,s\,\mathrm{dr}}\right)^2 v_a\Big|_Q + \frac{\pi T^2}{12}\frac{\partial^2}{\partial\epsilon^2}\left[\left(q_a^{c,s\;dr}\right)^2 v_a\right]\Big|_{\varepsilon(Q)=0}\right\}$$

- **Phase diagram: characteristic of Luttinger liquid**

D^c



D^s



Universal scaling laws at quantum criticality

- **II-IV**

$$D_\Lambda = \frac{2}{\sigma(0)}\left(\frac{q_\Lambda^{\mathrm{dr}}(0)}{2\pi}\right)^2\left(\frac{\varepsilon''(0)}{2}\right) \quad f_{1/2}$$

$$D_k = D^0\left\{1 - f_{1/2}\frac{\sigma(0)q_\Lambda^{\mathrm{dr}\prime}(0) - 2\sigma'(0)q_\Lambda^{\mathrm{dr}}(0)}{\rho^0 q_\mathrm{k}^{\mathrm{dr}0}}\right\} + \frac{\pi^2T^2}{6}\frac{\partial^2}{\partial\kappa^2}D^0\Big|_{\kappa(k)=0}$$

New Result

- **IV-V**

$$D_\mathrm{k} = \frac{2}{\rho(\pi)}\left(\frac{q_\mathrm{k}^{\mathrm{dr}}(\pi)}{2\pi}\right)^2\left(-\frac{\kappa''(\pi)}{2}\right) \quad k_{1/2}$$

$$D_\Lambda = D^0\left\{1 + k_{1/2}\frac{\rho(\pi)q_\mathrm{k}^{\mathrm{dr}\prime}(\pi) - 2\rho'(\pi)q_\mathrm{k}^{\mathrm{dr}}(\pi)}{\sigma^0 q_\Lambda^{\mathrm{dr}0}}\right\} + \frac{\pi^2T^2}{6}\frac{\partial^2}{\partial\varepsilon^2}D^0\Big|_{\varepsilon(\Lambda)=0}$$

General !

- **I-II** $$D_\mathrm{k} = \frac{2}{\rho(0)}\left(\frac{q_\mathrm{k}^{\mathrm{dr}}(0)}{2\pi}\right)^2\left(\frac{\kappa''(0)}{2}\right) \quad \bar{k}_{1/2}$$
- **II-III** $$D_\mathrm{k} = \frac{2}{\rho(\pi)}\left(\frac{q_\mathrm{k}^{\mathrm{dr}}(\pi)}{2\pi}\right)^2\left(-\frac{\kappa''(\pi)}{2}\right) \quad k_{1/2}$$
- **III-V** $$D_\Lambda = \frac{2}{\sigma(0)}\left(\frac{q_\Lambda^{\mathrm{dr}}(0)}{2\pi}\right)^2\left(\frac{\varepsilon''(0)}{2}\right) \quad f_{1/2}$$

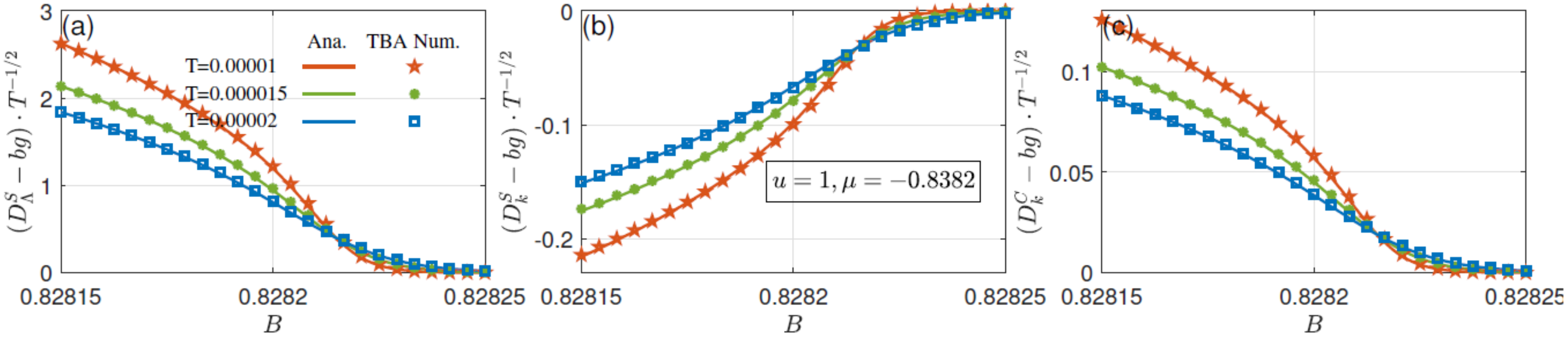
Also applied to other systems

$$f_{1/2} = -\frac{T^{1/2}}{2}\left(\frac{\varepsilon''(0)}{2}\right)^{-\frac{1}{2}}\pi^{\frac{1}{2}}\mathrm{Li}_{\frac{1}{2}}\left(-\mathrm{e}^{-\varepsilon(0)/T}\right)$$

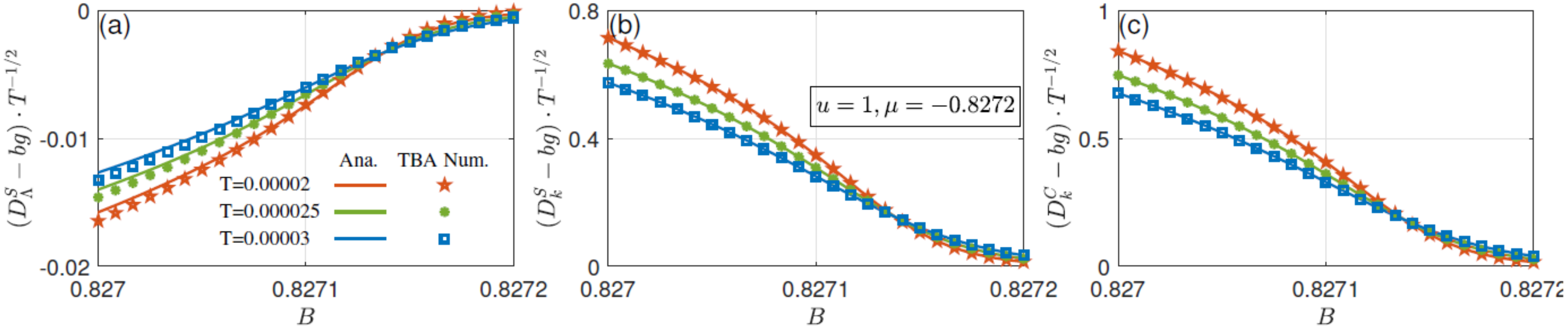
$$k_{1/2} = -\frac{T^{1/2}}{2}\left(-\frac{\kappa''(\pi)}{2}\right)^{-\frac{1}{2}}\pi^{\frac{1}{2}}\mathrm{Li}_{\frac{1}{2}}\left(-\mathrm{e}^{\kappa(\pi)/T}\right)$$

$$\bar{k}_{1/2} = -\frac{T^{1/2}}{2}\left(\frac{\kappa''(0)}{2}\right)^{-\frac{1}{2}}\pi^{\frac{1}{2}}\mathrm{Li}_{\frac{1}{2}}\left(-\mathrm{e}^{-\kappa(0)/T}\right)$$

Universal scaling for phase transition from II to IV



Universal scaling for phase transition from IV to V



Excellent agreement between numerical and analytical results!

Nonlinear Drude weight

- **Universal laws at ground state**

$$D^{(3)} = \sum \frac{\partial^2}{\partial\varepsilon^2}\underbrace{[2\rho^T g_1^4 \dot{\varepsilon}^3]}_{\mathbf{C}}\Big|_{\varepsilon=0} - \frac{\partial}{\partial\varepsilon}\underbrace{[12\rho^T (g_1^4 \dot{\varepsilon}\ddot{\varepsilon} + 2g_1^2 g_2 \dot{\varepsilon}^2)]}_{\mathbf{B}}\Big|_{\varepsilon=0}$$
$$+\underbrace{2\rho^T[(12g_2^2 + 24g_1g_3)\dot{\varepsilon} + 36g_1^2g_2\ddot{\varepsilon} + 4g_1^4\dddot{\varepsilon} + 3g_1^4\ddot{\varepsilon}^2/\dot{\varepsilon}]}_{\mathbf{A}}\Big|_{\varepsilon=0}$$

$$2\pi\rho^t g_1 = q^{dr},$$
$$g_n = g_1 \partial g_{n-1}/n$$

$\boldsymbol{x_1 = g_1\phi;\ x_n = g_n\phi^n/n!}$

$\boldsymbol{v, m, \lambda}$ ... $\boldsymbol{q, \dot{q}, \ddot{q}}$ ... $\boldsymbol{\rho, \dot{\rho}}$ **...for nonlinear DW**

$$C = \frac{q^4 v^3}{\pi}, B = \frac{6q^2 v}{\pi}\left(\frac{q^2}{m} + \frac{q\dot{q}v}{2\pi\rho}\right)$$

$$A = \frac{q^3}{\pi}\left(4q\lambda + \frac{3q}{m^2 v} + \frac{9\dot{q}}{\pi\rho m}\right) + \frac{q^2 v}{\pi(2\pi\rho)^2}\left(7\dot{q}^2 + 4q\ddot{q} - \frac{4q\dot{q}\dot{\rho}}{\rho}\right)$$

$\boldsymbol{v: d\varepsilon/dp}$ velocity
$\boldsymbol{m: d^2\varepsilon/dp^2}$ mass
$\boldsymbol{\lambda: d^3\varepsilon/dp^3}$?

Linear DW only depends on *q* and *v*

$$D^{(1)} = \sum \frac{1}{2\pi}\left(q_a^{\mathrm{dr}}\right)^2 v_a \Big|_Q$$

- **Universal laws in TLL area**

$$D^{(3)} = \underrightarrow{\left[A - \frac{\partial B}{\partial\varepsilon} + \frac{\partial^2 C}{\partial\varepsilon^2}\right]}\Big|_{\varepsilon=0} + \frac{\pi^2 T^2}{6}\frac{\partial^2}{\partial\epsilon^2}\underrightarrow{\left[A - \frac{\partial B}{\partial\varepsilon} + \frac{\partial^2 C}{\partial\varepsilon^2}\right]}\Big|_{\varepsilon=0}$$

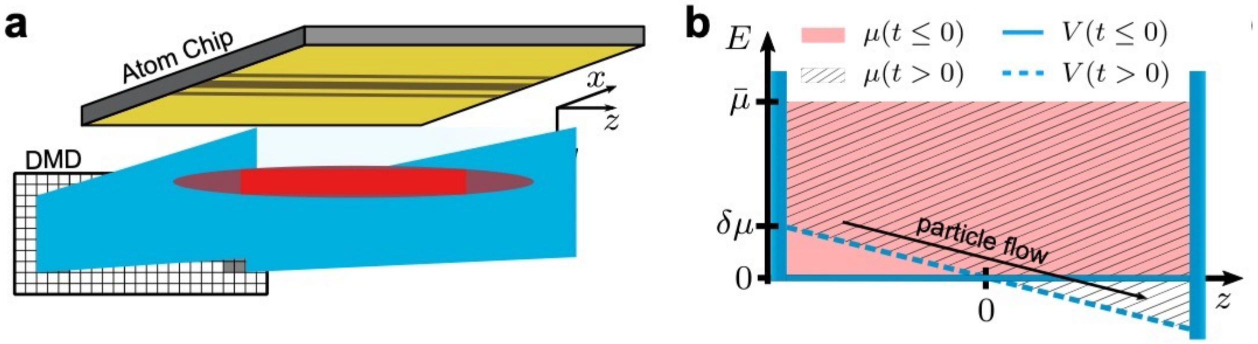
The general features in linear and nonlinear Drude weight

Conjecture

	linear	three order nonlinear	higher order nonlinear
T=0 results	$D_0^{(1)}$	$D_0^{(3)}$	$D_0^{(l)}, l > 3$
parameters	q^{dr}, v	$\boldsymbol{q^{dr}}, \dot{q}^{dr}, \ddot{q}^{dr}; \boldsymbol{v}, m, \lambda; \rho, \dot{\rho}$	$\partial^{l-1}q^{dr}, d^l\varepsilon/dp^l, \partial^{l-2}\rho$
TLL areas	$D_0^{(1)} + \frac{\pi^2T^2}{6}\frac{\partial^2}{\partial\epsilon^2}D_0^{(1)}$	$D_0^{(3)} + \frac{\pi^2T^2}{6}\frac{\partial^2}{\partial\epsilon^2}D_0^{(3)}$	$D_0^{(l)} + \frac{\pi^2T^2}{6}\frac{\partial^2}{\partial\epsilon^2}D_0^{(l)}$

Quantum transport in 1D Hubbard model

Measuring Drude weight in 1D Bose gas



At t=0, the bottom of the box trap is tilted, leading an accelerating the current of atoms across the center.

Schuttelkopt et. al. arXiv:2406.17569

dynamic process ⟷ equilibrium problem

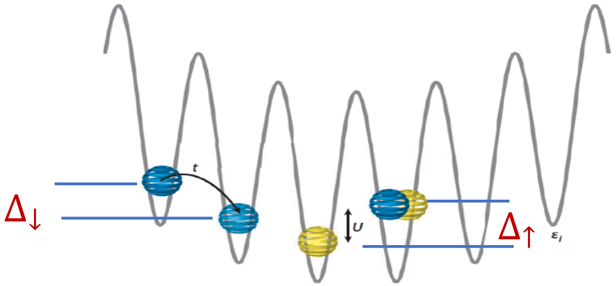
Breakdown integrability

Nichols et. al., Science 363, 383 (2019)

J^c : charge current

J^s : spin current

J^e : kinetic current



$$H_0 = -\sum_{j=1}^{L}\sum_{a=\uparrow\downarrow}\left(c_{j,a}^{+}\, c_{j+1,a} + c_{j+1,a}^{+} c_{j,a}\right)$$

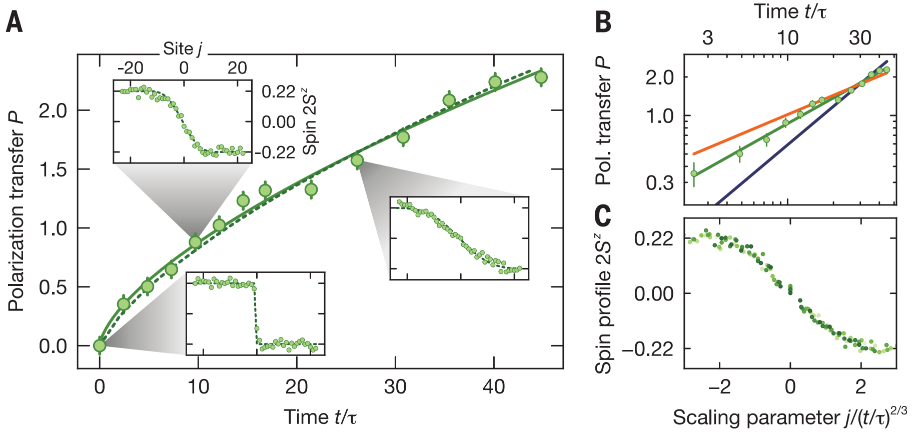
$$+u\sum_{j=1}^{L}(1-2n_{j\uparrow})(1-2n_{j\downarrow})$$

$$+\Delta_\downarrow \sum_{j} j\, n_{j\downarrow} + \Delta_\uparrow \sum_{j} j\, n_{j\uparrow}$$

New frontiers in quantum integrability: Super diffusive spin transport

Quantum gas $$H = -J\sum_{j=1}^{L}(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z)$$

Ferromagnetic $J = \frac{4\tilde{t}^2}{U}; \ \Delta \approx 1,$ **^{87}Rb atoms with hyperfine states**



Super diffusive transport in Heisenberg chain at high T

A: The polarization transfer for a domain wall initial state with a magnetization $\eta = 0.22$.
The insets show spin profiles $2S^z(t)$ at **t=0, 10, 26** J/h.
B: Polarization transfer in log-log plot. **C:** Spatial spin profiles at times **t=5-35** j/h, showing $z = 1.54(7)$

Kardar-Parisi-Zhang hydrodynamics!

Polarization: measuring polarization transfer

$$P(t) = (P_L(t) - P_R(t))/2 \propto t^{1/z}$$

$$P_{L,R}(t) = 2\sum_{i=L,R}(S_i^z(t) - S_i^z(0))$$

KPZ dynamics: z=3/2 for Δ≈ 1

Integrability and non-Abelian SU(2) symmetry

Generalized hydrodynamics approach

DCF $$C(x,t) \equiv < S_x^z(t) S_x^z(0) >$$

$$P(t) = \iint_{-\infty}^{0,x} dx\, dx' C(x',t)$$

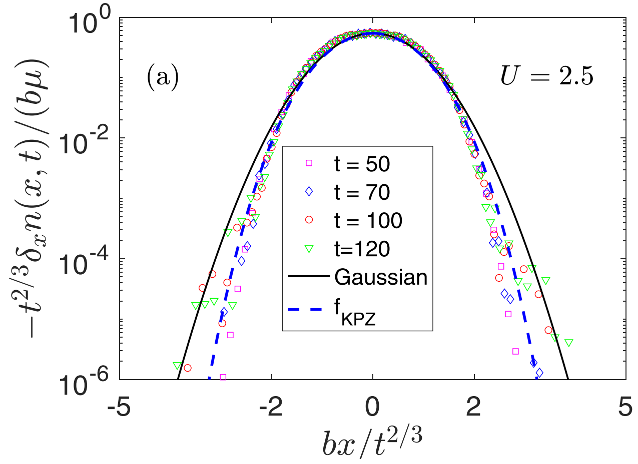
$$C(x,t) \sim t^{-\frac{1}{z}} C(x^z/t)$$

Wei, et. al. Science 376, 716 (2024)

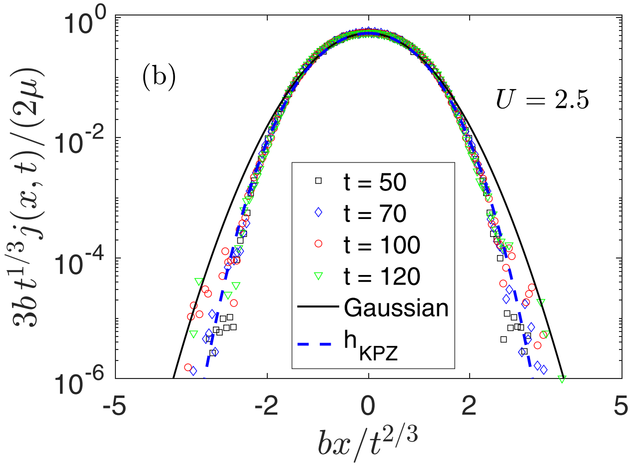
Superdiffusive in charge

Both spin and charge have a SU(2) symmetry

$H = 0,\ \mu = 0,\ n_c = 1$



Density gradient KPZ scaling



Current KPZ scaling

$$\dot{n}(x,t) + \partial_x j(x,t) = 0$$

$$\partial_x n(x,t) = -\frac{b\,\mu}{t^\alpha} f(bx/t^\alpha)$$

$$j(x,t) = \frac{\mu}{b}\,\alpha\, t^{\alpha-1} h(bx/t^\alpha)$$

$$\boldsymbol{\alpha = 3/2}$$

Spin and charge Drude weight vanish

Spin and charge superdiffusive transport

Density gradient

$$\boldsymbol{\delta_x n(x,t) = \delta n(x,t) - \delta n(x-1,t)}$$

Current:

$$j(x,t) = \mathrm{tr}\left\{\frac{i}{2}[c^\dagger_{x+1,\sigma} c_{x\sigma} - c^\dagger_{x\sigma} c_{x+1\sigma}]\rho(t)\right\}$$

Moca, et. al. Phys. Rev. B 108, 235139 (2023)

Kardar-Parisi-Zhang hydrodynamics!

$$\langle S^\mu(x,t) S^\mu(0,0)\rangle = \frac{\chi_h}{[\lambda^{(S)}_{\mathrm{KPZ}} t]^{2/3}} f_{\mathrm{KPZ}}\left(\frac{x}{[\lambda^{(S)}_{\mathrm{KPZ}} t]^{2/3}}\right)$$

$$\langle n(x,t) n(0,0)\rangle = \frac{\chi_\mu}{[\lambda^{(\eta)}_{\mathrm{KPZ}} t]^{2/3}} f_{\mathrm{KPZ}}\left(\frac{x}{[\lambda^{(\eta)}_{\mathrm{KPZ}} t]^{2/3}}\right)$$

Conclusion and discussion

1. The 1D repulsive Hubbard model exhibits novel phases of Luttinger liquids and phase transitions driven by either external potentials or interaction.

2. The spin and charge Drude weights at low temperature have been analytically obtained, showing universal ballistic transport with spin polarization.

3. We have built up exact relations between Luttinger parameters and dressed charges.

4. The universal scaling laws of the Drude weight at quantum criticality obtained shed light on non-Fermi liquid behaviour.

The decade-old 1D Hubbard model continues to yield new and exciting physics!

Thanks for your listening!