On rational solutions of the Painlevé equations

Mathematics and Physics of Integrability, MATRIX, July 2024

Clare Dunning, University of Kent

work with Peter Clarkson SAPM 152, 1 (2024) 453

Sanjida Islam and Ben Mitchell

Outline

- Painlevé equations
- Wronskian Hermite polynomials and Wronskian Laguerre polynomials
- Wronskian Laguerre polynomials in solutions of $\mathrm{P_V}$ and properties
- The role of the partition

Painlevé equations

The Painlevé equations

Discovered by Painlevé, Gambier and their colleagues in the late 19th/early 20th centuries while studying

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = F\left(z, w, \frac{\mathrm{d}w}{\mathrm{d}z}\right)$$

Painlevé I

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z$$

The Painlevé equations P_{II} to P_{VI}

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right)$$

Special solutions of Painlevé equations

- Painlevé III, V, VI have algebraic solutions for certain values of the parameters
- Painlevé II-V have solutions expressed in terms of the classical special functions (Airy, Bessel, parabolic cylinder, Kummer and hypergeometric functions) for certain values of the parameters
- All but Painlevé I have rational solutions

Rational solutions of Painlevé equations

Rational solutions typically appear as

$$w(z) = az + b + \frac{d}{dx} \ln \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials and a, b constants

The $P(z), Q(z)$ are usually determinants (often Wronskians) of known polynomials

$\mathrm{P_{IV}}$	Generalised Hermite polynomials	$H_{m,n}(z)$	$H_n(z)$
	Okamoto polynomials	$R_n(z), S_n(z)$	
	Generalised Okamoto polynomials	$Q_{m,n}(z)$	
$\mathrm{P_V}$	Generalised Laguerre polynomials	$T_{m,n}^{(\mu)}(z)$	$L_n^{(\mu)}(z)$
	Generalised Umemura polynomials	$U_{m,n}(z;\mu)$	

Wronskian Hermite Polynomials

Wronskian Hermite polynomials

Probabilists' Hermite polynomials

$$H_0 = 1 \quad , \quad H_1 = x \quad , \quad H_2 = x^2 - 1 \quad , \quad H_3 = x(x^2 - 3) \quad , \quad \ldots$$

Wronskian Hermite polynomials

$$\begin{aligned}
Wr[H_3, H_2] &= x^4 + 3 \\
Wr[H_7, H_5, H_2, H_1] &= x^3(x^6 - x^4 - 7x^2 + 35) \\
Wr[H_{12}, H_4, H_3, H_1] &= x^{14} - 29\,x^{12} + 183\,x^{10} + 105\,x^8 \\
&\quad -525\,x^6 - 4095\,x^4 + 2205\,x^2 + 315
\end{aligned}$$

where

$$Wr[f_1, f_2, \ldots f_r] = \mathrm{Det}\left(\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} f_i\right)_{1 \le i,j \le r}$$

Wronskian Hermite polynomials

Rather than considering the labels $h = (h_1, h_2, \ldots, h_r)$, define the Wronskian Hermite polynomial

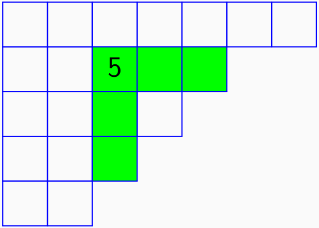
$$H_\lambda = Wr[H_{h_1}, H_{h_2}, \ldots, H_{h_r}]$$

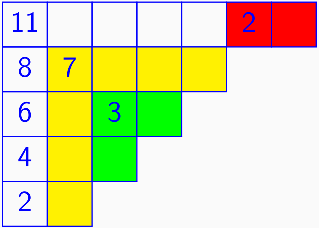
with $h_1 > h_2 > \cdots > h_r > 0$ in terms of the partition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) = (h_1 - r + 1, h_2 - r + 2, \ldots, h_r)$$

Young diagram and hook lengths of partition

$\lambda = (7, 5, 4, 3, 2)$ is a partition of $|\lambda| = 21$





Examples of WHP

h	λ	Hooks	H_λ
(3,2)	(2,2)	3 2 2 1	x^4+3
(5,4,2)	(3,3,2)	5 4 2 4 3 1 2 1	x^8+10x^4-15
(7,4,2,1)	(4,2,1,1)	7 4 2 1 4 1 2 1	x^8-14x^4-7
(7,5,2,1)	(4,3,1,1)	7 4 3 1 5 2 1 2 1	$x^3(x^6-x^4-7x^2+35)$

Generalised Hermite polynomials

Generalised Hermite polynomials

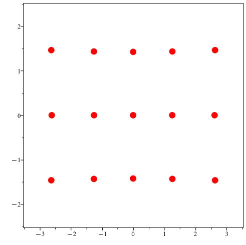
$$H_{m,n} = Wr[\{H_{m+j}\}_{j=0}^{n-1}]$$

are labelled by rectangular partitions $\lambda = (m^n)$

Example

$$H_{5,3} = Wr[H_5, H_6, H_7]$$

7	6	5	4	3
6	5	4	3	2
5	4	3	2	1



Wronskian Laguerre Polynomials

Generalised Laguerre polynomials

Define the generalised Laguerre polynomial $T_{m,n}^{(\mu)}(z)$ as the double Wronskian

$$T_{m,n}^{(\mu)}(z) = \det \left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}} L_{m+n}^{(\mu+1)}(z) \right]_{j,k=0}^{n-1}$$

where $L_n^{(\alpha)}(z)$ is the associated Laguerre polynomial

$$L_n^{(\alpha)}(z) = \frac{z^{-\alpha}\,\mathrm{e}^{z}}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(z^{n+\alpha}\,\mathrm{e}^{-z} \right), \qquad n \geq 0$$

It would be more convenient if there was a Wronskian representation.

Wronskian representation

From properties of Laguerre polynomials and discriminants

$$T_{m,n}^{(\mu)}(z) = \mathrm{Wr}\left(L_{m+1}^{(n+\mu)}(z), L_{m+2}^{(n+\mu)}(z), \ldots, L_{m+n}^{(n+\mu)}(z)\right)$$

The labelling partition is

$$\lambda = ((m+1)^n)$$

Schur representations

Define the elementary Schur polynomials $p_j(\mathbf{t})$ through

$$\sum_{j=0}^{\infty} p_j(\mathbf{t})\, x^j = \exp\left(\sum_{j=1}^{\infty} t_j\, x^j\right), \qquad p_j(\mathbf{t}) = 0 \quad , \quad j < 0$$

where $p_0(\mathbf{t}) = 1$ and $\mathbf{t} = (t_1, t_2, \ldots)$. The *Schur polynomial* is

$$S_\lambda(\mathbf{t}) = \det\left[p_{\lambda_j+k-j}(\mathbf{t})\right]_{j,k=1}^{r}$$

Then

$$T_{m,n}^{(\mu)}(z) = (-1)^{n(n-1)/2}\, S_\lambda(\mathbf{t}) \quad , \quad \lambda = ((m+1)^n)$$

with

$$t_j = \frac{\mu + n + 1}{j} - z, \qquad j = 1, 2, \ldots$$

and $p_j(\mathbf{t}) = L_j^{(\mu+n)}(-z), \qquad j = 0, 1, \ldots$

Generalised Laguerre polynomials

Define $\widehat{T}_{m,n}(z;\mu)$ as

$$\widehat{T}^{(\mu)}_{m,n}(z) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}} L^{(\mu+1)}_{m+n}(-z)\right]_{j,k=0}^{n-1}$$

We note that

$$T^{(\mu)}_{m,n}(-z) = \widehat{T}^{(\mu)}_{m,n}(z).$$

The generalised Laguerre polynomials also satisfy

$$T^{(\mu)}_{m,n}(z) = (-1)^{\lfloor (m+n+1)/2 \rfloor}\, \widehat{T}^{(-\mu-2n-2m-2)}_{n-1,m+1}(z),$$

which follows from properties of Schur polynomials

Differential-difference and discrete equations

$$T^{(\mu)}_{m+1,n-1}\,T^{(\mu)}_{m-1,n+1} = T^{(\mu)}_{m,n}\frac{\mathrm{d}^2\,T^{(\mu)}_{m,n}}{\mathrm{d}z^2} - \left(\frac{\mathrm{d}\,T^{(\mu)}_{m,n}}{\mathrm{d}z}\right)^2$$

$$T^{(\mu-1)}_{m,n+1}\,T^{(\mu+1)}_{m,n-1} = T^{(\mu-1)}_{m+1,n}\,T^{(\mu+1)}_{m-1,n} - \left(T^{(\mu)}_{m,n}\right)^2$$

$$T^{(\mu-1)}_{m,n+1}\,T^{(\mu+1)}_{m+1,n-1} = T^{(\mu-1)}_{m+1,n}\,T^{(\mu+1)}_{m,n} - T^{(\mu)}_{m+1,n}\,T^{(\mu)}_{m,n}$$

$$T^{(\mu-1)}_{m,n}\,T^{(\mu+1)}_{m,n-1} = T^{(\mu)}_{m,n}\,T^{(\mu)}_{m,n-1} - T^{(\mu)}_{m-1,n}\,T^{(\mu)}_{m+1,n-1}$$

and

$$\mathrm{D}_z\left(T^{(\mu+1)}_{m,n-1}\cdot T^{(\mu)}_{m,n}\right) = T^{(\mu)}_{m+1,n-1}\,T^{(\mu+1)}_{m-1,n}$$

where D_z is the Hirota bilinear operator

$$\mathrm{D}_z(f\cdot g) = \frac{\mathrm{d}f}{\mathrm{d}z}g - f\frac{\mathrm{d}f}{\mathrm{d}z}$$

Differential-difference and discrete equations

$$T^{(\mu)}_{m+1,n-1}\,T^{(\mu)}_{m-1,n+1} = T^{(\mu)}_{m,n}\frac{\mathrm{d}^2\,T^{(\mu)}_{m,n}}{\mathrm{d}z^2} - \left(\frac{\mathrm{d}\,T^{(\mu)}_{m,n}}{\mathrm{d}z}\right)^2$$

$$T^{(\mu-1)}_{m,n+1}\,T^{(\mu+1)}_{m,n-1} = T^{(\mu-1)}_{m+1,n}\,T^{(\mu+1)}_{m-1,n} - \left(T^{(\mu)}_{m,n}\right)^2$$

$$T^{(\mu-1)}_{m,n+1}\,T^{(\mu+1)}_{m+1,n-1} = T^{(\mu-1)}_{m+1,n}\,T^{(\mu+1)}_{m,n} - T^{(\mu)}_{m+1,n}\,T^{(\mu)}_{m,n}$$

$$T^{(\mu-1)}_{m,n}\,T^{(\mu+1)}_{m,n-1} = T^{(\mu)}_{m,n}\,T^{(\mu)}_{m,n-1} - T^{(\mu)}_{m-1,n}\,T^{(\mu)}_{m+1,n-1}$$

and

$$\mathrm{D}_z\left(T^{(\mu+1)}_{m,n-1}\cdot T^{(\mu)}_{m,n}\right) = T^{(\mu)}_{m+1,n-1}\,T^{(\mu+1)}_{m-1,n}$$

where D_z is the Hirota bilinear operator

$$\mathrm{D}_z(f\cdot g) = \frac{\mathrm{d}f}{\mathrm{d}z}g - f\frac{\mathrm{d}f}{\mathrm{d}z}$$

Rational solutions of $\mathrm{P_V}$

Rational solutions of $\mathrm{P_V}$: Case (i) (Kitaev, Law & McLeod (1994))

$$
\begin{aligned}
\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} \;&=\; \left(\frac{1}{2w}+\frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) \\
&\qquad + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}
\end{aligned}
$$

has rational solutions when $m, n, \in \mathbb{Z}, m \geq 1$, and μ constant for

$$
\alpha = \tfrac{1}{2}m^2,\ \beta = -\tfrac{1}{2}(m+2n+1+\mu)^2,\ \gamma = \mu,\ \delta = -1/2
$$

together with the solutions obtained through the symmetries

$$
\begin{aligned}
\mathcal{S}_1 : &\qquad w_1(z;\alpha,\beta,-\gamma,-\tfrac{1}{2}) = w(-z;\alpha,\beta,\gamma,-\tfrac{1}{2}) \\
\mathcal{S}_2 : &\qquad w_2(z;-\beta,-\alpha,-\gamma,-\tfrac{1}{2}) = \frac{1}{w(z;\alpha,\beta,\gamma,-\tfrac{1}{2})}
\end{aligned}
$$

Rational solutions of $\mathrm{P_V}$: Cases (ii) and (iii) (Kitaev, Law & McLeod (1994))

$\mathrm{P_V}$ also has rational solutions when

$$\alpha = \tfrac{1}{2}(m+\mu)^2,\ \beta = -\tfrac{1}{2}(n+\epsilon\mu)^2,\ \gamma = m+\epsilon n,\ \delta = -1/2$$

with $\epsilon = \pm 1$ and

$$\alpha = \tfrac{1}{2}(m+\tfrac{1}{2})^2,\ \beta = -\tfrac{1}{2}(n+\tfrac{1}{2})^2,\ \gamma = \mu,\ \delta = -1/2$$

provided that $m \neq 0$ or $n \neq 0$

Case (i) solutions are written in terms of generalised Laguerre polynomials

Cases (ii) and (iii) solutions are written in terms of generalised Umemura polynomials

Rational solutions of $\mathrm{P_V}$ in case (i)

The function

$$w_{m,n}(z;\mu) = \frac{T^{(\mu)}_{m-1,n}(z)\,T^{(\mu)}_{m-1,n+1}(z)}{T^{(\mu)}_{m,n}(z)\,T^{(\mu)}_{m-2,n+1}(z)}, \qquad m,n \geq 1$$

is a rational solution for the parameters

$$\alpha_{m,n} = \tfrac{1}{2}m^2, \qquad \beta_{m,n} = -\tfrac{1}{2}(m+2n+1+\mu)^2, \qquad \gamma_{m,n} = \mu$$

Alternatively

$$w_{m,n}(z;\mu) = \frac{z}{m}\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln \frac{T^{(\mu)}_{m-2,n+1}(z)}{T^{(\mu)}_{m,n}(z)}\right\} - \frac{z-m-2n-1-\mu}{m}$$

Rational solutions of $\mathrm{P_V}$ in case (i)

The function

$$w_{m,n}(z;\mu) = \frac{T^{(\mu)}_{m-1,n}(z)\,T^{(\mu)}_{m-1,n+1}(z)}{T^{(\mu)}_{m,n}(z)\,T^{(\mu)}_{m-2,n+1}(z)}, \qquad m,n \geq 1$$

is a rational solution for the parameters

$$\alpha_{m,n} = \tfrac{1}{2}m^2, \qquad \beta_{m,n} = -\tfrac{1}{2}(m+2n+1+\mu)^2, \qquad \gamma_{m,n} = \mu$$

When $n = 0$

$$w_{m,0}(z;\mu) = \frac{T^{(\mu)}_{m-1,1}(z)}{T^{(\mu)}_{m-2,1}(z)} = \frac{L^{(\mu+1)}_{m}(z)}{L^{(\mu+1)}_{m-1}(z)}, \qquad m \geq 1$$

The function

$$\widehat{w}_{m,n}(z;\mu) = \frac{\widehat{T}^{(\mu)}_{m-1,n}(z)\ \widehat{T}^{(\mu)}_{m-1,n+1}(z)}{\widehat{T}_{m,n}\mu(z)\ \widehat{T}^{(\mu)}_{m-2,n+1}(z)}, \qquad m,n \geq 1$$

is a rational solution for the parameters

$$\alpha_{m,n} = \tfrac{1}{2}m^2, \qquad \beta_{m,n} = -\tfrac{1}{2}(m+2n+1+\mu)^2, \qquad \gamma_{m,n} = -\mu$$

Non-uniqueness of rational solutions of $\mathrm{P_V}$

Kitaev, Law and McLeod (94): rational solutions of $\mathrm{P_V}$ are unique when $\mu \notin \mathbb{Z}$

Non-uniqueness of rational solutions of $\mathrm{P_V}$

Kitaev, Law and McLeod (94): rational solutions of $\mathrm{P_V}$ are unique when $\mu \notin \mathbb{Z}$

Theorem
If $\mu = k$ with $k \in \mathbb{Z}$ and $k \geq -n$, then

$$w_{m,n}(z;k) \text{ and } \widehat{w}_{m,n+k}(z;-k)$$

both satisfy $\mathrm{P_V}$ for the parameters

$$\alpha = \tfrac{1}{2}m^2, \qquad \beta = -\tfrac{1}{2}(m+2n+k+1)^2, \qquad \gamma = k$$

Example 1

The rational functions

$$w_{1,1}(z;1) = -\frac{(z-3)(z^2-8z+20)}{(z-2)(z-6)}$$

$$\widehat{w}_{1,2}(z;-1) = \frac{(z^2+4z+6)(z^3+9z^2+36z+60)}{z^4+12z^3+54z^2+96z+72},$$

are both solutions of $\mathrm{P_V}$ with parameters

$$\alpha = 1/2, \qquad \beta = -25/2, \qquad \gamma = 1$$

Example 2

The rational functions

$$w_{1,1}(z;-1) = -\frac{(z-1)(z^2-4z+6)}{(z^2-4z+2)}$$

$$\widehat{w}_{1,0}(z;1) = z+3$$

are both solutions of $\mathrm{P_V}$ with parameters

$$\alpha = 1/2, \qquad \beta = -9/2, \qquad \gamma = -1$$

Example 2 ctd

Solutions arise from a special function solution of $\mathrm{P_V}$ as a determinant of Bessel functions through appropriate choice of the arbitrary constants C_1, C_2 at the relevant parameter values:

$$w = \frac{-C_1^2(z^2-4z+2)e^{2z} - C_1C_2(z^3+z^2+4z-4)e^z - 2C_2^2}{C_1^2(z-1)(z^2-4z+6)e^{2z} - C_1C_2(z^2+8z-12)e^z - 2C_2^2(z+3)}$$

The partition is more than a label

Generalised Laguerre polynomials

Recall

$$T^{(\mu)}_{m,n}(z) = \mathrm{Wr}\left(L^{(n+\mu)}_{m+1}(z), L^{(n+\mu)}_{m+2}(z), \ldots, L^{(n+\mu)}_{m+n}(z)\right)$$

is associated with the partition $\lambda = ((m+1)^n)$

Example $m = 3, n = 5$

8	7	6	5
7	6	5	4
6	5	4	3
5	4	3	2
4	3	2	1

Coefficients of general Wronskian of Laguerre polynomials

Define

$$\Omega_\lambda^{(\alpha)}(z) = \mathrm{Wr}\left(L_{h_1}^{(\alpha)}(z), L_{h_2}^{(\alpha)}(z), \ldots, L_{h_{\ell(\lambda)}}^{(\alpha)}(z)\right),$$

$$\Omega_\lambda^{(\alpha)}(z) = \sum_{j=0}^{|\lambda|} r_j^{(\alpha)} \, z^{|\lambda|-j},$$

$$r_j^{(\alpha)} = \binom{|\lambda|}{j} \sum_{\widetilde{\lambda} <_j \lambda} \frac{F_{\widetilde{\lambda}} F_{\lambda/\widetilde{\lambda}}}{F_\lambda} \frac{\Psi_\lambda^{(\alpha)}}{\Psi_{\widetilde{\lambda}}^{(\alpha+\ell(\lambda)-\ell(\widetilde{\lambda}))}},$$

Continued

$$\Psi_{\rho}^{(\alpha)} = (-1)^{|\rho|+\mathsf{ht}(\mathrm{P})} \prod_{j=1}^{\ell(\rho)} \Bigg(\prod_{k=\ell(\rho)}^{h_{\rho_j}-1} \big(h_{\rho_j} - k + \alpha + \ell(\rho)\big) \times \prod_{k \in \{0,1,\ldots,\ell(\rho)-1\} \setminus h_{\rho}}^{j-1} \big(j - 1 - k - \alpha - \ell(\rho)\big) \Bigg)$$

Coefficients of $T_{m,n}^{(\mu)}(z)$

$$T_{\lambda}^{(\mu)}(z) = (z^{n(m+1)} - n(m+1)(\mu + m + n + 1)z^{n(m+1)-1} + \cdots + (-1)^{n(m+1)} \prod_{h \in \mathcal{H}_{m,n}} (\mu + n + h)$$

Hooks of $\lambda = ((m+1)^n$

Example $m = 3, n = 5$

8	7	6	5
7	6	5	4
6	5	4	3
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Hooks of $\lambda = ((m+1)^n$

Example $m = 3, n = 5$

8	7	6	5
7	6	5	4
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5	4	3	2
4	3	2	1

The hook multiset of $\lambda = ((m+1)^n$ is

$$\mathcal{H}_{m,n} = \begin{cases} \{k^{p_1}\}_{k=1}^{m} \cup \{k^{p_2}\}_{k=m+1}^{n} \cup \{k^{p_3}\}_{k=n+1}^{m+n}, & n > m, \\ \{k^{p_1}\}_{k=1}^{n} \cup \{k^{\widetilde{p}_2}\}_{k=n+1}^{m+1} \cup \{k^{p_3}\}_{k=m+2}^{m+n}, & n \leq m, \end{cases}$$

where

$$p_1 = k \quad , \quad p_2 = m+1 \quad , \quad \widetilde{p}_2 = n \quad , \quad p_3 = m+n+1-k$$

are the multiplicities of the hooks in each respective set

Discriminants

Recall that a monic polynomial $f(x)$

$$f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0,$$

with roots $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{C}$ has discriminant

$$\mathrm{Dis}(f) = \prod_{1 \le j < k \le d} (\alpha_j - \alpha_k)^2$$

generalising the case when $d = 2$ and

$$\mathrm{Dis}(f) = (\alpha_1 - \alpha_2)^2 = a_1^2 - 4a_0$$

since

$$\alpha_{1,2} = -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_0}}{2}$$

Discriminant of generalised Hermite polynomials

The discriminant $D_{m,n}$ of $H_{m,n}$ is (Roberts (2003))

$$D_{m,n} = \pm \prod_{k=1}^{m+n-1} k^{k\, e(m,n,k)}$$

where

$$e(m,n,k) = \begin{cases} k^2 - 2(m-k)(n-k) & k < \min\,(m,n) \\ \min\,(m,n)^2 & \min\,(m,n) \leq k \leq \max\,(m,n) \\ (m+n-k)^2 & \text{otherwise} \end{cases}$$

Example: Discriminant of $H_{3,5}$

The generalised Hermite polynomial

$$
\begin{aligned}
H_{3,5} &= x\big(x^{14} + 15x^{12} + 135x^{10} + 525x^8 + 675x^6 \\
&\qquad +4725x^4 - 7875x^2 - 23625\big)
\end{aligned}
$$

has discriminant

$$
D_{3,5} = (2)^{92}\,(3)^{51}\,(5)^{45}\,(7)^{7}
$$

Example: Discriminant of $H_{3,5}$

The generalised Hermite polynomial

$$H_{3,5} = x\big(x^{14} + 15x^{12} + 135x^{10} + 525x^8 + 675x^6 + 4725x^4 - 7875x^2 - 23625\big)$$

has discriminant

$$\begin{aligned} D_{3,5} &= (2)^{92}\,(3)^{51}\,(5)^{45}\,(7)^{7} \\ &= 2^{-12}\big(1^{1.1^2}\,2^{2.2^2}\,3^{3.3^2}\,4^{4.3^2}\,5^{5.3^2}\,6^{6.2^2}\,7^{7.1^2}\big) \end{aligned}$$

Example: Discriminant of $H_{3,5}$

The generalised Hermite polynomial

$$H_{3,5} = x\big(x^{14} + 15x^{12} + 135x^{10} + 525x^{8} + 675x^{6} + 4725x^{4} - 7875x^{2} - 23625\big)$$

has discriminant

$$D_{3,5} = (2)^{92}\,(3)^{51}\,(5)^{45}\,(7)^{7}$$
$$= 2^{-12}\big(1^{1.1^2}\,2^{2.2^2}\,3^{3.3^2}\,4^{4.3^2}\,5^{5.3^2}\,6^{6.2^2}\,7^{7.1^2}\big)$$

7	6	5
6	5	4
5	4	3
4	3	2
3	2	1

The hook multiset is

$$\mathcal{H} = \{1, 2^2, 3^3, 4^3, 5^3, 6^2, 7\}$$

Discriminant of the generalised Hermite polynomials

The discriminant of the generalised Hermite polynomial is

$$|D_{m,n}| = \frac{\prod_{k=1}^{m+n-1} k^{k\,p_k^2}}{\prod_{k=1}^{\min(m,n)} k^{2k(m-k)(n-k)}}$$

where a hook of length k has multiplicity p_k.

The hook multiset is

$$\mathcal{H} = \{k^k\}_{k=1}^{\min(m,n)-1} \cup \{k^{\min(m,n)}\}_{k=\min(m,n)}^{\max(m,n)} \cup \{k^{m+n-k}\}_{k=\max(m,n)+1}^{m+n-1}$$

Generalised Okamoto polynomials $Q_{m,n}$

Define

$$Q_{m,n} = Wr[\{H_{1+3j}\}_{j=0}^{m-1}, \{H_{2+3j}\}_{j=0}^{n-1}]$$

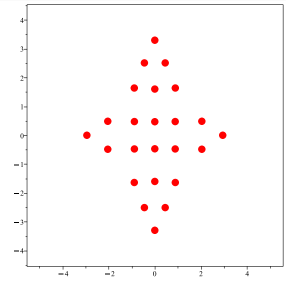
Then

Condition	Partition λ
$m \geq n$	$\left(\{n-1+2j\}_{j=1}^{m-n}; \{j^2\}_{j=1}^{n}\right)$
$m < n$	$\left(\{m+2j\}_{j=1}^{n-m}; \{j^2\}_{j=1}^{m}\right)$

Example

$$Q_{3,5} = W[H_1, H_4, H_7; H_2, H_5, H_8, H_{11}, H_{14}]$$

14	11	8	5	4	2	1
11	8	5	2	1		
8	5	2				
7	4	1				
5	2					
4	1					
2						
1						



Discriminant of the general Okamoto polynomials

Roberts proved that for $m \geq n$

$$D_{m,n} = \prod_{k=0}^{m-1}(1+3k)^{(1+3k)(m-k)^2}\prod_{k=0}^{n-1}(2+3k)^{(2+3k)(n-k)^2} \times \prod_{k=0}^{m-n-1}(2+3k)^{(2+3k)(m-n-1-k)^2}$$

We show that for all $m, n \geq 0$ the discriminant of $Q_{m,n}$ is

$$D_{m,n} = \prod_{k\in\mathcal{H}} k^{k\,p_k^2}$$

where $\mathcal{H}$ is the set of hooks k with multiplicity p_k

Discriminants of generalised Laguerre polynomials

Discriminants of $T_{m,n}^{(\mu)}(z)$:

$$\mathrm{Dis}_{1,1}(\mu) = (\mu+3)$$

$$\mathrm{Dis}_{1,2}(\mu) = (\mu+3)(\mu+4)^4(\mu+5)/2^4 3^3$$

$$\mathrm{Dis}_{1,3}(\mu) = (\mu+4)^2(\mu+5)^8(\mu+6)^4(\mu+7)/2^{24}3^8$$

$$\mathrm{Dis}_{2,1}(\mu) = (\mu+3)(\mu+4)^2/2^2 3$$

$$\mathrm{Dis}_{2,2}(\mu) = -(\mu+3)(\mu+4)^4(\mu+5)^8(\mu+6)^2/2^{24}3^8$$

$$\mathrm{Dis}_{2,3}(\mu) = -(\mu+4)^2(\mu+5)^8(\mu+6)^{16}(\mu+7)^8(\mu+8)^2/2^{60}3^{21}5^{11}$$

Discriminant of $T_{m,n}^{(\mu)}(z)$ for $n > m$

The discriminant of $T_{m,n}^{(\mu)}(z)$ for $n > m$ is

$$\begin{aligned}\mathrm{Dis}_{m,n}(\mu) = &\prod_{j=1}^{m} j^{j^3} \prod_{j=m+1}^{n} j^{j(m+1)^2} \\ &\times \prod_{j=n+1}^{m+n} j^{j(m+n-j+1)^2} \prod_{j=1}^{m} j^{2j(n-j)(j-1-m)} \prod_{j=1}^{m} (\mu+n+j)^{f(n-1,j)} \\ &\times \prod_{j=m+1}^{n} (\mu+n+j)^{f(m+n-j,m+1)} \prod_{j=n+1}^{m+n} (\mu+n+j)^{f(m,m+n+1-j)}\end{aligned}$$

where $f(m,p) = mp^2 - p(p-1)(p-2)/3$

Discriminant of $T_{m,n}^{(\mu)}(z)$ for $n > m$ in terms of hook lengths

The discriminant is

$$\mathrm{Dis}_{m,n}(\mu) = \prod_{k=1}^{m+n} k^{k\, p_k^2}$$
$$\times \prod_{k=1}^{m} k^{2k(n-k)(k-1-m)} \prod_{k=1}^{m} (\mu+n+k)^{f(n-1,p_1)}$$
$$\times \prod_{k=m+1}^{n} (\mu+n+k)^{f(m+n-k,p_2)} \prod_{k=n+1}^{m+n} (\mu+n+k)^{f(m,p_3)}$$

Factorisation of the generalised Laguerre polynomials

The generalised Laguerre polynomials have multiple roots at the origin when

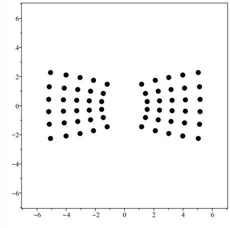
$$\mu = -n - j, \qquad j = 1, \ldots, m + n$$

Curiously

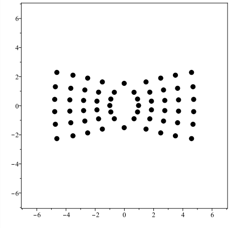
$$\begin{aligned} T_{m,n}^{(-n-j)}(z) &= c_1\, z^{nj}\, T_{m-j,n}^{(j-n)}(z) && j = 1, \ldots, m \\ T_{m,n}^{(-m-n-1)}(z) &= c_2\, z^{n(m+1)} && \\ T_{m,n}^{(-m-n-j)}(z) &= c_3\, z^{(m+1)(n+1-j)}\, T_{m,j-1}^{(-m-n-j)}(z) && j = 2, \ldots, n \end{aligned}$$

where c_1, c_2, c_3 are known constants

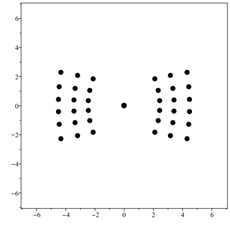
Root plots of $T_{6,4}^{(\mu)}(\frac{1}{2}z^2)$



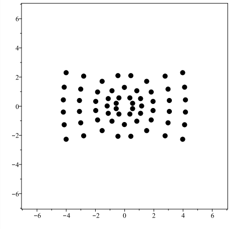
$\mu = -6$



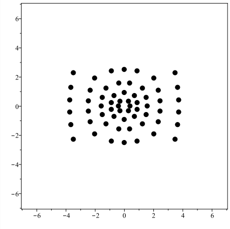
$\mu = -15/2$



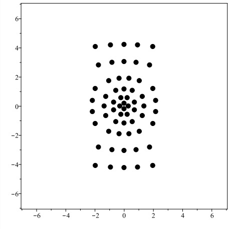
$\mu = -8$



$\mu = -17/2$

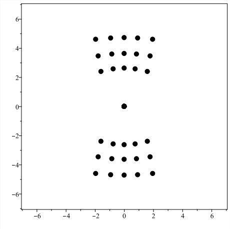


$\mu = -19/2$

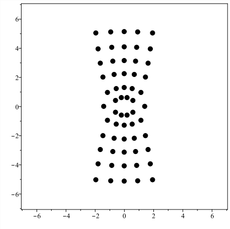


$\mu = -25/2$

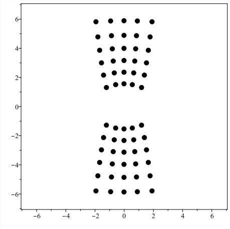
Root plots of $T_{6,4}^{(\mu)}(\frac{1}{2}z^2)$



$\mu = -14$

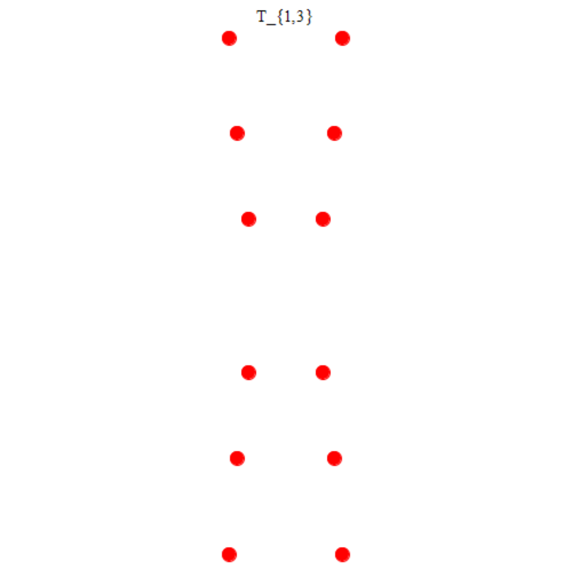


$\mu = -31/2$

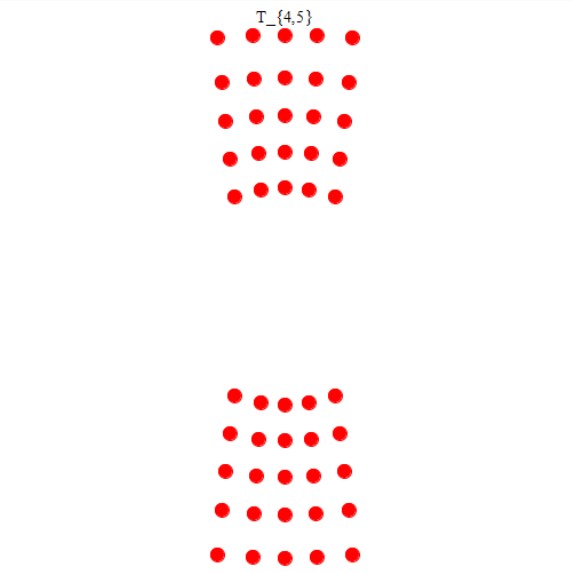


$\mu = -35/2$

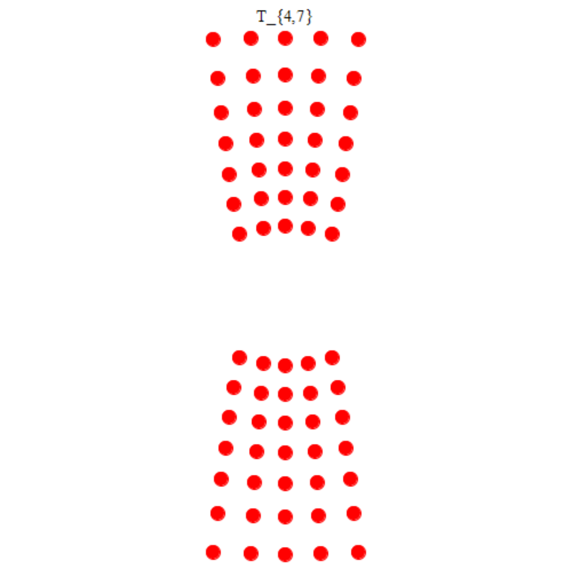
Roots of $T_{1,3}(z^2;\mu)$ for $\mu \in (-\infty, \infty)$ and $\lambda = (2^3)$



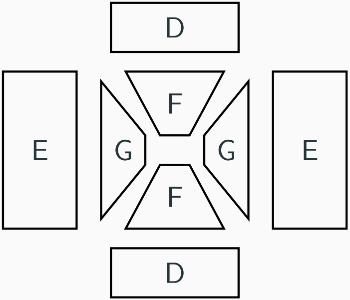
Roots of $T_{4,5}(z^2;\mu)$ for $\mu \in (-\infty,\infty)$ and $\lambda = (5^5)$

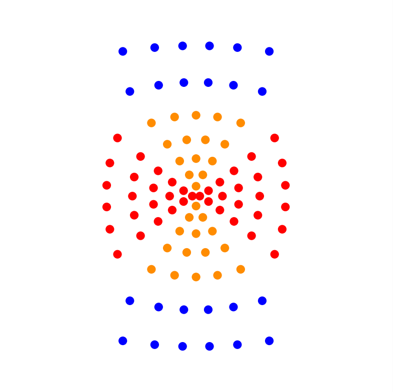


Roots of $T_{4,7}(z^2;\mu)$ for $\mu \in (-\infty,\infty)$ and $\lambda = (5^7)$

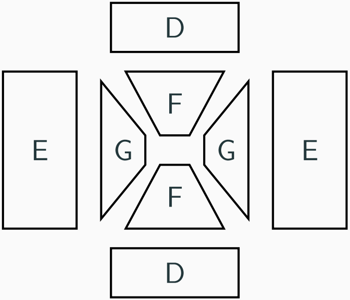


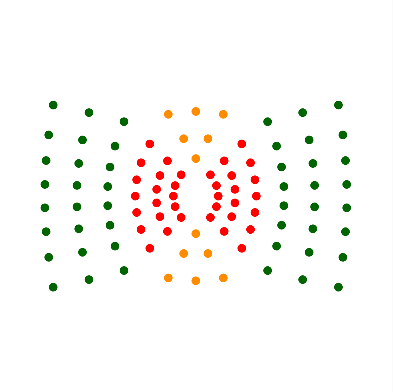
Allowed blocks and $T_{5,8}^{(-313/20)}(\frac{1}{2}z^2)$





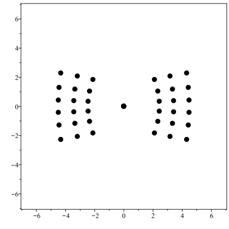
$T_{5,8}^{(-57/5)}\left(\frac{1}{2}z^2\right)$



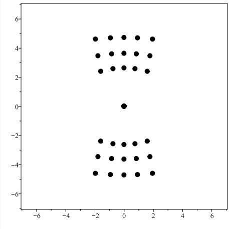


Classification of root blocks of $T^{(\mu)}_{m,n}(\frac{1}{2}z^2)$ when $n > m$ when there are zeros at origin

Condition		Number of zeros at origin	E rectangle	D rectangle
j	μ			
$1, \ldots, m+1$	$-n-j$	$2nj$	$m-j+1 \times n$	
$2, \ldots, n$	$-m-n-j$	$2(m+1)(n+1-j)$		$m+1 \times j-1$



$\mu = -8$



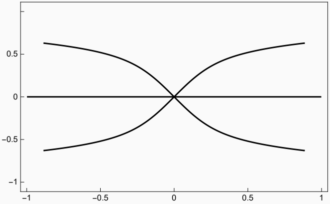
$\mu = -14$

Classification of root blocks of $T_{m,n}^{(\mu)}(\frac{1}{2}z^2)$ when $n > m$

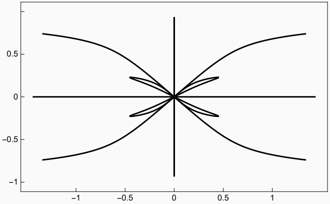
Condition $j = -n - \lceil \mu \rceil$	E rectangle	G trapezoid/ triangle	F triangle/ trapezoid	D rectangle
$j \leq 0$	$m+1 \times n$			
$1 < j < m+1$	$m+1-j \times n$	$n-1 \times n-j$	j	
$m+1 < j < n$		$m+n-j \times n-j$	m	$m+1 \times j-m$
$n < j < m+n$		$m+n-j$	$m \times j-n+1$	$m+1 \times j-m$
$j > m+n$				$m+1 \times n$

Root plot animation of $T_{5,3}(\frac{1}{2}z^2;\mu)$ for $\mu = -5$ to $\mu - = 6$

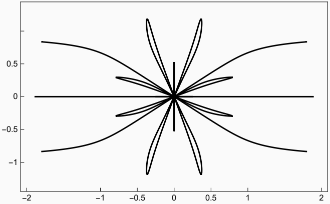
The movement of the roots of $T_{5,3}^{(\mu)}(\frac{1}{2}z^2)$ closest to the origin



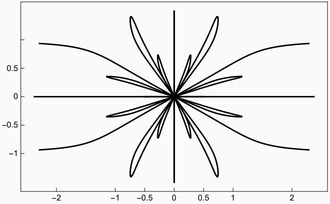
$\mu \in [-4, -3]$



$\mu \in [-5, -4]$

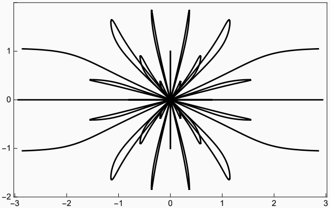


$\mu \in [-6, -5]$

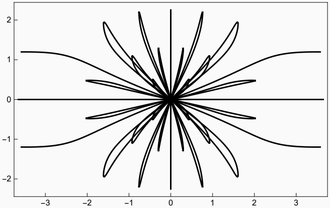


$\mu \in [-7, -6]$

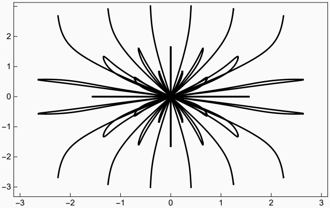
The movement of the roots of $T^{(\mu)}_{5,3}(\frac{1}{2}z^2)$ closest to the origin



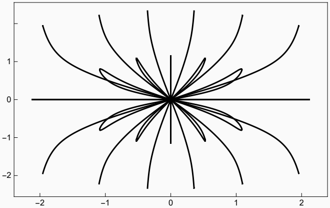
$\mu \in [-8, -7]$



$\mu \in [-9, -8]$

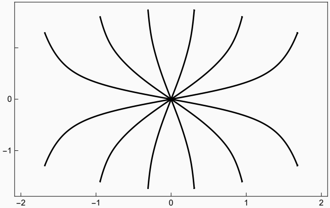


$\mu \in [-10, -9]$



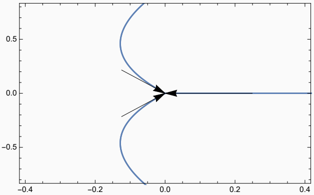
$\mu \in [-11, -10]$

The movement of the roots of $T_{5,3}^{(\mu)}(\frac{1}{2}z^2)$ closest to the origin

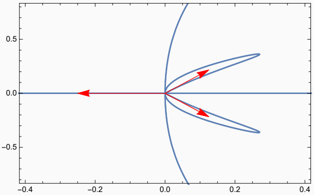


$\mu \in [-12, -11]$

Roots of $T_{2,3}^{(\mu)}(z)$ near origin

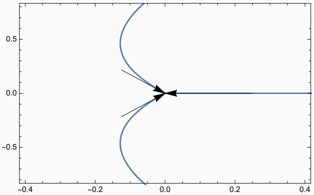


$\mu \in [-4, -3]$

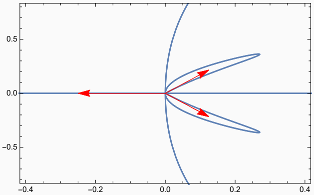


$\mu \in [-5, -4]$

Roots of $T_{2,3}^{(\mu)}(z)$ near origin



$\mu \in [-4, -3]$

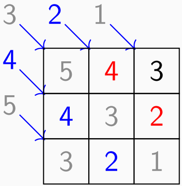


$\mu \in [-5, -4]$

j	μ	$\mu \to \mu^+$	$\mu \to \mu^-$
1	-4	$(z^3 - 1)$	$(z^3 + 1)$
2	-5	$(z^4 - 1)(z^2 + 1)$	$(z^4 + 1)(z^2 - 1)$
3	-6	$(z^5 - 1)(z^3 + 1)(z - 1)$	$(z^5 + 1)(z^3 - 1)(z + 1)$
4	-7	$(z^4 + 1)(z^2 - 1)$	$(z^4 - 1)(z^2 + 1)$
5	-8	$(z^3 - 1)$	$(z^3 + 1)$

Roots of $T_{2,3}^{(\mu)}$ near origin

j	μ	$\mu \to \mu^+$	$\mu \to \mu^-$
1	-4	$(z^3 - 1)$	$(z^3 + 1)$
2	-5	$(z^4 - 1)(z^2 + 1)$	$(z^4 + 1)(z^2 - 1)$
3	-6	$(z^5 - 1)(z^3 + 1)(z^1 - 1)$	$(z^5 + 1)(z^3 - 1)(z + 1)$
4	-7	$(z^4 + 1)(z^2 - 1)$	$(z^4 - 1)(z^2 + 1)$
5	-8	$(z^3 - 1)$	$(z^3 + 1)$



Rational solutions of Painlevé V

There are rational solutions of

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}$$

in terms of generalised Umemura polynomials $U_{m,n}(z;\mu)$ for the parameters

$$(\alpha,\beta,\delta,\gamma) = (\tfrac{1}{2}(m+\mu)^2, -\tfrac{1}{2}(n+\mu)^2, m+n, -1/2)$$

$$(\alpha,\beta,\delta,\gamma) = (\tfrac{1}{2}(m+\mu)^2, -\tfrac{1}{2}(n+\mu)^2, m-n, -1/2)$$

Generalised Umemura polynomials

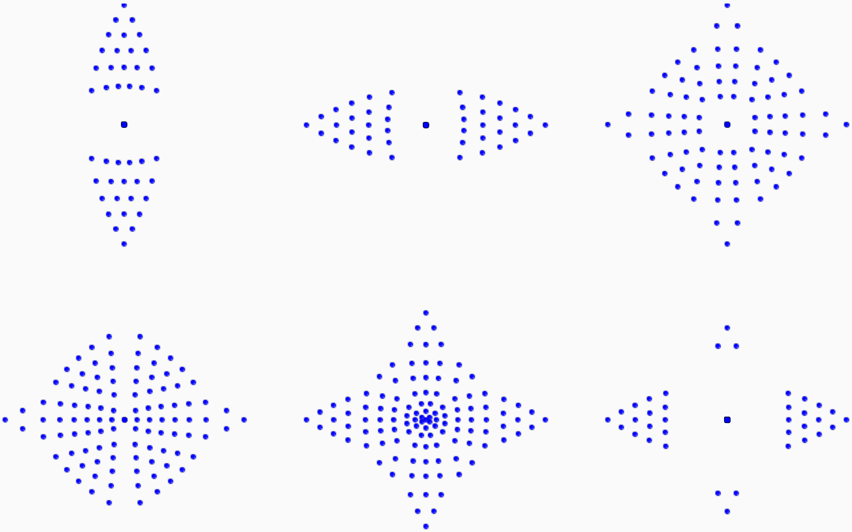
The polynomials $U_{m,n}(z;\mu)$ also have representations as a determinant or a Wronskian depending on two staircase partitions

$$\lambda_1 = (m, m-1, \ldots, 1) \quad , \quad \lambda_2 = (n, n-1, \ldots, 1)$$

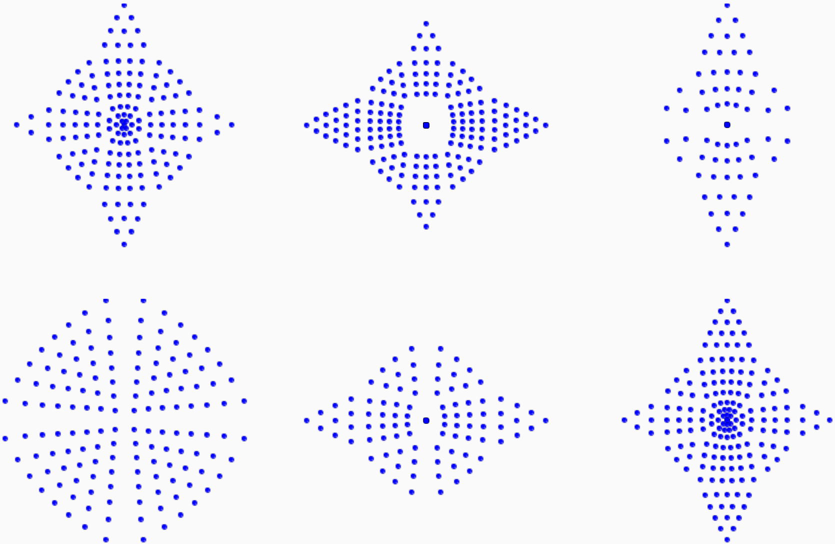
$$\begin{aligned} U_{m,n}(z;\mu) \;\; = \;\; & x^{n(m+\mu)} \operatorname{Wr}\big(L_m^{(\mu)}(z), L_{m+1}^{(\mu)}(z), \ldots, L_{2m-1}^{(\mu)}(z), \\ & x^{-\mu} L_n^{(\mu)}(z), x^{-\mu} L_{n+1}^{(\mu)}(z), \ldots, x^{-\mu} L_{2n-1}^{(\mu)}(z)\big) \end{aligned}$$

A Universal Character

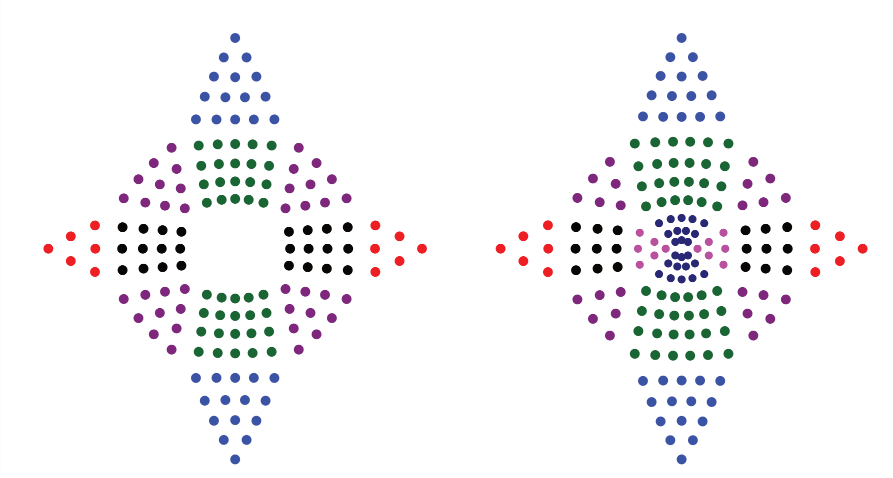
Generalised Umemura polynomials when $2\mu \in \mathbb{Z}$



More generalised Umemura polynomials when $2\mu \in \mathbb{Z}$



Generalised Umemura polynomials when $2\mu \in \mathbb{Z}$



Outlook

- Rational solutions of $\mathrm{P_V}$ are written in terms of generalised Laguerre polynomials or generalised Umemura polynomials
- Properties of generalised Laguerre polynomials given and some differential-difference and discrete equations found
- The dependence of the Wronskian polynomials on partitions is deeper than simply a means of labelling the constituent polynomials
- Why? How far does this extend to Universal Characters based on a generic pair of partitions