Set-theoretic YBE: quantum algebras & universal $\mathcal{R}$-matrices

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1. AD, arXiv:2405.04088.
2. AD, B. Rybolowicz, P. Stefanelli, arXiv:2401.12704.

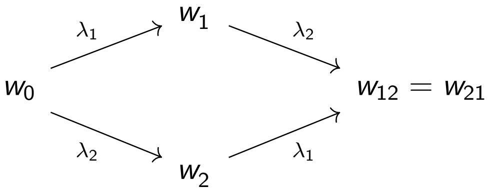
Review

- [*Drinfeld*] introduced the "Set-theoretic YBE".
- [*Hietiranta*] first to find examples of such solutions. [*Etingof, Shedler & Soloviev*] set-theoretic solutions & quantum groups for param. free R-matrices.
- Connections to: geometric crystals [*Berenstein & Kazhdan, Etingof*] and cellular automatons [*Hatayama, Kuniba & Takagi*]. Etingof rational solutions from geometric crystal theory.
- Classical discrete integrable systems (YB maps), quad-graph, discrete maps, solitons interactions: [*Veselov, Bobenko, Suris, Papageorgiou, Tongas,...*] **Parametric!**
- Set-theoretic involutive solutions of YBE from **braces**: [*Rump, Guarnieri, Vendramin, Gateva-Ivanova, Cedó, Jespers, Okniński, Smoktunowicz,...*]
- Connections to: *braid theory, Hopf algebras, knot theory, low dimensional topology, Hopf-Galois extensions, ternary structures, such as heaps & trusses ...*

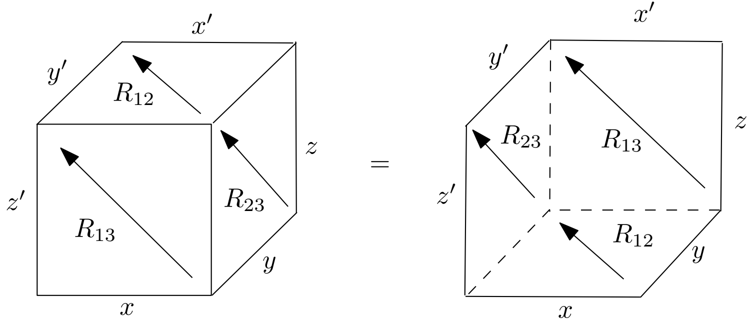
Motivation

- Non-parametric case: algebraic approach.
- Parametric case: discrete integrable systems and re-factorization problem (Bäcklund transform or discrete zero curvature condition), synonymous to Bianchi permutability: multi-solitons (soliton lattice). Also, **Cube or 3D consistency** condition 3D integrable discrete systems (time evolution).

Bianchi Permutability



3D Consistency Condition: YB Maps



Talk outline

- I will discuss the algebraic approach for the parametric case [*AD*]. Basic blueprint for the non-parametric case by [*AD, Rybolowicz, Stefanelli*].
- Introduce some preliminaries and motivations. Introduce the set-theoretic YBE and the notions of shelves, racks and quandles.
- Introduce the notions of parametric set-theoretic YBE and p-shelve and racks: parametric self-distributivity lead to solutions of the YBE
- Admissible Drinflel'd twist: all set theoretic solutions obtained form p-shelves (racks) and an admissible twist! Prototypical algebraic solutions presented.
- Formulate the underlying quasi-triangular Hopf-like algebraic structures. Well known examples of quantum algebras: Yangians and q-deformed algebras.
  A new paradigm of Quantum Algebra.

Preliminaries: Set theoretic-YBE

- Let a set $X = \{x_1, \ldots, x_{\mathcal{N}}\}$ and $\check{r} : X \times X \to X \times X$. Denote

Set-theoretic solution

$$\check{r}(x, y) = (\sigma_x(y), \tau_y(x))$$

1. $(X, \check{r})$ non-degenerate: σ_x and τ_y are bijective functions
2. $(X, \check{r})$ involutive: $\check{r}(\sigma_x(y), \tau_y(x)) = (x, y)$, $\check{r}^2 = \mathrm{id}$

- *Suppose $(X, \check{r})$ is an involutive, non-degenerate set-theoretic solution of the Braid equation*:

$$(\check{r} \times Id_X)(Id_X \times \check{r})(\check{r} \times Id_X) = (Id_X \times \check{r})(\check{r} \times Id_X)(Id_X \times \check{r}).$$

Matrices

- **Linearization:** $x_j \to e_{x_j}$, then $\mathbb{B} = \{e_{x_j}\}$, $x_j \in X$ is a basis of $V = \mathbb{C}X$ space of dimension equal to the cardinality of X. Recall, $e_{x,y} = e_x e_y^T$, $\mathcal{N} \times \mathcal{N}$ matrices. *Set-theoretic $\check{r}$ as $\mathcal{N}^2 \times \mathcal{N}^2$ matrix:*

Matrix form

$$\check{r} = \sum_{x,y \in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}$$

- **Baxterization for involutive solutions**: $\check{r} : V \otimes V \to V \otimes V$: $\check{r}^2 = I_{V \otimes V}$. Reps of the symmetric group. *Baxterization*:

$$\check{R}(\lambda) = \lambda \check{r} + \mathbb{I} \;\Rightarrow\; R(\lambda) = \lambda \mathbf{r} + \mathcal{P}$$

Define $r = \mathcal{P}\check{r}$. In the special case $\check{r} = \mathcal{P}$ ($r = \mathbb{I}$) we recover the **Yangian**. *If $\lambda = 0$ then $r = \mathcal{P} \to$ commuting Hamiltonians!*

Local Hamiltonians

- Results by [*AD & Smoktunowicz*] and [*AD*].

Local Hamiltonian

$$H = \sum_{n=1}^{N} \sum_{x,y \in X} e^{(n)}_{x,\sigma_x(y)}\ e^{(n+1)}_{y,\tau_y(x)}$$

Unlike Yangian, periodic Ham is *not* $\mathfrak{gl}_N$ **symmetric...***Surprise!*
(twisted Yangian coproduts, quasi bialgebra!).
Lyubashenko solution, $\sigma(y) = y+1,\ \tau(x) = x-1,\ mod\mathcal{N},\ x, y \in \{1, 2, \dots, \mathcal{N}\}$,

$$\boxed{H = \sum_{n=1}^{N} \sum_{x,y=1}^{\mathcal{N}} e^{(n)}_{x,y+1}\ e^{(n+1)}_{y,x-1}}$$

.

- Spectrum and eigenstates of commuting Hams challenging (symmetries of transfer matrix). Deriving Drinfeld twist key step (non-local maps [*Soloviev*])!
- q-deformed version of the involutive set-theoretic solutions has been constructed via an admissible Drinfeld twist.

Shelves, racks & quandles

- Shelves, racks & quandles [*Joyce, Matveev, Dehornoy,....*] satisfy axioms analogous to the Reidemeister moves used to manipulate knot diagrams. Link invariants, coloring of links a knot is tri-colored or not; Alexander's theorem: all links closed braids. $\to$ Special non-involutive set-theoretic solutions.

Definition

Let X be a non-empty set and $\triangleright$ a binary operation on X. Then, the pair $(X, \triangleright)$ is said to be a *left shelf* if $\triangleright$ is left self-distributive, namely, the identity

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$$

is satisfied, for all $a, b, c \in X$. Moreover, a left shelf $(X, \triangleright)$ is called

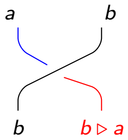
1. a *left rack* if $a\triangleright$ is bijective, for every $a \in X$.
2. a *quandle* if $(X, \triangleright)$ is a left rack and $a \triangleright a = a$, for all $a \in X$.

1. **Conjugation quandle.** Let $(X, \bullet)$ be a group and $\triangleright : X \times X \to X$, such that $a \triangleright b = a^{-1} \bullet b \bullet a$. Then $(X, \triangleright)$ is a quandle.
2. **Core quandle:** Let $(X, \bullet)$ be a group and $\triangleright : X \times X \to X$, such that $a \triangleright b = a \bullet b^{-1} \bullet a$. Then $(X, \triangleright)$ is a quandle.

Proposition

Let X be a non empty set, then the map $\check{r} : X \times X \to X \times X$, such that $\check{r}(a, b) = (b, b \triangleright a)$ is a solution of the braid equation if and only if $(X, \triangleright)$ is a shelve. The solution is invertible if and only if $(X, \triangleright)$ is a rack.

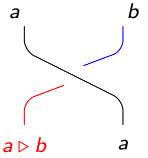
- Solutions from quandles **non-involutive**! All non-involutive set-theoretic solutions come from quandles by admissible Drinfeld twist [*AD, Rybolowicz, Stefanelli*].
- Extra motivation: q-deformed racks, quandles....from q braids.



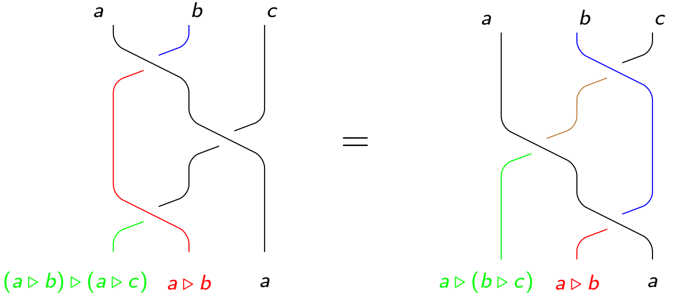
$$\check{r} = \sum_{a,b \in X} e_{a,b} \otimes e_{b, b \triangleright a}$$

- $\check{r}^{-1}(a, b) = (a \triangleright^{-1} b, a)$, $\check{r}(a, b) = (a \triangleright b, a)$ also solution of braid equ.

Self-distributivity - shelve solutions



$$\check{r} = \sum_{a,b \in X} e_{a, a \triangleright b} \otimes e_{b,a}$$



$$(\check{r} \times id)(id \times \check{r})(\check{r} \times id) \quad = \quad (id \times \check{r})(\check{r} \times id)(id \times \check{r})$$

Examples of quandles

- Let $i, j \in X := \{1, 2, \dots, n\}$ and define $i \triangleright j = 2i - j \ mod \ n$: $(X, \triangleright)$ is a quandle called the **dihedral quandle** (a core quandle).
- Special case [*Dehornoy*]. $n = 3$, $X = \{x_1, x_2, x_3\}$, $\triangleright : X \times X \to X$, such that:

$\triangleright$	x_1	x_2	x_3
x_1	x_1	x_3	x_2
x_2	x_3	x_2	x_1
x_3	x_2	x_1	x_3

- The 3D vector space. The canonical basis:

$$\hat{e}_{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \hat{e}_{x_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \hat{e}_{x_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Recall $\check{r} = \sum_{x,y\in X} e_{x,y} \otimes e_{y,y\triangleright x}$, where $e_{x,y}$ the elementary 3×3 matrix $e_{x,y} = e_x e_y^T$.
I.e. $\check{r} = \sum_{j=1}^{3} e_{x_j,x_j} \otimes e_{x_j,x_j} + e_{x_1,x_2} \otimes e_{x_2,x_3} + e_{x_2,x_1} \otimes e_{x_1,x_3} + \dots$

The $\check{r}$ matrix:

$$\check{r} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- $\check{r}^{-1} = \check{r}^T$. Unitary quantities from Twisted Yangian, [*AD*] in progress.
- Combinatorial matrices! [*Kauffman...*]: **qudits, topological quantum computing - braid gates.**

More quandles: Affine (or Alexander) quandles.

Let X be a non empty set equipped with two group operations, $+$ and $\circ$. Define $\triangleright : X \times X \to X$, such that for $z \in X$ and $\forall\ a, b \in X$, $a \triangleright b = -a \circ z + b \circ z + a$. Similar to a $\mathbb{Z}(t, t^{-1})$ ring module. (*For non-abelian* $(X, +)$ [*AD, Stefanelli, Rybolowicz*]).

KEY STATEMENTS.

1. All involutive set-theoretic solutions, $\check{r} = \sum_{a,b \in X} e_{a,\sigma_a(b)} \otimes e_{b,\tau_b(a)}$ come from the permutation operator via an *admissible Drilfenl'd twist* (similarity) [*AD*].
2. All generic **non**-involutive set-theoretic solutions come from quandle solutions operator via an *admissible Drilfenl'd twist* [*AD, Stefanelli, Rybolowicz*]. *To be generalized in the parametric case.*

Parametric set-theoretic YBE

- Let $X,\ Y \subseteq X$ be non-empty sets, $z_{i,j} \in Y,\ i,j \in \mathbb{Z}^+$ and $R^{z_{ij}} : X \times X \to X \times X$, such that for all $x, y \in X$, $R^{z_{ij}}(y,x) = \left(\sigma_x^{z_{ij}}(y), \tau_y^{z_{ij}}(x)\right)$. $(X, R^{z_{ij}})$ is a solution of the parametric, set-theoretic YBE if

Parametric set-theoretic YBE

$$R_{12}^{z_{12}}\ R_{13}^{z_{13}}\ R_{23}^{z_{23}} = R_{23}^{z_{23}}\ R_{13}^{z_{13}}\ R_{12}^{z_{12}}$$

$R_{12}^{z_{ij}}(c,b,a) = (\sigma_b^{z_{ij}}(c), \tau_c^{z_{ij}}(b), a)$, $R_{13}^{z_{ij}}(c,b,a) = (\sigma_a^{z_{ij}}(c), b, \tau_c^{z_{ij}}(a))$ and $R_{23}^{z_{ij}}(c,b,a) = (c, \sigma_a^{z_{ij}}(b), \tau_b^{z_{ij}}(a))$.

- $R^{z_{ij}}$ is a left non-degenerate if $\forall,\ z_{i,j} \in Y$, $\sigma_x^{z_{ij}}$ is a bijecton and non-degenerate if both $\sigma_x^{z_{ij}},\ \tau_y^{z_{ij}}$ are bijections. z_{ij} denotes dependence on (z_i, z_j).
- $R^{z_{ij}}$ is called "reversible" if $R_{21}^{z_{21}} R_{12}^{z_{12}} = \mathrm{id}$ [*Bobenko, Suris, Papageorgiou, Veselov*]. All solutions from discrete integrable systems are reversible.

- For the first time we present **non-unitary solutions** of the p set-theoretic YBE.
- Focus first on special type of solution $R^{z_{ij}} : X \times X \to X \times X$ such that $R^{z_{ij}}(a, b) = (a, a \triangleright_{z_{ij}} b)$.

Definition

Let $X,\ Y \subseteq X$ be non empty sets. We define for all $z_{i,j} \in Y$, the binary operation $\triangleright_{z_{ij}} : X \times X \to X$, $(a, b) \mapsto a \triangleright_{z_{ij}} b$. The pair $\left(X, \triangleright_{z_{ij}}\right)$ is said to be a *left parametric (p)-shelf* if $\triangleright_{z_{ij}}$ satisfies the generalized left p-self-distributivity:

$$a \triangleright_{z_{ik}} \left(b \triangleright_{z_{jk}} c\right) = \left(a \triangleright_{z_{ij}} b\right) \triangleright_{z_{jk}} \left(a \triangleright_{z_{ik}} c\right)$$

for all $a, b, c \in X$, $z_{i,j,k} \in Y$. Moreover, a left p-shelf $\left(X, \triangleright_{z_{ij}}\right)$ is called a left p-rack if the maps $L_a^{z_{ij}} : X \to X$ defined by $L_a^{z_{ij}}(b) := a \triangleright_{z_{ij}} b$, for all $a, b, \in X$, $z_{i,j} \in Y$, are bijective.

- Henceforth, whenever we say p-shelf or p-rack we mean left p-shelf or left p-rack.

Proposition

Let X, $Y \subseteq X$ be non empty sets. We define for $z_{i,j} \in Y$ the binary operation $\triangleright_{z_{ij}} : X \times X \to X$, $(a, b) \mapsto a \triangleright_{z_{ij}} b$. Then $R^{z_{ij}} : X \times X \to X \times X$, such that for all $a, b \in X$, $z_{i,j} \in Y$, $R^{z_{ij}}(b, a) = (b, b \triangleright_{z_{ij}} a)$ is a solution of the parametric set-theoretic Yang-Baxter equation if and only if $(X, \triangleright_{z_{ij}})$ is a p-shelf. If $R^{z_{ij}}$ invertible then $(X, \triangleright_{z_{ij}})$ is a p-rack.

Proof. Equating LHS and RHS of YBE.

Definition (skew braces)

[*Rump, Guarnieri & Vendramin*] A *left skew brace* is a set B together with two group operations $+, \circ : B \times B \to B$, the first is called addition and the second is called multiplication, such that for all $a, b, c \in B$,

$$a \circ (b + c) = a \circ b - a + a \circ c.$$

If $+$ is an abelian group operation B is called a *left brace*. Moreover, if B is a left skew brace and for all $a, b, c \in B$ $(b + c) \circ a = b \circ a - a + c \circ a$, then B is called a *two sided skew brace.*

- The additive identity of a skew brace B will be denoted by 0 and the multiplicative identity by 1. In every skew brace $0 = 1$. Braces $\to$ radical rings [*Rump, Smoktunowicz,...*]!

  From now on when we say skew brace we mean left skew brace.

Examples of braces

Example

1. Finite braces. Let $U(\mathbb{Z}/2^n\mathbb{Z}) =: U_n$ denote a set of odd integers mod 2^n, $n \in \mathbb{N}$. Define also $+_1 : U_n \times U_n \to U_n$, such that $a +_1 b := a - 1 + b$, for all $a, b \in U_n$. Moreover, $+$ is the usual addition and $\circ$ is the usual multiplication of integers. Then the triplet $(U_n, +_1, \circ)$ is a brace. For instance: 1. $n = 1$, $U_1 = \{1\}$, 2. $n = 2$, $U_2 = \{1,\ 3\}$, 3. $n = 3$, $U_2 = \{1,\ 3,\ 5,\ 7\}$ ...

Example

2. Infinite braces. Consider a set $O := \{\frac{2n+1}{2k+1} | n, k \in \mathbb{Z}\}$ together with two binary operations $+_1 : O \times O \to O$ such that $(a, b) \mapsto a - 1 + b$ and $\circ : O \times O \to O$ such that $(a, b) \mapsto a \circ b$, where $+$, $\circ$ are addition and multiplication of rational numbers, respectively. Then the triplet $(O, +_1, \circ)$ is a brace

Solutions from p-racks

Proposition

Let $(X, +, \circ)$ be a skew brace and $Y \subseteq X$, such that

- for all $a, b \in X$, $z \in Y$, $(a+b) \circ z = a \circ z - z + b \circ z$,
- $z \in Y$ are central in $(X, +)$.

Define also for all $z_{i,j} \in Y$ the binary operation $\triangleright_{z_{ij}} : X \times X \to X$, such that for all $a, b \in X$,

1. $a \triangleright_{z_{ij}} b = -a \circ z_i \circ z_j^{-1} + b + a \circ z_i \circ z_j^{-1}$.
2. $a \triangleright_{z_{ij}} b = -a \circ z_i \circ z_j^{-1} \circ z + b \circ z + a \circ z_i \circ z_j^{-1}, \;\; z \in Y$.

Then the map $R^{z_{ij}} : X \times X \to X \times X$, such that for all $a, b \in X$, $z_{i,j} \in Y$,

$$R^{z_{ij}}(a, b) = (a, a \triangleright_{z_{ij}} b)$$

is a solution of the parametric Yang-Baxter equation. The map $R^{z_{ij}}$ is invertible.

Proof. It suffices to show parametric self-disctributivity for $\triangleright_{z_{ij}}$, which indeed holds. Also, $\triangleright_{z_{ij}}$, is a bijection indeed.

- **Remark.** In the special case where $(X, +, \circ)$ is a brace, i.e. $(X, +)$ is an abelian group, then in cae 1, for all $a, b \in X$, $z_{i,j} \in Y$, $a \triangleright_{z_{ij}} b = b$, and hence $R^{z_{ij}} = \mathrm{id}$.

Generic solutions

- We focus on the generic solution of the set-theoretic YBE, $R^{z_{ij}} : X \times X \to X \times X$, such that for all $a, b \in X$, $z_{i,j} \in Y$,

$$R^{z_{ij}}(b, a) = (\sigma_a^{z_{ij}}(b), \tau_b^{z_{ij}}(a))$$

- In this case, p-biracks and p-biquandles (**two binary operations**). Biracks and biquandles: virtual links and braids (ribbons).
- Generic solution obtained via admisssible Drinfeld twist!!

Definition

Let $(X, \triangleright_{z_{ij}})$ be a p-shelf. We say that the twist $\varphi^{z_{ij}} : X \times X \to X \times X$, such that $\varphi^{z_{ij}}(a, b) := (a, \sigma_a^{z_{ji}}(b))$ for all $a, b \in X$, $z_{i,j} \in Y$ is admissible, if for all $a, b, c \in X$, $z_{i,j,k} \in Y$: $(\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) = \sigma^{z_{ij}}_{\sigma_a^{z_{jk}}(b)}(\sigma^{z_{ik}}_{\tau_b^{z_{jk}}(a)}(c))$ & $\sigma_c^{z_{ik}}(b) \triangleright_{z_{ij}} \sigma_c^{z_{jk}}(a) = \sigma_c^{z_{jk}}(b \triangleright_{z_{ij}} a)$.

Admissible twists & general solutions

Theorem

Let $(X, \triangleright_{z_{ij}})$ be a p-shelf and $\varphi^{z_{ij}} : X \times X \to X \times X$, such that $\varphi^{z_{ij}}(a,b) := (a, \sigma_a^{z_{ji}}(b))$ for all $a, b \in X$, $z_{i,j} \in Y$. Then, the map $R^{z_{ij}} : X \times X \to X \times X$ defined by

$$R^{z_{ij}}(a,b) = \Big(\sigma_a^{z_{ij}}(b)\,,\, (\sigma^{z_{ji}}_{\sigma_a^{z_{ij}}(b)})^{-1}(\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a)\Big)$$

for all $a, b \in X$, $z_{i,j} \in Y$ is a solution if and only if $\varphi^{z_{ij}}$ is an admissible twist.

Proof. The proof is involved based on the (1), (2) of the Definition of the adm. twist and the fundamental relations from the YBE. $R^{z_{ij}} = (\varphi^{z_{ij}})^{-1} S^{z_{ij}} (\varphi^{z_{ji}})^{(op)}$, where $S^{z_{ij}}(x,y) = (x, x \triangleright_{z_{ij}} y)$.

• **Conclusion.** *The problem of generic solutions of the p set-theoretic Yang-Baxter equation is reduced to the classification of p-shelve/rack solutions & admissible twists.*

• Explicit solutions derived [*AD*].

- Back to the linearized version, recall:
  1. $R^{z_{ij}} = \sum_{a,d\in X} e_{b,\sigma_a^{z_{ij}}(b)} \otimes e_{a,\tau_b^{z_{ij}}(a)}$, generic set-theoretic solutions:
  2. $R^{z_{ij}} = \sum_{a,b\in X} e_{b,a} \otimes e_{a,b\triangleright_{z_{ij}} a}$, p-shelves solutions,
- Linearization formally generalizes to infinite countable sets & for compact sets, use of functional analysis and study of kernels of integral operators required.
- We establish the algebraic framework in the tensor product formulation. This naturally provides solutions to the parametric set-theoretic YBE, thus the linearized version is essential in what follows.
- Next, explore algebraic structures that provide universal $\mathcal{R}$-matrices associated to p-rack and general set-theoretic solutions of the YBE.

p-rack algebras

Definition

Let $Y \subseteq X$ be non-empty sets. We define for all $z_{i,j,k} \in Y$, the binary operation, $\triangleright_{z_{ij}} : X \times X \to X$, $(a, b) \mapsto a \triangleright_{z_{ij}} b$. Let also $(X, \triangleright_{z_{ij}})$ be a finite magma, or such that $a \triangleright_{z_{ij}}$ is surjective, for every $a \in X$, $z_{i,j} \in Y$. We say that the unital, associative algebra $\mathcal{Q}$, over a field k generated by, $1_{\mathcal{Q}}$, $q_a^{z_{ij}}$, $(q_a^{z_{ij}})^{-1}$, $h_a \in \mathcal{Q}$ $(h_a = h_b \Leftrightarrow a = b)$ and relations for all $a, b \in X$, $z_{i,j,k} \in Y$:

$$q_a^{z_{ij}}(q_a^{z_{ij}})^{-1} = (q_a^{z_{ij}})^{-1}q_a^{z_{ij}} = 1_{\mathcal{Q}}, \quad q_a^{z_{jk}}q_b^{z_{ik}} = q_b^{z_{ik}}q_{b \triangleright_{z_{ij}} a}^{z_{jk}},$$

$$h_a h_b = \delta_{a,b} h_a, \quad q_b^{z_{ij}} h_{b \triangleright_{z_{ij}} a} = h_a q_b^{z_{ij}}$$

is a p-rack algebra.

The choice of the name p-rack algebra is justified by the following result.

Proposition

Let $\mathcal{Q}$ be the p-rack algebra, then for all $a, b, c \in X$ and $z_{i,j,k} \in Y$,
$c \triangleright_{z_{ik}} (b \triangleright_{z_{jK}} a) = (c \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (c \triangleright_{z_{ik}} a)$, i.e. $(X, \triangleright_{z_{ij}})$ is a p-rack.

Proof. We compute $h_a q_b^{z_{jk}} q_c^{z_{ik}}$ using the **associativity** of the algebra, also due to invertibility of $q_a^{z_{ij}}$ for all $a \in X$, $z_{i,j} \in Y$:

$$h_{c \triangleright_{z_{ik}} (b \triangleright_{z_{jk}} a)} = h_{(c \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (c \triangleright_{z_{ik}} a)} \;\Rightarrow\; c \triangleright_{z_{ik}} (b \triangleright_{z_{jk}} a) = (c \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (c \triangleright_{z_{ik}} a).$$

Also, $a \triangleright_{z_{ij}}$ is bijective and thus $(X, \triangleright_{z_{ij}})$ is a p-rack.

The universal R-matrix

Proposition

Let $\mathcal{Q}$ be the p-rack algebra and $\mathcal{R}^{z_{ij}} \in \mathcal{Q} \otimes \mathcal{Q}$ be an invertible element, such that $\mathcal{R}^{z_{ij}} = \sum_a h_a \otimes q_a^{z_{ij}}$, $z_{i,j} \in Y$. Then $\mathcal{R}^{z_{ij}}$ satisfies the parametric Yang-Baxter equation

$$\mathcal{R}_{12}^{z_{12}} \mathcal{R}_{13}^{z_{13}} \mathcal{R}_{23}^{z_{23}} = \mathcal{R}_{23}^{z_{23}} \mathcal{R}_{13}^{z_{13}} \mathcal{R}_{12}^{z_{12}}$$

$\mathcal{R}_{12}^{z_{12}} = \sum_{a \in X} h_a \otimes q_a^{z_{12}} \otimes 1_{\mathcal{Q}}$, $\mathcal{R}_{13}^{z_{13}} = \sum_{a \in X} h_a \otimes 1_{\mathcal{Q}} \otimes q_a^{z_{13}}$, and $\mathcal{R}_{23}^{z_{23}} = \sum_{a \in X} 1_{\mathcal{Q}} \otimes h_a \otimes q_a^{z_{23}}$. The inverse $\mathcal{R}$-matrix is $(\mathcal{R}^{z_{ij}})^{-1} = \sum_{a \in X} h_a \otimes (q_a^{z_{ij}})^{-1}$.

Proof. From YBE and p-rack algebra relations. Also, $(\mathcal{R}^{z_{ij}})^{-1} = \sum_{a \in X} h_a \otimes (q_a^{z_{ij}})^{-1}$.

- **Fundamental representation:** Recall, $e_{i,j}$, $n \times n$ matrices with elements $(e_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$. Let $\mathcal{Q}$ be the p-rack algebra and $\rho : \mathcal{Q} \to \mathrm{End}(V)$, defined by $q_a^{z_{ij}} \mapsto \sum_{x \in X} e_{x, a \triangleright_{z_{ij}} x}$, $\quad h_a \mapsto e_{a,a}$. Then $\mathcal{R}^{z_{ij}} \mapsto R^{z_{ij}} = \sum_{a,b \in X} e_{b,b} \otimes e_{a, b \triangleright_{z_{ij}} a}$: the linearized p-rack solution.

Definition

A p-rack algebra $\mathcal{Q}$ is called a restricted p-rack algebra if for all $z_{i,j} \in Y$ there exits a binary operation $\bullet_{z_{ij}} : X \times X \to X$, $(a, b) \mapsto a \bullet_{z_{ij}} b$, such that, $a \bullet_{z_{ij}}$, is a bijection and $a \bullet_{z_{ji}} b = b \bullet_{z_{ij}} (b \triangleright_{z_{ij}} a)$, for all $a, b \in X$, $z_{i,j} \in Y$.

- **NOTE.** In the parameter free case: motivated by pre-Lie algebras (chronological algebras) [*Agrachev, Gerstenhaber.... *] introduce the **pre-Lie skew brace**. Identified families of affine quandles that generate pre-Lie skew braces [*AD, Rybolowicz, Stefanelli*].

Theorem

Let $\mathcal{Q}$ be the restricted p-rack algebra and $\mathcal{R}^{z_{ij}} = \sum_a h_a \otimes q_a^{z_{ij}} \in \mathcal{Q} \otimes \mathcal{Q}$ be a solution of the Yang-Baxter equation. Moreover, assume that for all $z_{i,j,k} \in Y$, $a, b \in X$, $(b \triangleright_{z_{ij}} a_1) \bullet_{z_{jk}} (b \triangleright_{z_{ik}} a_2) = b \triangleright_{z_{ij}} (a_1 \bullet_{z_{jk}} a_2)$. We also define for $z_{i,j,k} \in Y$, $\Delta_{z_{ij}} : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$, such that for all $a \in X$,

$$\Delta_{z_{jk}}((q_a^{z_{ik}})^{\pm 1}) := (q_a^{z_{ij}})^{\pm 1} \otimes (q_a^{z_{ik}})^{\pm 1}, \quad \Delta_{z_{ij}}(h_a) := \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \bullet_{z_{ij}} c = a}.$$

Then the following statements hold:

1. $\Delta_{z_{ij}}$ is a $\mathcal{Q}$ algebra homomorphism for all $z_{i,j} \in Y$.
2. $\mathcal{R}^{z_{jk}} \Delta_{z_{jk}}(y) = \Delta^{(op)}_{z_{kj}}(y) \mathcal{R}^{z_{jk}}$, for all $z_{j,k} \in Y$, $y \in \{h_a, \ q_a^{z_{ik}}\}$. Recall $\Delta^{(op)}_{z_{ij}} := \pi \circ \Delta_{z_{ij}}$, where π is the flip map.

Parametric co-associativity

- **Proposition.** Let $\mathcal{Q}$ be the restricted p-rack algebra, assume also that for all $a, b, c \in X$ and $z_{i,j,k} \in Y$, $(b \triangleright_{z_{ij}} a) \bullet_{z_{jk}} (b \triangleright_{z_{ik}} c) = b \triangleright_{z_{ik}} (a \bullet_{z_{jk}} c)$ and $(a \bullet_{z_{ij}} b) \bullet_{z_{jk}} c = a \bullet_{z_{ik}} (b \bullet_{z_{jk}} c)$.

  We also define for $z_{i,1,2,\ldots,n} \in Y$, $\Delta^{(n)}_{z_{12\ldots n}} : \mathcal{Q} \to \mathcal{Q}^{\otimes n}$, such that

$$\Delta^{(n)}_{z_{12\ldots n}}((q_a^{z_{in}})^{\pm 1}) = (q_a^{z_{i1}})^{\pm 1} \otimes (q_a^{z_{i2}})^{\pm 1} \otimes \ldots (\otimes q_a^{z_{in}})^{\pm 1},$$
$$\Delta^{(n)}_{z_{12\ldots n}}(h_a) := \sum_{a_1,\ldots,a_n \in X} h_{a_1} \otimes h_{a_2} \otimes \ldots \otimes h_{a_n}\Big|_{\Pi_{z_{1\ldots n}}(a_1,a_2,\ldots,a_n)=a},$$

where for all $a_1, a_2, \ldots, a_n \in X$, $z_1, \ldots, z_n \in Y$:

$$\begin{aligned}
\Pi_{z_{12}}(a_1, a_2): &= a_1 \bullet_{z_{12}} a_2 \\
\Pi_{z_{12\ldots n}}(a_1, a_2, \ldots, a_n): &= a_1 \bullet_{z_{1n}} (a_2 \bullet_{z_{2n}} (a_3 \ldots \bullet_{z_{n-2n}} (a_{n-1} \bullet_{z_{n-1n}} a_n) \ldots)) \\
&= ((\ldots((a_1 \bullet_{z_{12}} a_2) \bullet_{z_{23}} a_3) \ldots a_{n-1}) \bullet_{z_{n-1n}} a_n, \quad n > 2.
\end{aligned}$$

Then:

1. For all $z_{i,1,2,\ldots n} \in Y$,

$$\Delta^{(n)}_{z_{12\ldots n}} := (\Delta^{(n-1)}_{z_{12\ldots n-1}} \otimes \mathrm{id})\Delta_{z_{n-1n}} = (\mathrm{id} \otimes \Delta^{(n-1)}_{z_{23\ldots n}})\Delta_{z_{1n}}.$$

2. For all $a, b \in X$, $z_{i,1,2,\ldots n} \in Y$, $\Delta^{(n)}_{z_{12\ldots n}}$ is an algebra homomorphism.

Example

Consider the binary operations $\bullet_{z_{ij}}, \ \triangleright_{z_{ij}} : X \times X \to X$ such as $a \bullet_{z_{ij}} b = a \circ z_i + b \circ z_j$ and $a \triangleright_{z_{ij}} b = -a \circ z_i \circ z_j^{-1} + b + a \circ z_i \circ z_j^{-1}$, then for all $a, b, c \in X$, $z_{i,j,k} \in Y$,

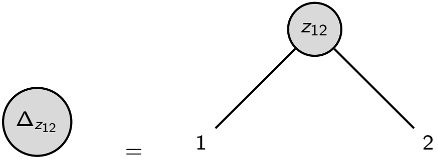
$$(a \triangleright_{z_{ij}} b) \bullet_{z_{jk}} (b \triangleright_{z_{ik}} c) = a \triangleright_{z_{io}} (b \bullet_{z_{jk}} c), \quad (a \bullet_{z_{ij}} b) \bullet_{z_{ok}} c = a \bullet_{z_{io}} (b \bullet_{z_{jk}} c),$$

where $z_o = 1$ and

$$\Pi_{z_1 \ldots z_n}(a_1, a_2, \ldots, a_n) = a_1 \circ z_1 + a_2 \circ z_2 + \ldots + a_{n-1} \circ z_{n-1} + a_n \circ z_n.$$

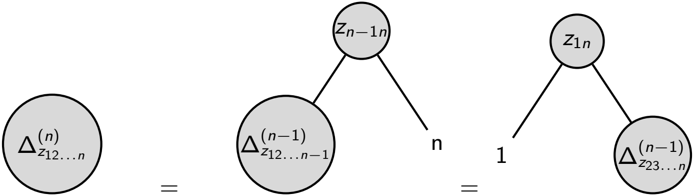
Binary Trees

Graphical representation of the parametric co-product $\Delta_{z_{12}}$:



The n^{th} coproduct $\Delta^{(n)}_{z_{12\ldots n}}$, $a \in X$, $z_{k,1,2,\ldots n} \in Y$ is depicted by 2^{n-2} equivalent diagrams.

$$\Delta^{(n)}_{z_{12\ldots n}} := (\Delta^{(n-1)}_{z_{12\ldots n-1}} \otimes \mathrm{id})\Delta_{z_{n-1n}} = (\mathrm{id} \otimes \Delta^{(n-1)}_{z_{23\ldots n}})\Delta_{z_{1n}}.$$



Unfolding $\Delta^{(n-1)}$ in the LHS and RHS produces 2^{n-2} binary tree diagrams.

The parameter free case: quasi-triangular Hopf algebra

- The p-rack algebra reduces to a *rack algebra* in the parameter free case. In this case one recovers a quasi-triangular Hopf algebra if $(X, \bullet)$ is a group [*AD, Rybolowicz, Stefanelli*].

Theorem

Let $\mathcal{A}$ be a rack algebra, with $(X, \bullet, e)$ being a group. Let also $\mathcal{R} = \sum_{a \in X} h_a \otimes q_a$ be a solution of the Yang-Baxter equation and $q_a \in \mathcal{A}$ are such that $q_a q_b = q_{a \bullet b}$. Then the structure $(\mathcal{A}, \Delta, \epsilon, S, \mathcal{R})$ is a quasi-triangular Hopf algebra:

- Co-product. $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\Delta(q_a^{\pm 1}) = q_a^{\pm 1} \otimes q_a^{\pm 1}$ and $\Delta(h_a) = \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \bullet c = a}$.
- Co-unit. $\epsilon : \mathcal{A} \to k$, $\epsilon(q_a^{\pm 1}) = 1$, $\epsilon(h_a) = \delta_{a,e}$.
- Antipode. $S : \mathcal{A} \to \mathcal{A}$, $S(q_a^{\pm 1}) = q_a^{\mp 1}$, $S(h_a) = h_{a^*}$, where a^* is the inverse in $(X, \bullet)$ for all $a \in X$.

- Relevant: Pointed Hopf Algebras from racks [*Andruskiewitsch & Grana*].

The p-decorated algebra

1. Let $\mathcal{Q}$ be the p-rack algebra. Let also $\sigma_a^{z_{ij}},\ \tau_b^{z_{ij}} : X \to X$, and $\sigma_a^{z_{ij}}$ be a bijection for all $a \in X$, $z_{i,j} \in Y$. We say that the unital, associative algebra $\hat{\mathcal{Q}}$ over k, generated by intedeterminates $q_a^{z_{ij}}, (q_a^{z_{ij}})^{-1}, h_a, \in \mathcal{Q}$ and $w_a^{z_{ij}}, (w_a^{z_{ij}})^{-1} \in \hat{\mathcal{A}}$, $a \in X$, $1_{\hat{\mathcal{Q}}} = 1_{\mathcal{Q}}$ is the unit element and relations, for $a, b \in X$, $z_{i,j,k} \in Y$:

Decorated p-rack algebras

$$q_a^{z_{ij}}(q_a^{z_{ij}})^{-1} = (q_a^{z_{ij}})^{-1} q_a^{z_{ij}} = 1_{\hat{\mathcal{Q}}}, \quad q_a^{z_{jk}} q_b^{z_{ik}} = q_b^{z_{ik}} q_{b \triangleright_{z_{ij}} a}^{z_{jk}}, \quad h_a h_b = \delta_{a,b} h_a,$$

$$q_b^{z_{ij}} h_{b \triangleright_{z_{ij}} a} = h_a q_b^{z_{ij}} \quad w_a^{z_{ij}}(w_a^{z_{ij}})^{-1} = 1_{\hat{\mathcal{A}}}, \quad w_a^{z_{ki}} w_b^{z_{ji}} = w^{z_{ji}}_{\sigma_a^{z_{jk}}(b)} w^{z_{ki}}_{\tau_b^{z_{kj}}(a)}$$

$$w_a^{z_{ji}} h_b = h_{\sigma_a^{z_{ij}}(b)} w_a^{z_{ji}}, \quad w_a^{z_{kj}} q_b^{z_{ij}} = q^{z_{ij}}_{\sigma_a^{z_{ik}}(b)} w_a^{z_{kj}}$$

is a *decorated p-rack algebra*.

- **Proposition.** Let $\hat{\mathcal{Q}}$ be the decorated p-rack algebra, then for all $a, b, c \in X$, $z_{i,j,k} \in Y$:

$$\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) = \sigma^{z_{ij}}_{\sigma_a^{z_{jk}}(b)}(\sigma^{z_{ik}}_{\tau_b^{z_{jk}}(a)}(c)) \quad \& \quad \sigma_c^{z_{ik}}(b) \triangleright_{z_{ij}} \sigma_c^{z_{jk}}(a) = \sigma_c^{z_{jk}}(b \triangleright_{z_{ij}} a).$$

Proof. Follow from the algebra associativity. **These are the conditions of the Def of an admissible twist!**

- **Proposition.** Let $\hat{\mathcal{Q}}$ be the decorated p-rack algebra and $\mathcal{R}^{z_{ij}} = \sum_a h_a \otimes q_a^{z_{ij}} \in \mathcal{Q} \otimes \mathcal{Q}$ be a solution of the Yang-Baxter equation. We also define for $z_{i,j,k} \in Y$, $\Delta_{z_{ij}} : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$, such that for all $a \in X$,

$$\Delta_{z_{jk}}((y_a^{z_{ik}})^{\pm 1}) := (y_a^{z_{ij}})^{\pm 1} \otimes (y_a^{z_{ik}})^{\pm 1}, \quad \Delta_{z_{ij}}(h_a) := \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \bullet_{z_{ij}} c = a}.$$

$y_a^{z_{ik}} \in \{q_a^{z_{ik}},\ w_a^{z_{ik}}\}$.

Then the following statements hold:

1. $\Delta_{z_{ij}}$ is a $\hat{\mathcal{Q}}$ algebra homomorphism for all $z_{i,j} \in Y$.
2. $\mathcal{R}^{z_{jk}} \Delta_{z_{jk}}(y_a^{z_{ik}}) = \Delta^{(op)}_{z_{kj}}(y_a^{z_{ik}}) \mathcal{R}^{z_{jk}}$, for $y_a^{z_{ik}} \in \{q_a^{z_{ik}},\ w_a^{z_{ik}}\}$, $a \in X$, $z_{i,j,k} \in Y$. Recall, $\Delta^{(op)}_{z_{ij}} := \pi \circ \Delta_{z_{ij}}$, where π is the flip map.

Universal R-matrix by twisting

- **Proposition.** Let $\mathcal{R}^{z_{ij}} = \sum_{a\in X} h_a \otimes q_a^{z_{ij}} \in \mathcal{Q}\otimes\mathcal{Q}$ be the p-rack universal $\mathcal{R}-$matrix. Let also $\hat{\mathcal{Q}}$ be the decorated p-rack algebra and $\mathcal{F}^{z_{ij}} \in \hat{\mathcal{Q}}\otimes\hat{\mathcal{Q}}$, such that $\mathcal{F}^{z_{ij}} = \sum_{b\in X} h_b \otimes (w_b^{z_{ij}})^{-1}$ (invertible) for all $z_{i,j} \in Y$ then $\mathcal{F}$ is an admissible twist. *This guarantees that if $\mathcal{R}$ is a solution of the YBE then $\mathcal{R}^f$ also is!*

- The twisted $\mathcal{R}-$matrix:

$$\mathcal{R}^{Fz_{12}} = (\mathcal{F}^{z_{21}})^{(op)}\mathcal{R}^{z_{12}}(\mathcal{F}^{z_{12}})^{-1}$$

- The twisted coproducts: for $z_{12} \in Y$, $\Delta^F_{z_{12}}(y) = \mathcal{F}^{z_{12}}\Delta_{z_{12}}(y)(\mathcal{F}^{z_{12}})^{-1}$, $y \in \hat{\mathcal{Q}}$. Moreover it follows that $\mathcal{R}^{Fz_{21}}\Delta^F_{z_{12}}(y) = \Delta^{F(op)}_{z_{12}}(y)\mathcal{R}^{Fz_{12}}$, $y \in \hat{\mathcal{Q}}$, $z_{1,2} \in Y$.

- **Fundamental representation & the set-theoretic solution:**
  Let $\hat{\mathcal{Q}}$ be the decorated p-rack algebra, $\rho : \hat{\mathcal{Q}} \to \mathrm{End}(V)$, such that

$$q_a^{z_{ij}} \mapsto \sum_a e_{x, a \triangleright_{z_{ij}} x}, \quad h_a \mapsto e_{a,a}, \quad w_a^{z_{ij}} \mapsto \sum_{b \in X} e_{\sigma_a^{z_{ji}}(b), b},$$

  then $\mathcal{R}^{Fz_{ij}} \mapsto R^{Fz_{ij}} = \sum_{a,b \in X} e_{b, \sigma_a^{z_{ij}}(b)} \otimes e_{a, \tau_b^{z_{ij}}(a)}$, where
  $\tau_b^{z_{ij}}(a) := \sigma^{z_{ji}}_{(\sigma_a^{z_{ij}})^{-1}(b)}(\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a)$.

  $R^{Fz_{ij}}$ is the linearized version of the set-theoretic solution.

- The associated quantum algebra (non-parametric case) is a *quasi-triangular quasi Hopf algebra* [*AD, Vlaar, Ghionis*].