Bethe state preparation

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Introduction

Quantum state preparation problem:

?

How to prepare a given quantum state (say, an $\boxed{\text{eigenstate of a Hamiltonian}}$) on a quantum computer?

Applications: computing e.g. correlation functions in this state

Attractive candidate: Bethe state (eigenstate of an *integrable* Hamiltonian)

- better understood
- non-trivial

Goal of today's talk: algorithm for preparing Bethe states of the Heisenberg quantum spin chain

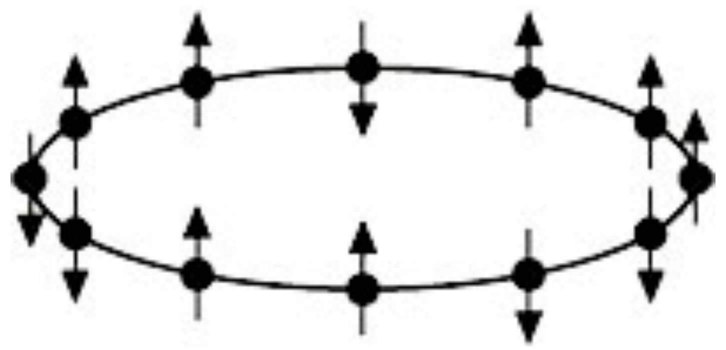
David Raveh, RN arXiv: 2403.03283

Outline

1. Bethe ansatz review
2. Dicke state preparation
3. Bethe state preparation
4. Bethe roots from VQE
5. Outlook

1. Bethe ansatz review

closed isotropic (XXX) spin-1/2 Heisenberg quantum spin chain



Deceptively simple model of magnetism (1928)

- L spin-1/2 spins (qubits) arranged in a circle
- Each spin interacts isotropically with its neighbor

Hamiltonian

$$\boxed{H = -\frac{1}{2}\sum_{n=1}^{L}(\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - I)}$$

$$\vec{\sigma}_{L+1} \equiv \vec{\sigma}_1$$

periodic boundary conditions

$$\vec{\sigma}_n = \underset{1}{\mathbb{I}} \otimes \cdots \mathbb{I} \otimes \underset{n}{\vec{\sigma}} \otimes \mathbb{I} \cdots \otimes \underset{L}{\mathbb{I}}$$

Pauli spin matrices

$$\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z) \qquad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$2^L \times 2^L$ matrix

Can be realized experimentally;
connections with CFT, AdS/CFT, …

Problem: find eigenvalues and eigenvectors of H

$$H|v\rangle = E|v\rangle$$

Brute force doesn't go very far…

Coordinate Bethe ansatz (1931)

Remarkable solution!

Reduces the problem to solving a system of polynomial equations   “Bethe equations”

Eigenvectors are multi-particle ("magnon") states

ground (0-particle) state:

$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes L} = |\uparrow \cdots \uparrow\rangle$$

$$H|\psi_0\rangle = 0$$

1-particle state:

$$|\psi(k)\rangle = \sum_{x=1}^{L} e^{ikx} |\uparrow \cdots \underset{x}{\downarrow} \cdots \uparrow\rangle$$

$$H|\psi(k)\rangle = e(k)|\psi(k)\rangle \qquad \boxed{e(k) = 4\sin^2(\tfrac{k}{2})} \qquad \text{1-particle energy}$$

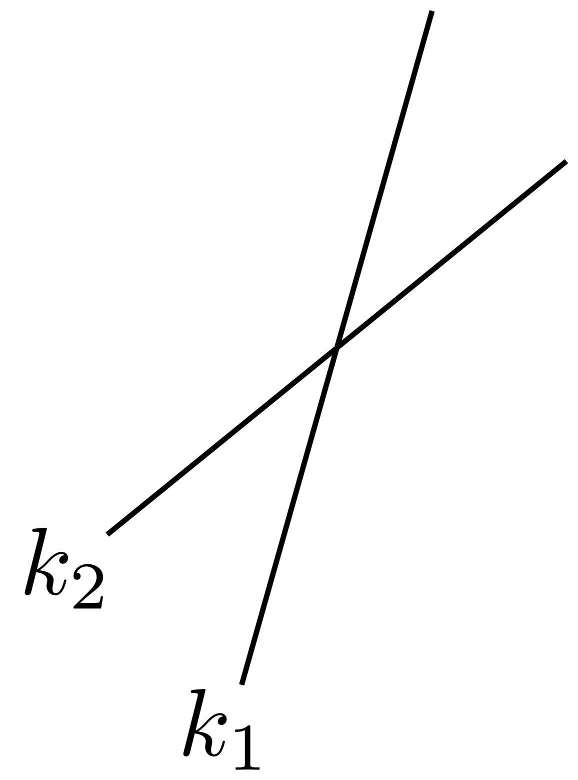
provided $\quad e^{ikL} = 1$

2-particle state:

$$|\psi(k_1,k_2)\rangle = \sum_{1\le x_1<x_2\le L} \left[s(k_2,k_1)e^{i(k_1x_1+k_2x_2)} - s(k_1,k_2)e^{i(k_2x_1+k_1x_2)} \right] | \uparrow \cdots \underset{x_1}{\downarrow} \cdots \underset{x_2}{\downarrow} \cdots \uparrow \rangle$$

$$\boxed{s(k,k') = 1 - 2e^{ik'} + e^{i(k+k')}}$$

$$S(k_2,k_1) = -\frac{s(k_1,k_2)}{s(k_2,k_1)}$$



2-particle S-matrix

$$H|\psi(k_1,k_2)\rangle = E|\psi(k_1,k_2)\rangle \qquad E = e(k_1) + e(k_2)$$

provided $\begin{cases} e^{ik_1L} = S(k_1,k_2) \\ e^{ik_2L} = S(k_2,k_1) \end{cases}$

M-particle state:

$$\boxed{|\psi(k_1,\ldots,k_M)\rangle = \sum_{1\le x_1<\ldots<x_M\le L} \; \sum_{\sigma\in \mathrm{Perm}(1,\ldots,\,\mathrm{M})} \varepsilon(\sigma)\, A(k_{\sigma(1)},\cdots,k_{\sigma(M)})\, e^{i\sum_{j=1}^{M} k_{\sigma(j)}x_j} |\uparrow\cdots\underset{x_1}{\downarrow}\cdots\underset{x_M}{\downarrow}\cdots\uparrow\rangle}$$

Bethe state

$$A(k_1,\ldots,k_M) = \prod_{1\le j<l\le M} s(k_l,k_j) \qquad \varepsilon(\sigma)=\pm 1 \quad \text{signature of } \sigma$$

$$H|\psi(k_1,\ldots,k_M)\rangle = E|\psi(k_1,\ldots,k_M)\rangle \qquad E=\sum_{j=1}^{M} e(k_j)$$

provided $\boxed{e^{ik_jL} = \prod_{l=1;l\neq j}^{M} S(k_j,k_l)\,, \qquad j=1,\ldots,M}$ Bethe equations

$$\boxed{\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^L = \prod_{\substack{l\neq j\\ l=1}}^{M} \frac{u_j-u_l+i}{u_j-u_l-i}} \qquad u_j=u(k_j) \qquad u(k)=\tfrac{1}{2}\cot(\tfrac{k}{2})$$

SU(2) symmetry $\Rightarrow$ degeneracy $L - 2M + 1$

$$M \leq \frac{L}{2}$$

Example: $L = 4$ $\quad M = 0, 1, 2$

M	u_j	E	degeneracy
0	-	0	5
1	$\frac{1}{2}$	2	3
1	$-\frac{1}{2}$	2	3
1	0	4	3
2	$\pm\frac{i}{2}$	2	1
2	$\pm\frac{1}{2\sqrt{3}}$	6	1

total: $16 = 2^4 = 2^L$ "complete" $\checkmark$

2. Dicke state preparation

Dicke states

$|D_k^n\rangle$: completely symmetric state of $|1\rangle$'s and $|0\rangle$'s

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$|1\rangle$'s: k $\quad$ $|0\rangle$'s: $n-k$ $\quad$ total # qubits = n

Ex:

$$|D_2^4\rangle = \frac{1}{\sqrt{6}}\left(|1100\rangle + |1010\rangle + |0110\rangle + |1001\rangle + |0101\rangle + |0011\rangle\right)$$

$$|0011\rangle = |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle$$

$$|D_k^n\rangle \propto \left(\mathbb{S}^-\right)^k |0\rangle^{\otimes n} \qquad \mathbb{S}^- = \mathbb{S}^x - i\mathbb{S}^y \qquad \vec{\mathbb{S}} = \sum_{i=1}^{n} \vec{S}_i \qquad \vec{S}_i = \tfrac{1}{2}\vec{\sigma}_i$$

Exact ground states of ferromagnetic Heisenberg & Lipkin-Meshkov-Glick Hamiltonians

$$-\sum_i \vec{S}_i \cdot \vec{S}_{i+1} \qquad -\vec{\mathbb{S}}^{\,2} = -\sum_{i,j} \vec{S}_i \cdot \vec{S}_j$$

How to prepare on quantum computer?

Cannot implement $|D_k^n\rangle \propto \left(\mathbb{S}^-\right)^k |0\rangle^{\otimes n}$ $\mathbb{S}^-$ is not unitary

Instead:

$$\boxed{|\mathrm{e}_k^n\rangle = |0\rangle^{\otimes(n-k)}|1\rangle^{\otimes k}}$$

"reference" state (product)

We seek:

$$\boxed{U_n\,|\mathrm{e}_k^n\rangle = |D_k^n\rangle}$$

"Dicke operator"

- unitary
- independent of k

key idea: recursion! [Bärtschi, Eidenbenz 2019]

Ex:
$$|D_2^4\rangle = \frac{1}{\sqrt{6}}\big(\underbrace{|1100\rangle + |1010\rangle + |0110\rangle}_{(|110\rangle + |101\rangle + |011\rangle)\otimes|0\rangle} + \underbrace{|1001\rangle + |0101\rangle + |0011\rangle}_{(|100\rangle + |010\rangle + |001\rangle)\otimes|1\rangle}\big)$$

$$|D_2^4\rangle = \sqrt{\frac{1}{2}}\,|D_2^3\rangle \otimes |0\rangle + \sqrt{\frac{1}{2}}\,|D_1^3\rangle \otimes |1\rangle$$

$$\boxed{|D_k^n\rangle = \sqrt{\frac{n-k}{n}}\,|D_k^{n-1}\rangle \otimes |0\rangle + \sqrt{\frac{k}{n}}\,|D_{k-1}^{n-1}\rangle \otimes |1\rangle}$$

Use $|D_k^n\rangle = U_n\,|\mathrm{e}_k\rangle$ on both sides:

$$U_n|\mathrm{e}_k^n\rangle = (U_{n-1}\otimes\mathbb{I})\left(\underbrace{\sqrt{\frac{n-k}{n}}\,|\mathrm{e}_k^{n-1}\rangle \otimes |0\rangle + \sqrt{\frac{k}{n}}\,|\mathrm{e}_{k-1}^{n-1}\rangle \otimes |1\rangle}_{\equiv W_n|\mathrm{e}_k^n\rangle}\right)$$

- unitary
- independent of k

$$\boxed{W_n|\mathrm{e}_k^n\rangle = \sqrt{\frac{n-k}{n}}\,|\mathrm{e}_k^{n-1}\rangle \otimes |0\rangle + \sqrt{\frac{k}{n}}\,|\mathrm{e}_{k-1}^{n-1}\rangle \otimes |1\rangle}$$

$$U_n = (U_{n-1} \otimes \mathbb{I})\, W_n \qquad \Rightarrow \qquad \boxed{U_n = \prod_{m=2}^{\overset{\curvearrowright}{n}} \left(W_m \otimes \mathbb{I}^{\otimes(n-m)} \right)}$$

Suffices to construct W_m's !

Constructing W_m's

Strategy: look for operators $\mathrm{I}_{m,l}$ such that

$$|\mathrm{e}_l^m\rangle = |0\rangle^{\otimes(m-l)}|1\rangle^{\otimes l}$$

$$\mathrm{I}_{m,l'}\,|\mathrm{e}_l^m\rangle = \begin{cases} |\mathrm{e}_l^m\rangle & l' < l \\ W_m\,|\mathrm{e}_l^m\rangle & l' = l \end{cases}$$

perform $W_m\,|\mathrm{e}_l^m\rangle$ for fixed l

and

$$\mathrm{I}_{m,l'}\;(\mathrm{I}_{m,l}\,|\mathrm{e}_l^m\rangle) = (\mathrm{I}_{m,l}\,|\mathrm{e}_l^m\rangle) \quad \text{for} \quad l' > l$$

ordered so as to not interfere with each other

Then

$$\boxed{W_m = \prod_{l=1}^{\overset{\curvearrowleft}{m-1}} \mathrm{I}_{m,l}}$$

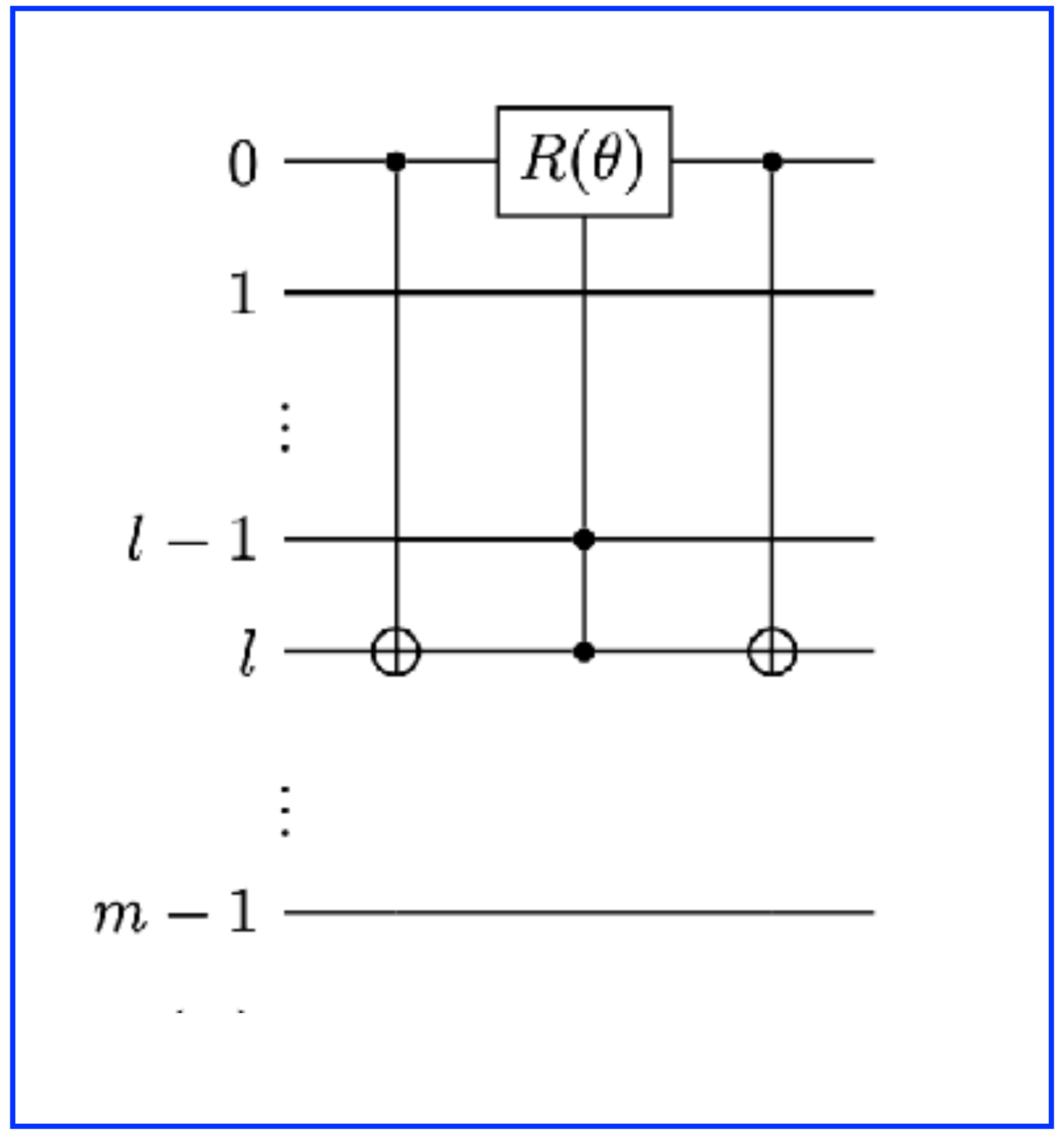
Suffices to construct $\mathrm{I}_{m,l}$'s !

Constructing $\mathrm{I}_{m,l}$'s

$$\mathrm{I}_{m,l}|\mathrm{e}_l^m\rangle = W_m\,|\mathrm{e}_l^m\rangle$$
$$= W_m\,|0\rangle^{\otimes(m-l)}|1\rangle^{\otimes l} = \sqrt{\frac{m-l}{m}}\,|0\rangle^{\otimes(m-l-1)}|1\rangle^{\otimes l}\otimes|0\rangle + \sqrt{\frac{l}{m}}\,|0\rangle^{\otimes(m-l)}|1\rangle^{\otimes(l-1)}\otimes|1\rangle$$

$$\overset{\substack{l \\ \downarrow}}{}\quad \overset{\substack{l-1 \\ \downarrow}}{}\quad \overset{\substack{0 \\ \downarrow}}{}$$
$$|0\cdots 0\;1\cdots 1\;1\rangle \qquad |0\cdots 1\;1\cdots 1\;0\rangle \qquad |0\cdots 0\;1\cdots 1\;1\rangle$$

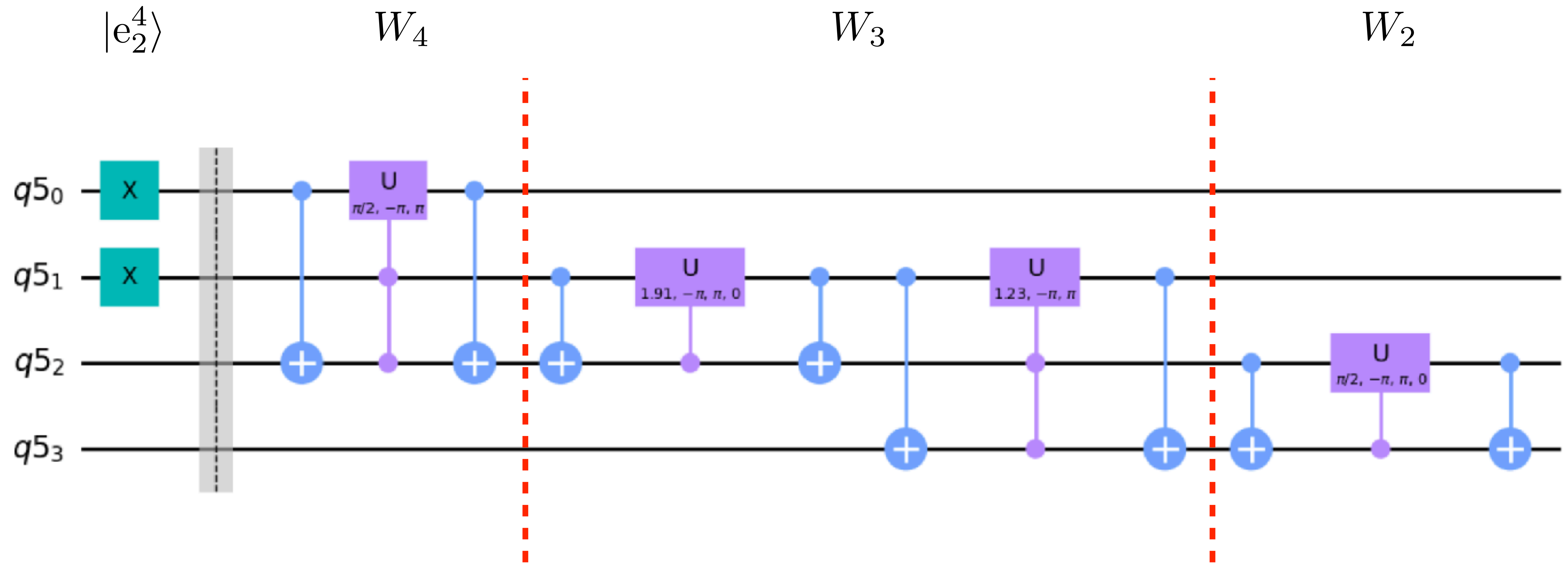
$$\mathrm{I}_{m,l}: \quad |0\rangle_l\,|1\rangle_{l-1}\,|1\rangle_0 \mapsto \underbrace{\sqrt{\frac{m-l}{m}}}_{-\sin(\frac{\theta}{2})}\,|1\rangle_l\,|1\rangle_{l-1}\,|0\rangle_0 + \underbrace{\sqrt{\frac{l}{m}}}_{\cos(\frac{\theta}{2})}\,|0\rangle_l\,|1\rangle_{l-1}\,|1\rangle_0$$



$$R(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

- circuit size = $\mathcal{O}\left(M(L-M)\right)$

Ex: $|D_2^4\rangle$



Generalizations:

• higher spin	RN, Ravanini, Raveh	arXiv: 2402.03233
• higher rank	RN, Raveh	arXiv: 2301.04989
• q-deformation	Raveh, RN	arXiv: 2308.08392

3. Bethe state preparation

Closed periodic spin-1/2 XXZ chain

$$\mathcal{H} = -\tfrac{1}{2}\sum_{n=1}^{L}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\left(\sigma_n^z\sigma_{n+1}^z - \mathbb{I}\right)\right), \qquad \vec{\sigma}_{L+1} = \vec{\sigma}_1$$

Assume Bethe roots $\{k_1, \ldots, k_M\}$ are known.

How to prepare corresponding Bethe state $|B_M^L\rangle$ on quantum computer?

Previous:

- Van Dyke, Barron, Mayhall, Barnes, Economou 2021 — coordinate BA — probabilistic; ancillas, real Bethe roots
- Sopena, Gordon, García-Martín, Sierra, López 2022 — algebraic BA — deterministic; QR decompositions
- Ruiz, Sopena, Gordon, Sierra, López 2023 — algebraic/coordinate BA — proposed analytical formulae for unitaries

Today: D. Raveh, RN 2024 — coordinate BA — deterministic; no ancillas or QR decomps

set of all permutations of M 1's and L-M 0's

Bethe state $$\boxed{|B_M^L\rangle \propto \sum_{w\in P(L,M)} f(w)\,|w\rangle}$$

Ex:
$$\begin{aligned}|B_2^4\rangle \propto\, & f(0011)\,|0011\rangle + f(0101)\,|0101\rangle + f(1001)\,|1001\rangle \\ & + f(0110)\,|0110\rangle + f(1010)\,|1010\rangle + f(1100)\,|1100\rangle\end{aligned}$$

Closed periodic spin-1/2 XXZ chain

$$\mathcal{H} = -\tfrac{1}{2}\sum_{n=1}^{L}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\left(\sigma_n^z\sigma_{n+1}^z - \mathbb{I}\right)\right) , \qquad \vec{\sigma}_{L+1} = \vec{\sigma}_1$$

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Bethe state

$$\boxed{|B_M^L\rangle \propto \sum_{w\in P(L,M)} \boxed{f(w)}\,|w\rangle}$$

$$f(w) = \sum_{\sigma\in \mathrm{Perm}(1,\ldots,\ \mathrm{M})} \varepsilon(\sigma)\, A(k_{\sigma(1)}, \cdots, k_{\sigma(M)})\, e^{i\sum_{j=1}^{M} k_{\sigma(j)}\boxed{x_j}}$$

$x_j \in \{1, \ldots, L\}$ positions of 1's in w

Ex:

$$\begin{aligned}|B_2^4\rangle \propto\, & f(0011)\,|0011\rangle + f(0101)\,|0101\rangle + f(1001)\,|1001\rangle \\ & + f(0110)\,|0110\rangle + f(1010)\,|1010\rangle + f(1100)\,|1100\rangle\end{aligned}$$

Closed periodic spin-1/2 XXZ chain

$$\mathcal{H} = -\tfrac{1}{2}\sum_{n=1}^{L}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\left(\sigma_n^z\sigma_{n+1}^z - \mathbb{I}\right)\right)\,, \qquad \vec{\sigma}_{L+1} = \vec{\sigma}_1$$

Assume Bethe roots $\{k_1, \ldots, k_M\}$ are known.

How to prepare corresponding Bethe state $|B_M^L\rangle$ on quantum computer?

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Bethe state
$$\boxed{|B_M^L\rangle \propto \sum_{w\in P(L,M)} \boxed{f(w)}\,|w\rangle} \qquad f(w) = \sum_{\sigma\in \mathrm{Perm}(1,\ldots,\,\mathrm{M})} \varepsilon(\sigma)\, A(k_{\sigma(1)},\cdots,k_{\sigma(M)})\, e^{i\sum_{j=1}^{M} k_{\sigma(j)}x_j}$$

$\binom{L}{M}$ known!

Ex:
$$\begin{aligned}|B_2^4\rangle \propto\, & f(0011)\,|0011\rangle + f(0101)\,|0101\rangle + f(1001)\,|1001\rangle \\ & + f(0110)\,|0110\rangle + f(1010)\,|1010\rangle + f(1100)\,|1100\rangle\end{aligned}$$

$$\boxed{|\psi(b)\rangle = \frac{1}{F(b)} \sum_{\substack{w \in P(L,M) \\ w = ab}} f(w)\,|w\rangle}$$

"b-tail" of $|B_M^L\rangle$ (ends in b)

permutations w of the form a b
(concatenation)

$$|w\rangle = |a\rangle|b\rangle$$

Ex: For $|B_2^4\rangle \propto f(0011)\,|0011\rangle + f(0101)\,|0101\rangle + f(1001)\,|1001\rangle$
$+f(0110)\,|0110\rangle + f(1010)\,|1010\rangle + f(1100)\,|1100\rangle$

$$|\psi(0)\rangle = \frac{1}{F(0)}\left(f(0110)\,|0110\rangle + f(1010)\,|1010\rangle + f(1100)\,|1100\rangle\right)$$

$$\boxed{|B_M^L\rangle = |\psi(\{\})\rangle}$$

$b = \{\}$

We'll do recursion over length of b!

Define:

$$\boxed{U_m \left|0\right\rangle^{\otimes(m-l)} \left|1\right\rangle^{\otimes l} \left|b\right\rangle_{L-m} = \left|\psi(b)\right\rangle_L}$$

"Bethe operator"

- unitary
- independent of l, b

$b = \{\}$:

$$\boxed{\left|B^L_M\right\rangle = U_L \left|0\right\rangle^{\otimes(L-M)} \left|1\right\rangle^{\otimes M}}$$

Recursion:

$$\boxed{\left|\psi(b)\right\rangle = G(0b)\left|\psi(0b)\right\rangle + G(1b)\left|\psi(1b)\right\rangle}$$

$$G(ib) = \frac{F(ib)}{F(b)}\,, \quad i = 0, 1$$

Ex: For $\left|\psi(0)\right\rangle = \frac{1}{F(0)}\left(f(1100)\left|1100\right\rangle + f(0110)\left|0110\right\rangle + f(1010)\left|1010\right\rangle\right)$

Define:

$$\boxed{\underset{\substack{\uparrow}}{U_m} \, |0\rangle^{\otimes(m-l)} \, |1\rangle^{\otimes l} \, |b\rangle_{L-m} = |\psi(b)\rangle_L}$$

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Recursion:

$$\boxed{|\psi(b)\rangle = G(0b) \, |\psi(0b)\rangle + G(1b) \, |\psi(1b)\rangle}$$

$$G(ib) = \frac{F(ib)}{F(b)} \,, \quad i = 0, 1$$

Ex: For $|\psi(0)\rangle = \frac{1}{F(0)} \boxed{(f(1100) \, |1100\rangle} + \boxed{f(0110) \, |0110\rangle + f(1010) \, |1010\rangle)}$

$$|\psi(0)\rangle = \frac{G(00)}{F(00)} \underbrace{f(1100) \, |1100\rangle}_{= \, |\psi(00)\rangle} + \frac{G(10)}{F(10)} \underbrace{(f(0110) \, |0110\rangle + f(1010) \, |1010\rangle)}_{= \, |\psi(10)\rangle}$$

$$|\psi(0)\rangle = G(00) \, |\psi(00)\rangle + G(10) \, |\psi(10)\rangle \qquad \checkmark$$

Define:

$$\boxed{U_m \, |0\rangle^{\otimes(m-l)} \, |1\rangle^{\otimes l} \, |b\rangle_{L-m} = |\psi(b)\rangle_L} \qquad (1)$$

"Bethe operator"

- unitary
- independent of l, b

$b = \{\}$:

$$\boxed{|B_M^L\rangle = U_L \; |0\rangle^{\otimes(L-M)} \, |1\rangle^{\otimes M}}$$

Recursion:

$$\boxed{|\psi(b)\rangle = G(0b) \, |\psi(0b)\rangle + G(1b) \, |\psi(1b)\rangle} \qquad (2)$$

$$G(ib) = \frac{F(ib)}{F(b)} \, , \qquad i = 0, 1$$

Use (1) on both sides of (2) to obtain recursion for U_m

$$U_m = U_{m-1} W_m \qquad \Rightarrow \qquad \boxed{U_L = \prod_{m=2}^{\overset{\curvearrowright}{L}} W_m}$$

$$\boxed{\begin{aligned} W_m \left|0\right\rangle^{\otimes(m-l)} \left|1\right\rangle^{\otimes l} \left|b\right\rangle_{L-m} = {} & G(0b) \left|0\right\rangle^{\otimes(m-l-1)} \left|1\right\rangle \left|1\right\rangle^{\otimes(l-1)} \left|0\right\rangle \left|b\right\rangle_{L-m} \\ & + G(1b) \left|0\right\rangle^{\otimes(m-l-1)} \left|0\right\rangle \left|1\right\rangle^{\otimes(l-1)} \left|1\right\rangle \left|b\right\rangle_{L-m} \end{aligned}}$$

Suffices to construct W_m's !

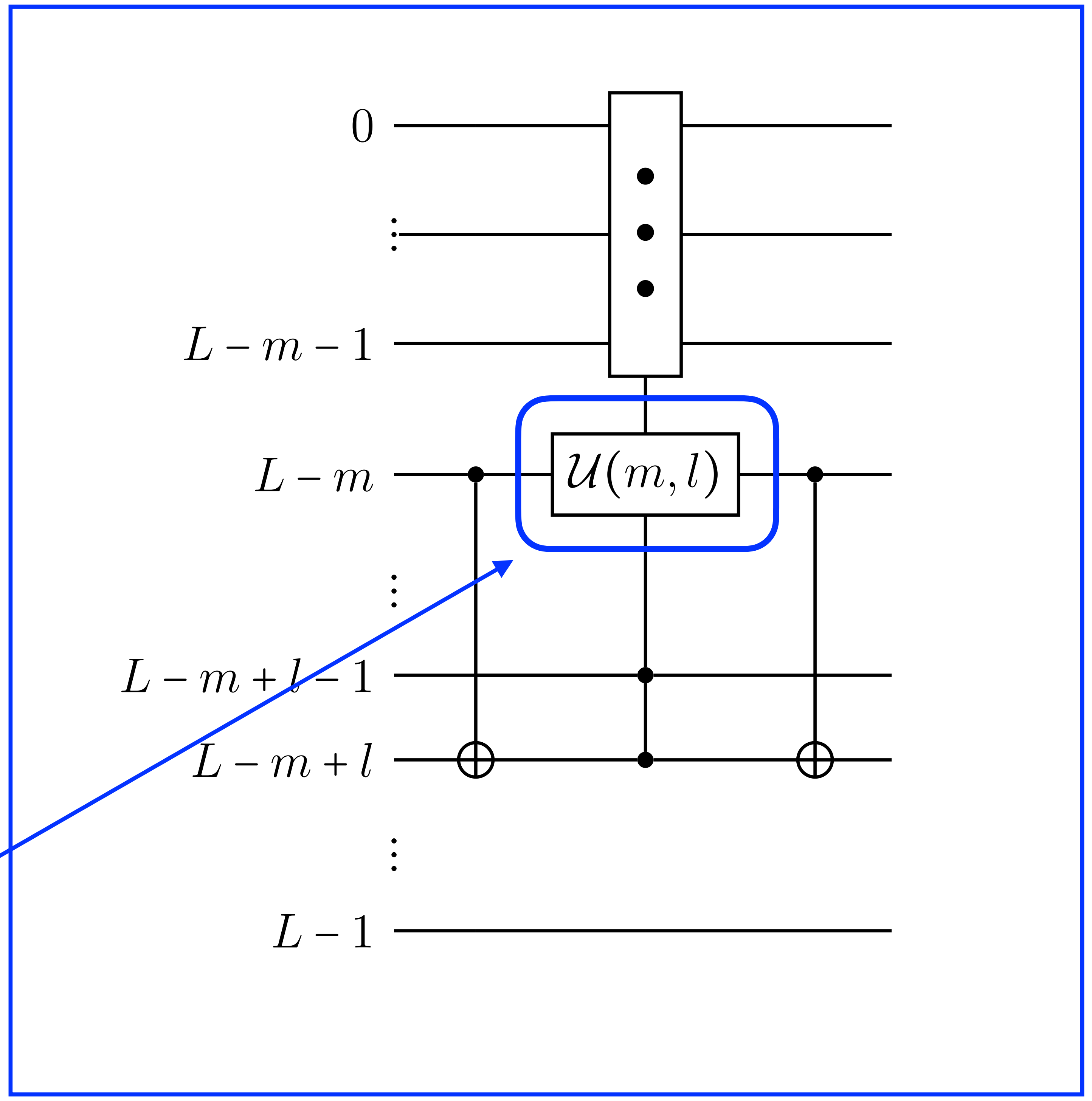
Strategy: look for operators $\mathrm{I}_{m,l}$ such that

- perform $W_m \left|0\right\rangle^{\otimes(m-l)} \left|1\right\rangle^{\otimes l} \left|b\right\rangle_{L-m}$ for fixed l for all $\left|b\right\rangle_{L-m}$
- ordered so as to not interfere with each other

Then $$\boxed{W_m = \prod_{l=1}^{\overset{\curvearrowleft}{m-1}} \mathrm{I}_{m,l}}$$

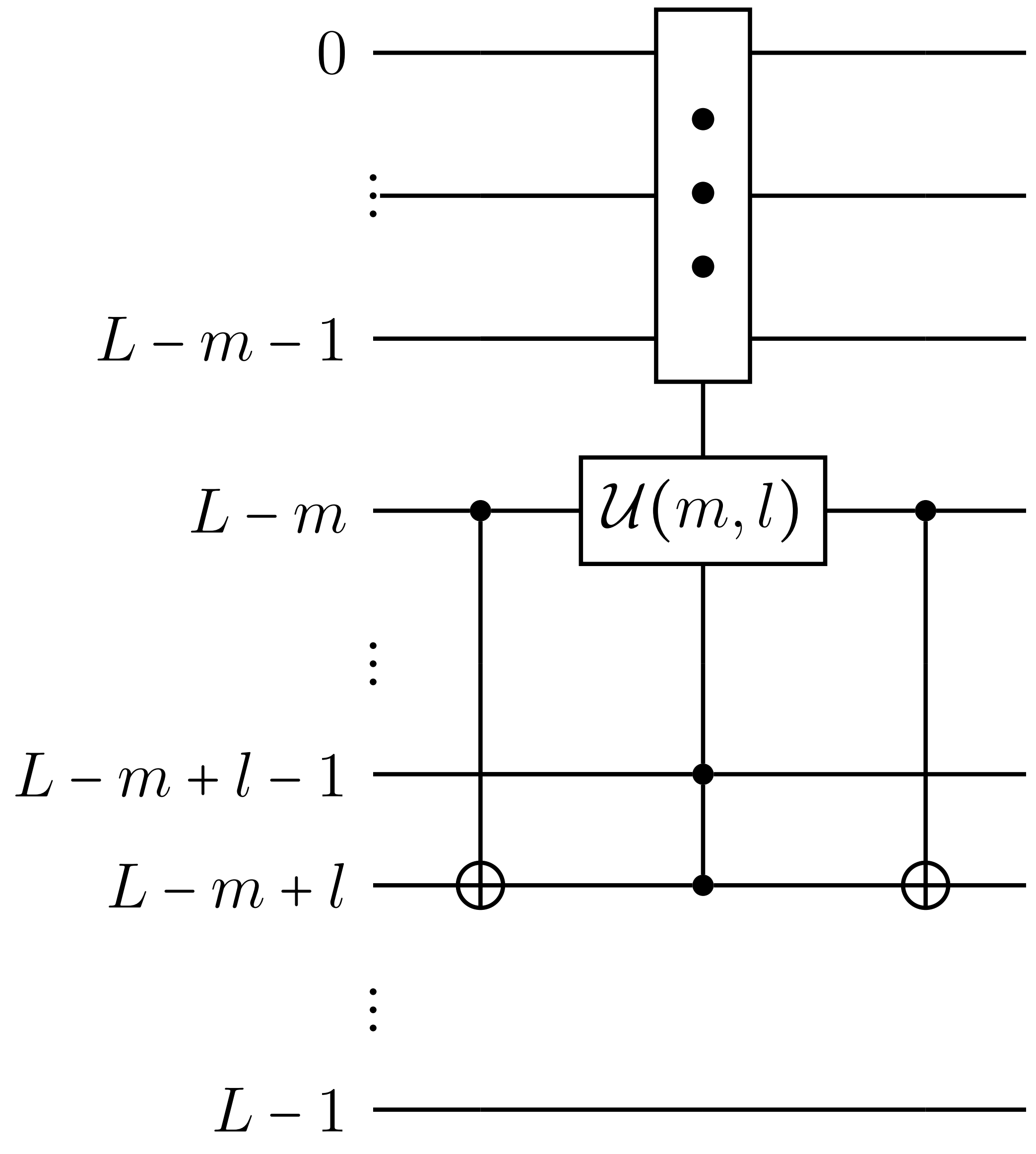
Suffices to construct $\mathrm{I}_{m,l}$'s !

$\mathrm{I}_{m,l}$:



$$\mathcal{U}(m,l) = \prod_{b \in P(L-m,M-l)} u(m,l,b)$$

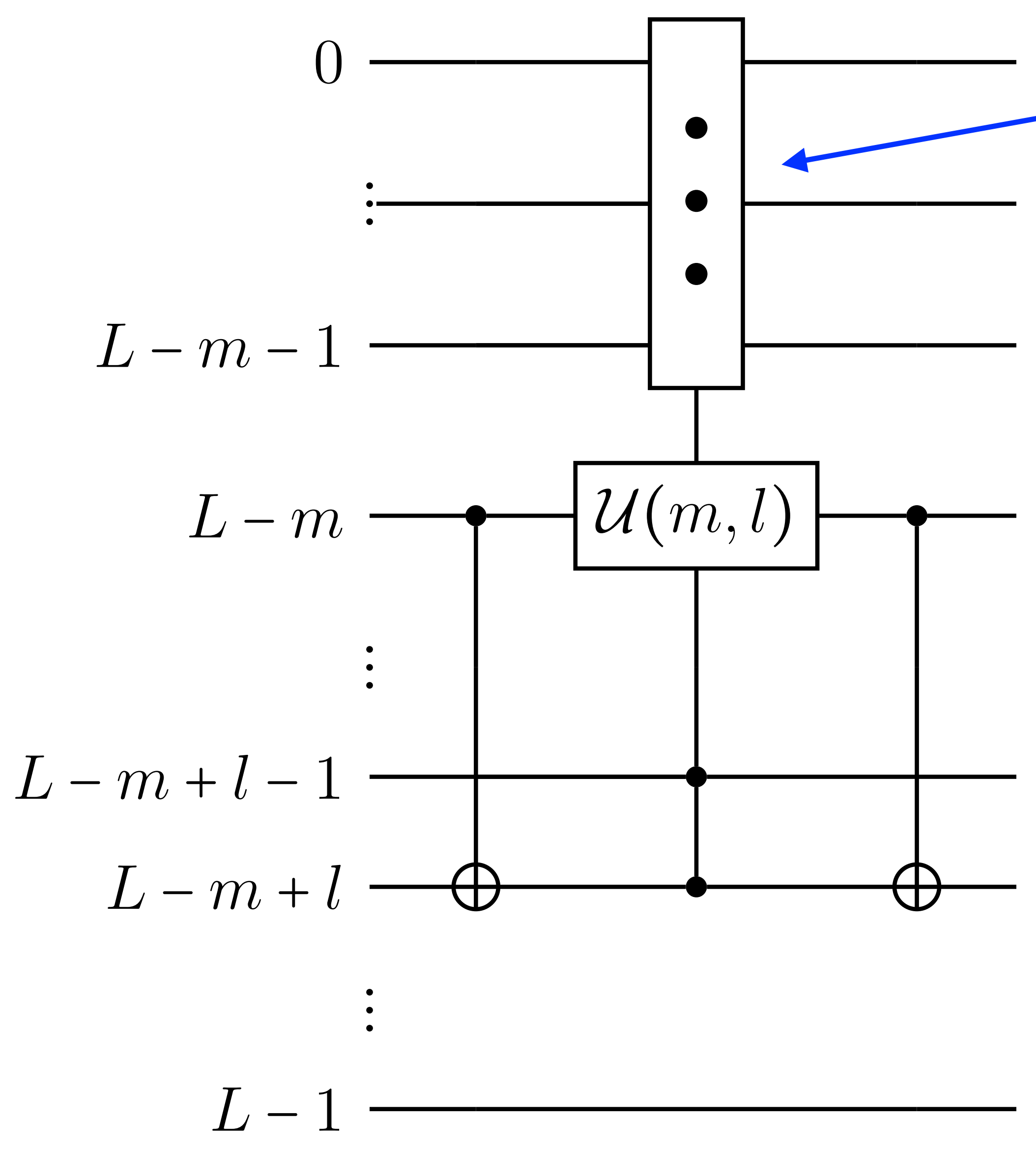
$\mathrm{I}_{m,l}$:



$$\mathcal{U}(m,l) = \prod_{\boxed{b \in P(L-m, M-l)}} u(m,l,b)$$

all possible b's for given m, l

$\mathrm{I}_{m,l}$:



controls where b's have 1's

$$\mathcal{U}(m,l) = \prod_{\boxed{b \in P(L-m, M-l)}} u(m,l,b)$$

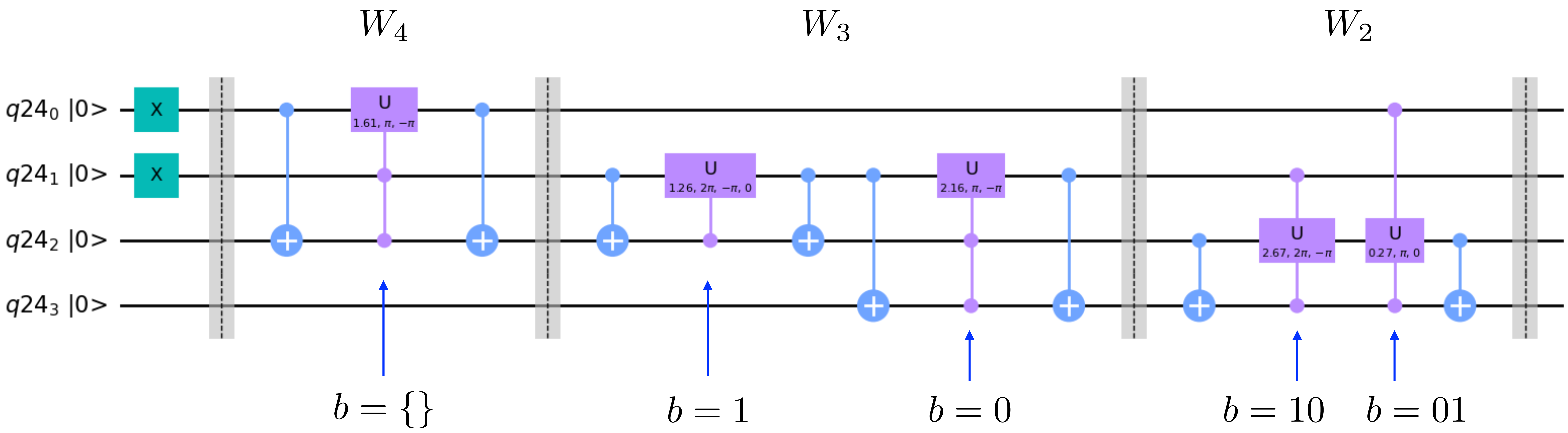
all possible b's for given m, l

$$u(m,l,b) = \begin{pmatrix} \cos(\frac{\theta}{2}) & -e^{i\lambda}\sin(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) & e^{i(\phi+\lambda)}\cos(\frac{\theta}{2}) \end{pmatrix}$$

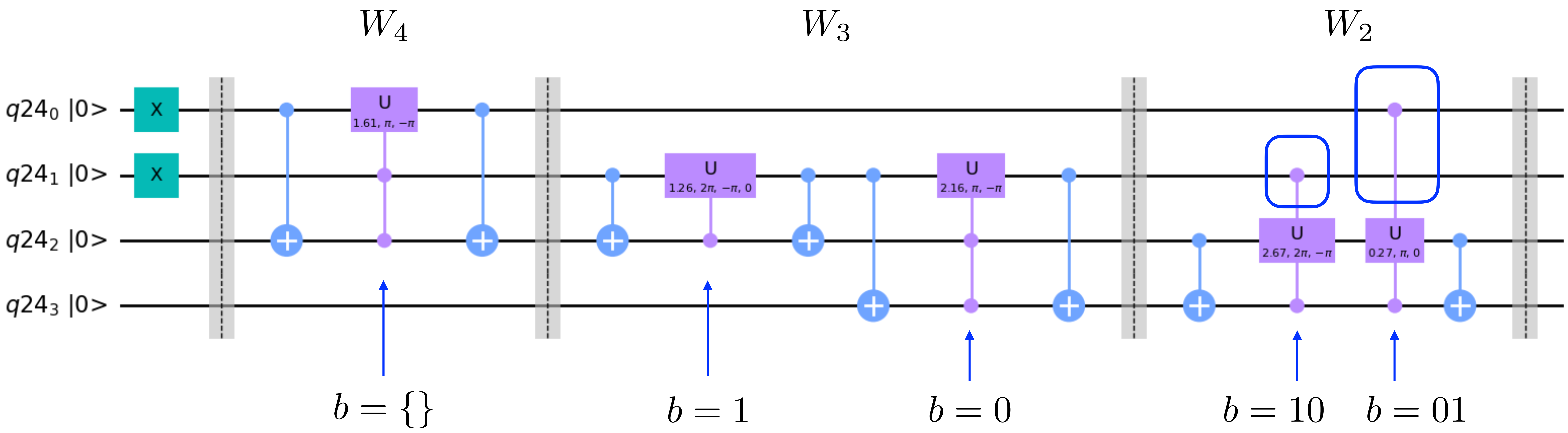
$$\theta = 2\arccos\left(|G(1b)|\right) , \quad \lambda = \arg(G(0b)) - \pi , \quad \phi = \arg(G(1b)) - \lambda$$

■

Ex: (L,M) = (4,2)

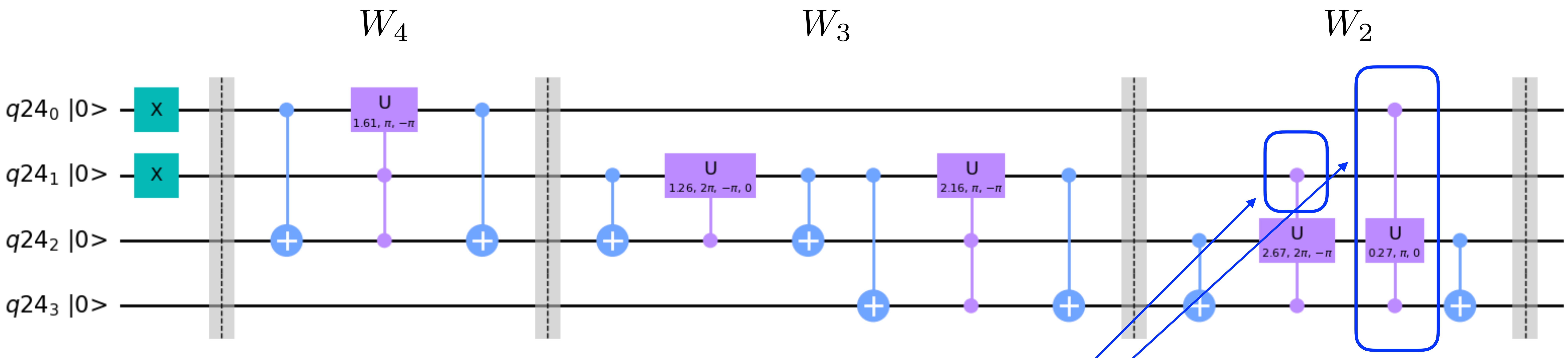

W_4
W_3
W_2
$q24_0$ |0>
$q24_1$ |0>
$q24_2$ |0>
$q24_3$ |0>
X
X
U
1.61, π, −π
U
1.26, 2π, −π, 0
U
2.16, π, −π
U
2.67, 2π, −π
U
0.27, π, 0
$b = \{\}$
$b = 1$
$b = 0$
$b = 10$
$b = 01$

Ex: (L,M) = (4,2)



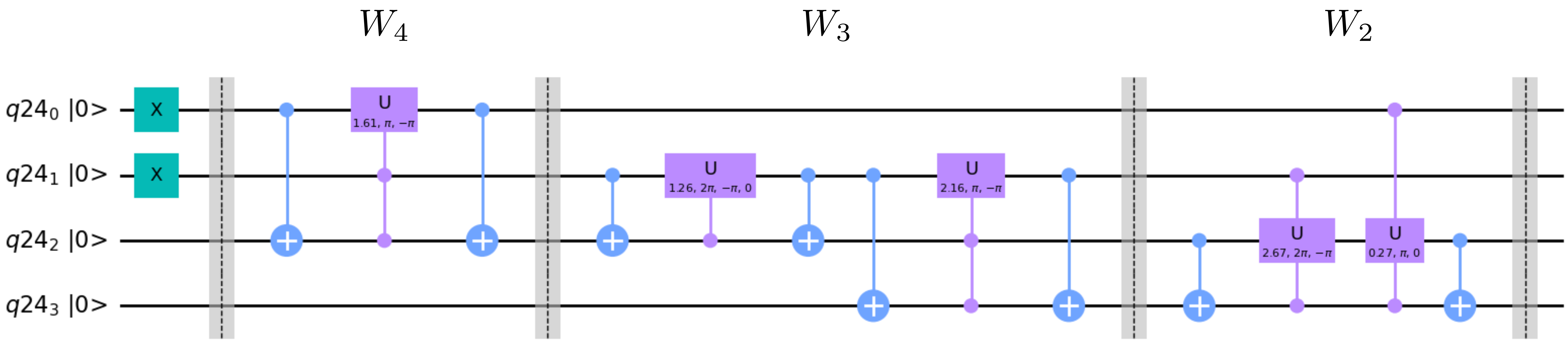
controls differentiate different b’s

Ex: (L,M) = (4,2)



Same as circuit for Dicke state $|D_2^4\rangle$ except for angles & these gates

Ex: (L,M) = (4,2)



circuit size = $\mathcal{O}\left(\binom{L}{M}\right)$ ~ # $f(w)$'s $\therefore$ expect ~ optimal

For M=L/2 $\binom{L}{L/2} \sim \frac{2^L}{\sqrt{\pi L/2}}$ ☹️

4. Bethe roots from VQE

Bethe equations are generally hard to solve. Can quantum computers help?

Variational Quantum Eigensolver (VQE): hybrid quantum/classical algorithm for estimating the ground-state energy E_0 of a Hamiltonian $\mathcal{H}$ using the variational theorem

normalized trial state $|\Psi(\vec{\theta})\rangle$ $\vec{\theta}$ parameters

iteration: $\vec{\theta}^{(0)}$

classical simulators

~~quantum~~ $\langle\Psi(\vec{\theta}^{(n)})|\mathcal{H}|\Psi(\vec{\theta}^{(n)})\rangle$

classical $\vec{\theta}^{(n)} \to \vec{\theta}^{(n+1)}$

variational theorem $\Rightarrow$ $\langle\mathcal{H}\rangle \geq E_0$

Estimate Bethe roots using VQE, taking *exact Bethe states* as *trial states*, and treating *Bethe roots* $\vec{k}$ *as variational parameters*

Raveh, RN 2404.18244

To test this idea, we instead used classical simulators.

Classical simulators:

- Qiskit Statevector simulator: performs matrix arithmetic to compute *exact* expectation values
- Qiskit Aer simulator: noiseless simulation using 10,000 shots (trials)

XXZ models:

- Closed chain $$\mathcal{H} = \frac{1}{4}\sum_{n=1}^{L}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\,\sigma_n^z\sigma_{n+1}^z\right)\,, \qquad \vec{\sigma}_{L+1} = \vec{\sigma}_1$$
- Open chain $$\mathcal{H} = \frac{1}{4}\sum_{n=1}^{L-1}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\,\sigma_n^z\sigma_{n+1}^z\right) + \frac{1}{4}\left(\boxed{h}\,\sigma_1^z + \boxed{h'}\,\sigma_L^z\right)$$

boundary magnetic fields

The state-preparation algorithm works for *any* U(1)-invariant model, including open XXZ

[Alcaraz, Barber, Batchelor, Baxter, Quispel (1987)]

Ground-state Bethe roots:

- Closed chain $\Delta = 2$ Newton's method - Mathematica

L	M	Energy	True roots
2	1	-2	3.14159
4	2	-2.73205	± 1.94553
6	3	-3.85577	$\pm 1.49862, 3.14159$

Bethe roots real

Ground-state Bethe roots:

- Closed chain $\Delta = 2$

L	M	Energy	True roots	Statevector roots	Aer roots
2	1	-2	3.14159	3.1415	3.1487
4	2	-2.73205	± 1.94553	$1.9455, -1.9455$	$1.9623, -1.9503$
6	3	-3.85577	$\pm 1.49862, 3.14159$	$1.4986, -1.4986, 3.1416$	$1.5477, -1.4830, 3.1796$

Bethe roots real

Ground-state Bethe roots:

- Closed chain $\Delta = 2$

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Bethe roots real

- Open chain $\Delta = 1/2\,, \quad h = 3\,, \quad h' = 3/10$

L	M	Energy	True roots	Statevector roots	Aer roots
2	1	-0.965015	$3.14159 + 0.882174i$	$3.1415 + 0.8822i$	$3.0298 + 0.8785i$
3	2	-1.49506	$3.14159 + 0.908996i, 1.69883$	$3.1412 + 0.9071i, 1.6988$	$3.1537 + 0.9014i, 1.6930$
4	2	-1.76803	$3.14159 + 0.91503i, 2.11689$	$3.1412 + 0.9142i, 2.1169$	$3.0944 + 0.9929i, 2.1473$
5	3	-2.22762	$3.14159 + 0.916011i, 1.49569, 2.31576$	$3.1399 + 0.9144i, 1.4958, 2.3157$	$3.3120 + 1.5000i, 1.5449, 2.3291$
6	3	-2.53682	$3.14159 + 0.916239i, 1.82675, 2.47141$	$3.1419 + 0.9131i, 1.8266, 2.4712$	$3.0864 + 0.9053i, 2.0142, 2.3022$

One complex Bethe root

Excited-state Bethe roots:

Instead of minimizing $\langle \mathcal{H} \rangle$, we now minimize the *variance* [Zhang, Chen, Yuan, Yin (2020)]

$$\langle (\mathcal{H} - \langle \mathcal{H} \rangle)^2 \rangle = \langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2 \geq 0 \qquad (= 0 \text{ for exact eigenstate})$$

- Closed chain $\Delta = 2$

(selected)

L	M	Energy	True roots	Statevector roots	Aer roots
2	1	0	0	0	0.0019
3	1	−1	2.0944	2.0943	2.0938
4	2	0.732051	$\pm 0.831443i$	$\pm 0.8314i$	$\pm 0.8557i$
5	2	0.716341	$0.628319 \pm 0.835459i$	$0.6276 \pm 0.8349i$	$0.6263 \pm 0.8858i$
6	2	−1.75395	1.37766, 2.81114	1.3776, 2.8109	1.3998, 2.8293
6	3	1.18614	$0.244998 \pm 1.41247i$, 1.6044	$0.2451 \pm 1.4120i$, 1.6023	$0.2451 \pm 1.3341i$, 1.3152

- Open chain $\Delta = 1/2\,, \quad h = 3\,, \quad h' = 3/10$

(selected)

multiple complex Bethe roots

L	M	Energy	True roots	Statevector roots	Aer roots
2	1	0.715015	1.30258	1.3026	1.3025
3	1	−0.869852	$3.14159 + 0.911371i$	$3.1417 + 0.9113i$	$3.1683 + 0.8783i$
4	2	−0.224189	$3.14159 + 0.916221i$, $0.2264i$	$3.1413 + 0.9164i$, $0.2284i$	$3.0311 + 0.9359i$, $0.2148i$
4	3	−0.128194	$3.14159 + 0.916237i$, $0.93789, 0.245389i$	$3.1416 + 0.9174i$, $0.9382, 0.2474i$	$3.0528 + 0.9138i$, $0.9046, 0.2146i$
5	4	−1.61607	$3.14159 + 0.916185i$, 0.514675, 1.16211, 2.43263	$3.1421 + 0.9027i$, 0.5044, 1.1618, 2.4332	$3.0030 + 0.7578i$, 0.4451, 1.0290, 2.4882
$\boxed{6 \quad 5}$		0.21968	$3.14159 + 0.916291i$, $0.667057, 0.32195i$, $1.12044 \pm 0.160175i$	$3.1412 + 0.9156i$, $0.6252, 0.3029i$, $1.1347 \pm 0.1956i$	$3.1218 + 0.7523i$, $0.5055, 0.5615i$, $1.0101 \pm 0.2531i$

5. Outlook

- Deterministic algorithm for preparing exact Bethe states

  Works for *any* U(1)-eigenstate

- Used for estimating Bethe roots

- Exploit integrability?

- Reduce circuit size and/or depth? ancillas, measurements, feedforward operations Piroli, Styliaris, Cirac 2024

  Mao, Tian, Sun 2024

- Higher spin? Higher rank?

- Models without U(1) symmetry?

Thank you for your attention!