Alternating Sign MATRIXes & Integrability

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MATRIX

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Plan

I. Introduce alternating sign matrices (ASMs) & state a formula for their enumeration.

II. Introduce odd-order diagonally and antidiagonally symmetric alternating sign matrices (DASASMs) & state a formula for their enumeration.

III. Introduce diagonally symmetric alternating sign matrices (DSASMs) & state a formula for their enumeration.

I′. Outline a proof of the ASM formula, involving the integrable six-vertex model on a square with domain-wall boundary conditions *(Kuperberg 1996)*.

II′. Outline a proof of the odd-order DASASM formula, involving the integrable six-vertex model on an isosceles triangle with certain boundary conditions *(RB, Fischer, Konvalinka 2017)*.

III′. Outline a proof of the DSASM formula, involving the integrable six-vertex model on an "equilateral" triangle with certain boundary conditions *(RB, Fischer, Koutschan 2023)*.

I. Alternating Sign Matrices (ASMs)

ASM: square matrix for which

- each entry is 0, 1 or -1
- each row & column contains at least one 1
- along each row & column, the nonzero entries alternate in sign, starting & ending with a 1

e.g. $$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

History:

- Arose during study of Dodgson condensation algorithm for determinant evaluation *(Mills, Robbins, Rumsey 1982; Robbins, Rumsey 1986)*
- Many subsequent appearances in combinatorics, algebra, mathematical physics, . . .

Observations:
- first/last row/column of an ASM contains a single 1 & all other entries 0
- acting on an ASM with any symmetry operation of the square (reflection in 4 symmetry axes, rotation by 0°, 90°, 180° or 270°) gives another ASM
- any permutation matrix (exactly one 1 in each row & column, all other entries 0) is an ASM

Elementary Bounds on Number of $n\times n$ ASMs

$$\begin{aligned}\{n\times n \text{ permutation matrices}\} &\subseteq \{n\times n \text{ ASMs}\}\\ &\subseteq \{n\times n \text{ matrices with each entry 0, 1 or } -1\}\end{aligned}$$

implies

$$\begin{aligned}(\#\text{ of } n\times n \text{ permutation matrices}) &\le (\#\text{ of } n\times n \text{ ASMs})\\ &\le (\#\text{ of } n\times n \text{ matrices with each entry 0, 1 or } -1)\end{aligned}$$

implies

$$n! \le (\#\text{ of } n\times n \text{ ASMs}) \le 3^{n^2}$$

Exact Number A_n of $n \times n$ ASMs

n=1

$$(1) \quad \Rightarrow \quad A_1 = 1$$

n=2

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad A_2 = 2$$

n=3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad A_3 = 7$$

n=4

- $4! =$ 24 matrices without any -1's (permutation matrices)
- 4 matrices with one -1 at position 2,2:

$$\begin{pmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}$$

  Similarly:

  4 matrices with one -1 at 2,3

  4 matrices with one -1 at 3,2

  4 matrices with one -1 at 3,3

  So, 16 matrices with one -1

- 2 matrices with two -1's:

$$\begin{pmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&-1&1\\1&-1&1&0\\0&1&0&0\end{pmatrix}$$

$\Rightarrow A_4 = 24 + 16 + 2 = 42$

General Case

$$\# \text{ of } n\times n \text{ ASMs: } A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429, 7436, \ldots$$

- Recurrence: $\binom{2n}{n} A_{n+1} = \binom{3n+1}{n} A_n$
- Conjectured: *Mills, Robbins, Rumsey 1982*
- First proved:
  - *Zeilberger 1996* using constant term identities & known enumeration of totally symmetric self-complementary plane partitions (84 pages).
  - *Kuperberg 1996* using connections with integrable six-vertex model (12 pages).
- Several subsequent proofs – e.g. *Fischer, Konvalinka 2021* using signed bijections.
- No simple combinatorial proof (i.e. using elementary counting arguments) currently known.
- Book: D. Bressoud *Proofs & Confirmations: The Story of the ASM Conjecture* Cambridge University Press (1999), 274 pages.

- Outline of Kuperberg proof (more details later):
  - Obtain bijection between $n \times n$ ASMs & configurations of *six-vertex model* on $n \times n$ square with *domain-wall boundary conditions*.
  - Introduce spectral parameter-dependent vertex weights & consider weighted sum over all configurations of model, i.e. (inhomogeneous) *partition function*.
  - Use *Yang–Baxter equation* & other properties to obtain *Izergin–Korepin formula* for partition function as $n \times n$ determinant.
  - Evaluate determinant at certain values of parameters for which all weights are 1 (homogeneous limit).

II. Diagonally and Antidiagonally Symmetric Alternating Sign Matrices (DASASMs)

DASASM: ASM which is invariant under

- reflection in the diagonal &
- reflection in the antidiagonal

e.g. $$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

History:

- Arose during studies of classes of ASMs invariant under the action of subgroups of the symmetry group of the square *(Robbins 1985; Stanley 1986)*

Observations:

- any DASASM is also invariant under rotation by 180°
- any DASASM is uniquely determined by its entries in an isosceles triangle bounded by diagonal and antidiagonal
- central entry of an *odd-order* DASASM is ± 1

Number D_n of $(2n+1)\times(2n+1)$ DASASMs

& numbers $D_n^{\pm}$ of $(2n+1)\times(2n+1)$ DASASMs with central entry ± 1

n=0

$$(1)$$

$$\Rightarrow \quad D_0 = 1$$

$$\& \quad \frac{D_0^-}{D_0^+} = \frac{0}{1}$$

n=1

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad D_1 = 3$$

$$\& \quad \frac{D_1^-}{D_1^+} = \frac{1}{2}$$

n=2

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix},$$

$$\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\1&-1&0&1&0\\0&0&1&0&0\\0&1&0&-1&1\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\0&1&0&-1&1\\0&0&1&0&0\\1&-1&0&1&0\\0&1&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&1&0&0\\0&1&-1&1&0\\1&-1&1&-1&1\\0&1&-1&1&0\\0&0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&-1&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0&1\\0&0&1&0&0\\0&1&-1&1&0\\0&0&1&0&0\\1&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0&0\\0&1&0&0&0\\1&0&-1&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\1&0&-1&0&1\\0&1&0&0&0\\0&0&1&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\1&-1&1&0&0\\0&1&-1&1&0\\0&0&1&-1&1\\0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1&0\\0&0&1&-1&1\\0&1&-1&1&0\\1&-1&1&0&0\\0&1&0&0&0\end{pmatrix}$$

$$\Rightarrow \quad D_2 = 15$$

$$\& \quad \frac{D_2^-}{D_2^+} = \frac{6}{9} = \frac{2}{3}$$

General Case

of $(2n+1)\times(2n+1)$ DASASMs: $D_n = \prod_{i=0}^{n} \frac{(3i)!}{(n+i)!} = 1, 3, 15, 126, 1782, \ldots$

- Recurrrence: $\binom{2n-1}{n} D_n = \binom{3n}{n} D_{n-1}$
- Conjectured: *Robbins 1985*
- Proved: *RB, Fischer, Konvalinka 2017*
- No simple combinatorial proof currently known.

#'s of $(2n+1)\times(2n+1)$ DASASMs with central entry ±1: $\frac{D_n^-}{D_n^+} = \frac{n}{n+1}$

- Conjectured: *Stroganov 2008*
- Proved: *RB, Fischer, Konvalinka 2017*
- No simple combinatorial proof currently known.

III. Diagonally Symmetric Alternating Sign Matrices (DSASMs)

DSASM: ASM which is invariant under reflection in the main diagonal, i.e. under matrix transposition

e.g.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observation:

- any DSASM is uniquely determined by its upper triangular part

Number T_n of $n \times n$ DSASMs

n=1

$$(1) \quad \Rightarrow \quad T_1 = 1$$

n=2

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad T_2 = 2$$

n=3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad T_3 = 5$$

General Case

of $n\times n$ DSASMs:

$$T_n = \operatorname*{Pfaffian}_{0\,\text{or}\,1\le i,j\le n-1}\left(\sum_{k=0}^{\min(i,j)}(3-\delta_{k,0})\left(\binom{i+j-2k-1}{i-k}-\binom{i+j-2k-1}{j-k}\right)\right)$$

$$= 1, 2, 5, 16, 67, 368, 2630, 24376, 293770, 4610624, \ldots$$

- Range for i, j starts at 0 for n even, 1 for n odd.
- Proved: *RB, Fischer, Koutschan 2023*
- T_n for $n = 1, 2, \ldots, 1000$ has been computed using this formula.
- No simpler formula currently known.
- Product formula (i.e. product of ratios of factorials similar to formulae for # of ASMs or # of odd-order DASASMs) unlikely to exist, since prime factors of T_n do not seem bounded above by a polynomial in n.

Some Related Results

- Product formulae conjectured for numbers of ASMs in certain other symmetry classes *Robbins 1985*

- All these formulae now proved:
  - Odd-order vertically symmetric ASMs *Kuperberg 2002*
  - Odd-order vertically & horizontally symmetric ASMs *Okada 2006*
  - Even-order half-turn symmetric ASMs *Kuperberg 2002*
  - Odd-order half-turn symmetric ASMs *Razumov, Stroganov 2006*
  - Order 0 mod 4 quarter-turn symmetric ASMs *Kuperberg 2002*
  - Odd-order quarter-turn symmetric ASMs *Razumov, Stroganov 2006*
  - Odd-order diagonally & antidiagonally symmetric ASMs (DASASMs) *RB, Fischer, Konvalinka 2017*

- Diagonally symmetric ASMs (DSASMs) the only case for which a product formula is not known, but an alternative formula is known *RB, Fischer, Koutschan 2023*

- No formulae known or conjectured for numbers of ASMs in remaining nonempty symmetry classes:
  - Even-order diagonally & antidiagonally symmetric ASMs (DASASMs)
  - Odd-order totally symmetric ASMs

- Formulae also known for #'s of ASMs with fixed positions of 1's in first/last row/column:
  - SINGLE 1: *Zeilberger 1996*
  - TWO 1's on opposite or adjacent boundaries: *Colomo, Pronko, Stroganov 2005*
  - THREE or FOUR 1's: *Ayyer, Romik 2013; RB 2013*

- No simple combinatorial proofs of any non-trivial ASM enumeration formula currently known.

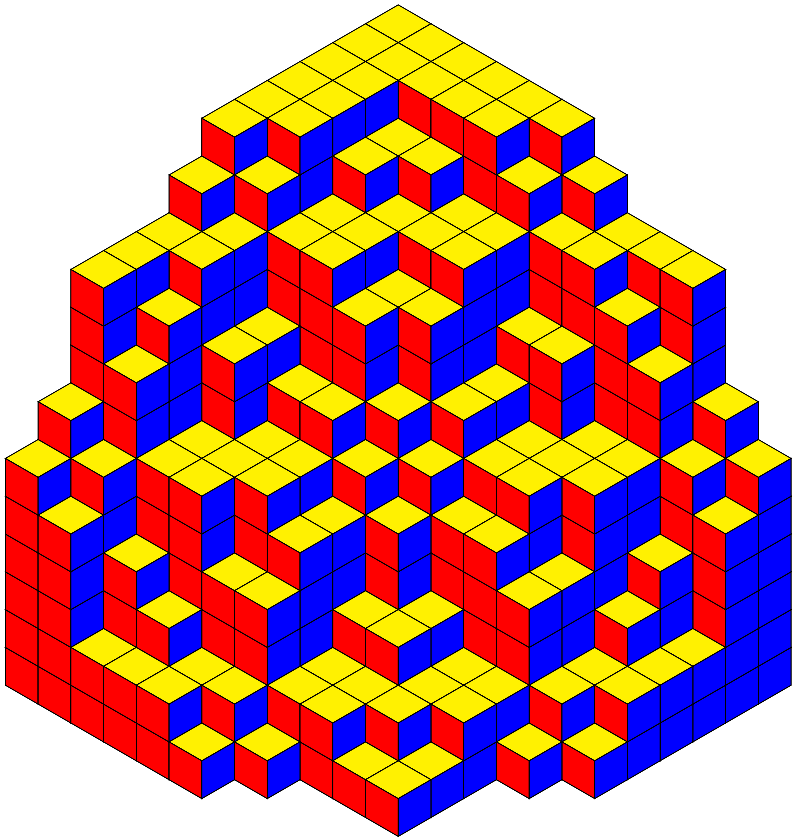
- Many other aspects of ASMs also still not properly understood, e.g., the relationship between ASMs and totally symmetric self-complementary plane partitions (TSSCPPs).

Totally Symmetric Self-Complementary Plane Partitions (TSSCPPs)

TSSCPP: aligned stack of unit cubes in a box, which is invariant under reflections, rotations & box-complementation

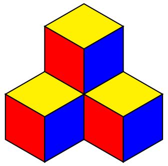
- Introduced: *Stanley 1986*

  e.g. TSSCPP in $12 \times 12 \times 12$ box:



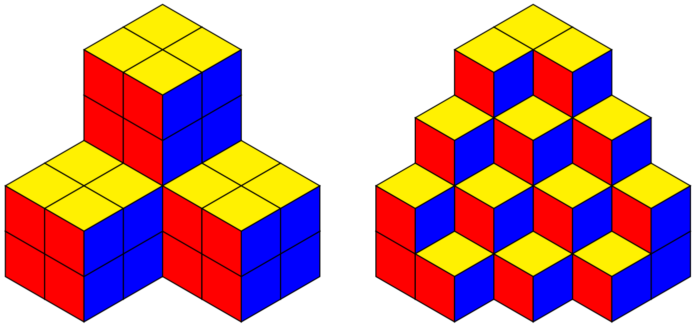
Number P_n of TSSCPPs in a $2n \times 2n \times 2n$ box

n=1



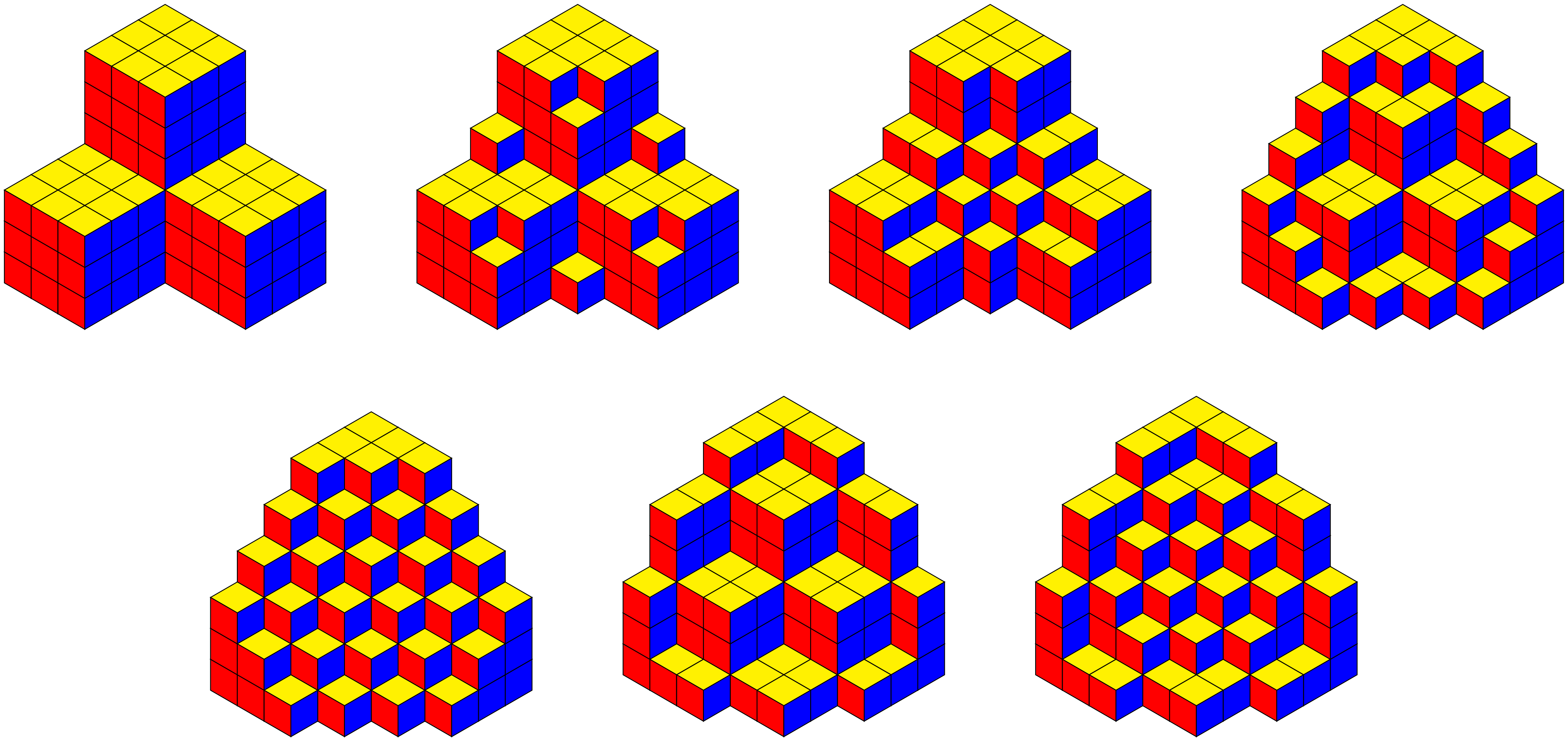
$\Rightarrow$ $P_1 = 1$

n=2



$\Rightarrow$ $P_2 = 2$

n=3



$$\Rightarrow \quad P_3 = 7$$

General Case

(# of TSSCPPs in $2n \times 2n \times 2n$ box): $P_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429, 7436, \ldots$

- Conjectured: *Mills, Robbins, Rumsey 1986*
- Proved: *Andrews 1994*
- Proof: Use sequences of nonintersecting lattice paths, "Lindström–Gessel–Viennot theorem" & determinant evaluation

- Therefore

(# of TSSCPPs in $2n \times 2n \times 2n$ box) $=$ (# of $n \times n$ ASMs)

- No explicit bijection currently known between {TSSCPPs in $2n \times 2n \times 2n$ box} & {$n \times n$ ASMs} for arbitrary n

- *"This is one of the most intriguing open problems in the area of bijective proofs." (R. Stanley 2009)*

- *"The greatest, still unsolved, mystery concerns the question of what plane partitions have to do with alternating sign matrices." (C. Krattenthaler 2016)*

Some ASM Review References

- D. Robbins *The story of 1, 2, 7, 42, 429, 7436, . . .* Math. Intelligencer **13** (1991)
- D. Bressoud & J. Propp *How the alternating sign matrix conjecture was solved* Notices Amer. Math. Soc. **46** (1999)
- D. Bressoud *Proofs and confirmations: the story of the alternating sign matrix conjecture* Cambridge University Press (1999) 274pp.
- J. Propp *The many faces of alternating sign matrices* Disc. Math. and Theor. Comp. Sci. Proc. **AA** (2001)
- J. de Gier *Loops, matchings and alternating-sign matrices* Discrete Math. **298** (2005)
- RB *Multiply-refined enumeration of alternating sign matrices* Adv. Math. **245** (2013)

Main References for Forthcoming Proofs

- A. Izergin *Partition function of the six-vertex model in a finite volume* Soviet Phys. Dokl. **32** (1987)
- G. Kuperberg *Another proof of the alternating-sign matrix conjecture* Internat. Math. Res. Notices **3** (1996)
- S. Okada *Enumeration of symmetry classes of alternating sign matrices & characters of classical groups* J. Algebraic Combin. **23** (2006)
- RB, I. Fischer & M. Konvalinka *Diagonally & antidiagonally symmetric alternating sign matrices of odd order* Adv. Math. **315** (2017)
- RB, I. Fischer & C. Koutschan *Diagonally symmetric alternating sign matrices* **arXiv:2309.08446** (2023)
- A. Garbali, J. de Gier, W. Mead & M. Wheeler *Symmetric functions from the six-vertex model in half-space* **arXiv:2312.14348** (2023)

I$'$. Outline of Proof of ASM Formula

$$\text{(\# of } n \times n \text{ ASMs)} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

(1) Obtain bijection between $n \times n$ ASMs & configurations of *six-vertex model* on $n \times n$ square with *domain-wall boundary conditions*.

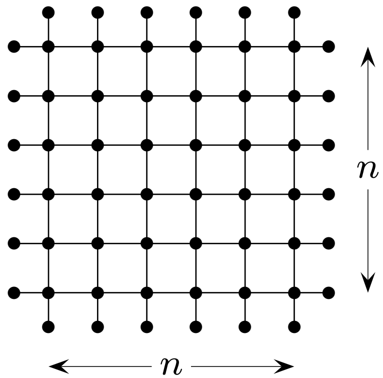
(2) Introduce spectral parameter-dependent vertex weights & consider weighted sum over all configurations of model, i.e. (inhomogeneous) *partition function*.

(3) Use *Yang–Baxter equation* & other properties to obtain *Izergin–Korepin formula* for partition function as $n \times n$ determinant.

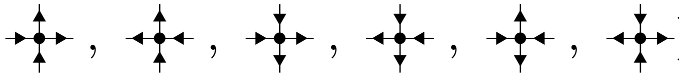
(4) Evaluate determinant at certain values of parameters for which all weights are 1 (homogeneous limit).

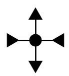
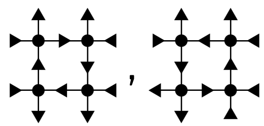
Configurations of Six-Vertex Model on a Square with Domain-Wall Boundary Conditions

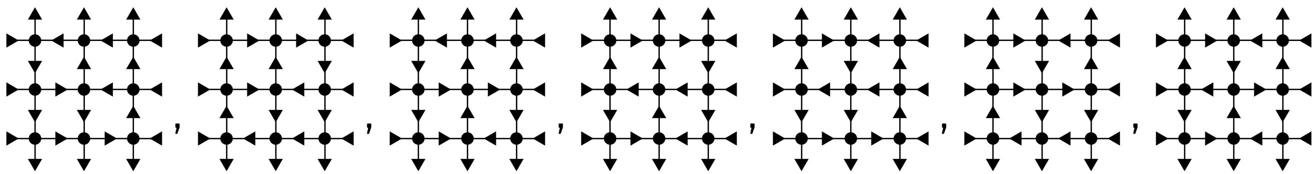
- Consider the $n \times n$ grid graph $\mathcal{G}_n$:=



- The set of config'ns of the six-vertex model on $\mathcal{G}_n$ with domain-wall boundary conditions is

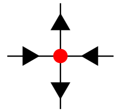
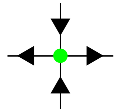
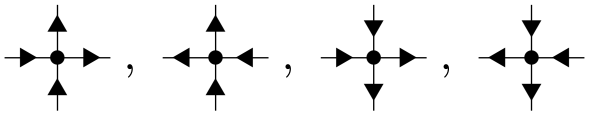
6V(n) := { edge orientations of $\mathcal{G}_n$ | • 2 in & 2 out arrows at each degree-4 vertex (⇒ 6 options: ) • all arrows out of $\mathcal{G}_n$ at top & bottom boundaries • all arrows into $\mathcal{G}_n$ at left & right boundaries }.

- e.g. 6V(1) = {  }, 6V(2) = {  },

6V(3) = {  }.

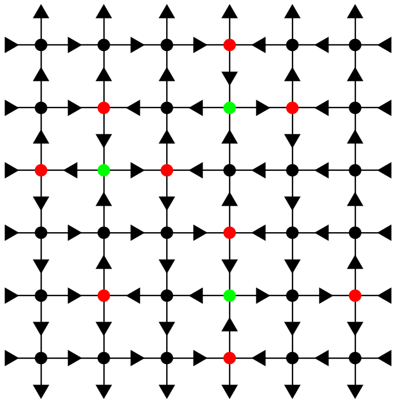
⇒ |6V(1)| = 1, |6V(2)| = 2, |6V(3)| = 7.

- Simple bijection between 6V(n) & $\{n \times n$ ASMs$\}$ obtained by associating six-vertex model local configurations at degree-4 vertices with ASM entries according to:

six-vertex model		ASM
	$\longleftrightarrow$	1
	$\longleftrightarrow$	-1
	$\longleftrightarrow$	0

(Elkies, Kuperberg, Larsen, Propp 1992)

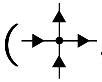
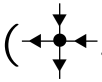
- e.g.

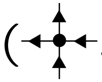
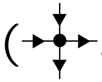


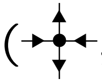
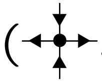
$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Vertex Weights

- Associate weights with six-vertex model local configurations at degree-4 vertices:

$W($$, u) = W($$, u) = \frac{\sigma(q\,u)}{\sigma(q^2)},$

$W($$, u) = W($$, u) = \frac{\sigma(q\,u^{-1})}{\sigma(q^2)},$

$W($$, u) = W($$, u) = 1,$

where $\sigma(x) := x - x^{-1}$, $u =$ spectral parameter, $q =$ crossing parameter.

- Simple properties of weights:
  - At $u = 1$ & $q = e^{i\pi/3}$: $W\Big(a \overset{d}{\underset{b}{+}} c\,, 1\Big)\big|_{q=e^{i\pi/3}} = 1$.
  - At $u = q^{\pm 1}$: $W\Big(a \overset{d}{\underset{b}{+}} c\,, q^{-1}\Big) = \delta_{ab'}\,\delta_{cd'}$ & $W\Big(a \overset{d}{\underset{b}{+}} c\,, q\Big) = \delta_{ad'}\,\delta_{bc'}$,

    where a' denotes reversal of orientation a (i.e. in $\leftrightarrow$ out).

- Weights satisfy the Yang–Baxter Equation (YBE):

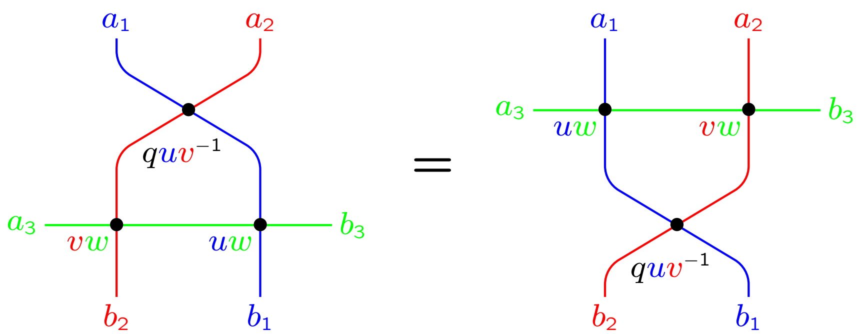
  For all a_1, b_1, a_2, b_2, a_3, b_3 $\in \{\text{in}, \text{out}\}$,

$$\sum_{c_1,c_2,c_3} W\left(\begin{smallmatrix} & a_1 & \\ c_2 & \bullet & a_2 \\ & c_1 & \end{smallmatrix}, q\,u\,v^{-1}\right) W\left(\begin{smallmatrix} & c_2 & \\ a_3 & \bullet & c_3 \\ & b_2 & \end{smallmatrix}, v\,w\right) W\left(\begin{smallmatrix} & c_1 & \\ c_3 & \bullet & b_3 \\ & b_1 & \end{smallmatrix}, u\,w\right)$$

$$= \sum_{c_1,c_2,c_3} W\left(\begin{smallmatrix} & a_1 & \\ a_3 & \bullet & c_3 \\ & c_1 & \end{smallmatrix}, u\,w\right) W\left(\begin{smallmatrix} & a_2 & \\ c_3 & \bullet & b_3 \\ & c_2 & \end{smallmatrix}, v\,w\right) W\left(\begin{smallmatrix} & c_1 & \\ b_2 & \bullet & c_2 \\ & b_1 & \end{smallmatrix}, q\,u\,v^{-1}\right),$$

  where sums are over all $c_1, c_2, c_3 \in \{\text{in}, \text{out}\}$ with 2 in & 2 out arrows at each vertex.

- YBE depicted as:



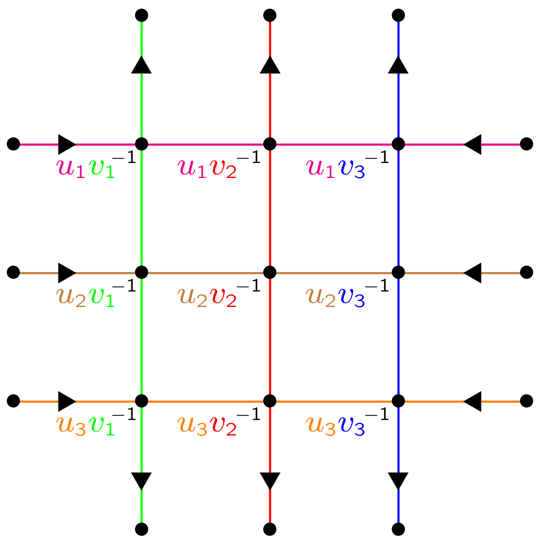
Partition Function

- Define the *partition function* as

$$Z(u_1,\ldots,u_n,v_1,\ldots,v_n) \;:=\; \sum_{C\in 6\mathsf{V}(n)} \prod_{i,j=1}^{n} W(C_{ij}, u_i v_j^{-1}),$$

where C_{ij} = local configuration at vertex in row i & column j of grid $\mathcal{G}_n$.

- e.g. $Z(u_1,u_2,u_3,v_1,v_2,v_3)$ = sum of 7 terms, each consisting of a product of 9 weights

" = " 

- Since $W(C_{ij},1)|_{q=e^{i\pi/3}} = 1$ for all i,j & $6\mathsf{V}(n)$ is in bijection with $\{n\times n$ ASMs$\}$:

$$Z(\underbrace{1,\ldots,1}_{2n})|_{q=e^{i\pi/3}} \;=\; |6\mathsf{V}(n)| \;=\; (\#\text{ of } n\times n \text{ ASMs}).$$

Izergin–Korepin Formula

- The *Izergin–Korepin Formula* is

$$Z(u_1,\ldots,u_n,v_1,\ldots,v_n)$$
$$= \frac{\prod_{i,j=1}^{n}\sigma(q\,u_iv_j^{-1})\,\sigma(q\,u_i^{-1}v_j)}{\sigma(q^2)^{n(n-1)}\prod_{1\le i<j\le n}\sigma(u_iu_j^{-1})\sigma(v_i^{-1}v_j)}\det_{1\le i,j\le n}\left(\frac{1}{\sigma(q\,u_iv_j^{-1})\,\sigma(q\,u_i^{-1}v_j)}\right) \quad \textit{(Izergin 1987)}$$

<u>Proof outline:</u>

(a) Show that a function $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ which satisfies the following conditions is uniquely determined:

(i) $X(u_1,v_1)=1$. *(Initial condition)*

(ii) $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is a Laurent polynomial in u_1 of lower degree $\geq -n+1$ & upper degree $\leq n-1$. *(Laurent polynomial condition)*

(iii) If $u_1v_1^{-1}=q^{\pm1}$, then

$$X(u_1,\ldots,u_n,v_1,\ldots,v_n)=\frac{\prod_{i=2}^{n}\sigma(q^{\pm1}u_1v_i^{-1})\sigma(q^{\pm1}u_iv_1^{-1})}{\sigma(q^2)^{2n-2}}X(u_2,\ldots,u_n,v_2,\ldots,v_n).$$

(Recurrence condition)

(iv) $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is symmetric in $v_1,\ldots,v_n$. *(Symmetry condition)*

(b) Show that LHS of Izergin–Korepin formula satisfies conditions (i)–(iv).

(c) Show that RHS of Izergin–Korepin formula satisfies conditions (i)–(iv).

Subsections of Proof of Izergin–Korepin Formula

(a) If $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ satisfies the following conditons, then it is uniquely determined:

(i) $X(u_1,v_1)=1$.

(ii) $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is a Laurent polynomial in u_1 of lower degree $\geq -n+1$ & upper degree $\leq n-1$.

(iii) If $u_1v_1^{-1}=q^{\pm 1}$, then

$$X(u_1,\ldots,u_n,v_1,\ldots,v_n) = \frac{\prod_{i=2}^n \sigma(q^{\pm 1}u_1v_i^{-1})\sigma(q^{\pm 1}u_iv_1^{-1})}{\sigma(q^2)^{2n-2}} X(u_2,\ldots,u_n,v_2,\ldots,v_n).$$

(iv) $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is symmetric in $v_1,\ldots,v_n$.

Proof:

By (ii), $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is uniquely determined if it is known at $2n-1$ values of u_1.

Combining (iii) & (iv) gives expressions at $2n$ values of u_1, i.e.
$X(q^{\pm 1}v_i,u_2,\ldots,u_n,v_1,\ldots,v_n) = \ldots\ X(u_2,\ldots,u_n,v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_n)$ for $i=1,\ldots,n$.

Required result now follows using recursion on n, together with (i). □

(b) The LHS of the Izergin-Korepin formula, i.e. the partition function $Z(u_1,\ldots,u_n,v_1,\ldots,v_n)$, satisfies:

(i) $Z(u_1,v_1)=1$.

(ii) $Z(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is a Laurent polynomial in u_1 of lower degree $\geq -n+1$ & upper degree $\leq n-1$.

(iii) If $u_1v_1^{-1}=q^{\pm 1}$, then

$$Z(u_1,\ldots,u_n,v_1,\ldots,v_n) = \frac{\prod_{i=2}^{n}\sigma(q^{\pm 1}u_1v_i^{-1})\sigma(q^{\pm 1}u_iv_1^{-1})}{\sigma(q^2)^{2n-2}}\,Z(u_2,\ldots,u_n,v_2,\ldots,v_n).$$

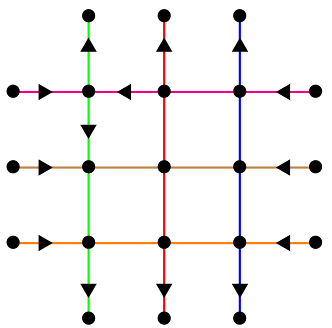
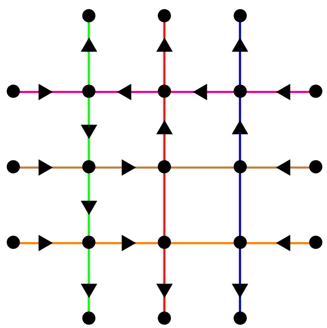
(iv) $Z(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is symmetric in $v_1,\ldots,v_n$.

Proof:

(i) & (ii) follow straightforwardly from definitions of weights & partition function.

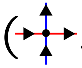
As an example of (iii), let $n=3$ & $u_1{v_1}^{-1}=q^{-1}$.

Then $Z(u_1,u_2,u_3,v_1,v_2,v_3)$

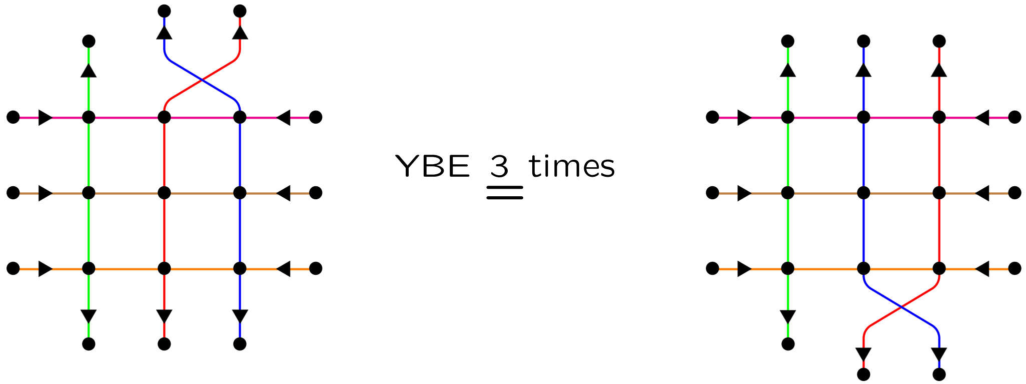
$=$  $=$ 

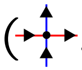
$$= \frac{\sigma(q^{-1}u_1{v_2}^{-1})\sigma(q^{-1}u_1{v_3}^{-1})\sigma(q^{-1}u_2{v_1}^{-1})\sigma(q^{-1}u_3{v_1}^{-1})}{\sigma(q^2)^4}\,Z(u_2,u_3,v_2,v_3).$$

As an example of (iv), symmetry of $Z(u_1, u_2, u_3, v_1, v_2, v_3)$ in v_2 & v_3 follows by using Yang-Baxter equation 3 times:

$W($  $, q{v_2}^{-1}v_3)\, Z(u_1, u_2, u_3, v_1, v_2, v_3)$

$=$



$=$ $W($  $, q{v_2}^{-1}v_3)\, Z(u_1, u_2, u_3, v_1, v_3, v_2).$

(c) The RHS of the Izergin–Korepin formula, i.e.

$$Y(u_1,\ldots,u_n,v_1,\ldots,v_n) := \frac{\prod_{i,j=1}^{n}\sigma(q\,u_iv_j^{-1})\,\sigma(q\,u_i^{-1}v_j)}{\sigma(q^2)^{n(n-1)}\prod_{1\le i<j\le n}\sigma(u_iu_j^{-1})\sigma(v_i^{-1}v_j)}\det_{1\le i,j\le n}\left(\frac{1}{\sigma(q\,u_iv_j^{-1})\,\sigma(q\,u_i^{-1}v_j)}\right),$$

satisfies:

(i) $Y(u_1,v_1)=1$.

(ii) $Y(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is a Laurent polynomial in u_1 of lower degree $\ge -n+1$, upper degree $\le n-1$.

(iii) If $u_1v_1^{-1}=q^{\pm1}$, then

$$Y(u_1,\ldots,u_n,v_1,\ldots,v_n) = \frac{\prod_{i=2}^{n}\sigma(q^{\pm1}u_1v_i^{-1})\sigma(q^{\pm1}u_iv_1^{-1})}{\sigma(q^2)^{2n-2}}\,Y(u_2,\ldots,u_n,v_2,\ldots,v_n).$$

(iv) $Y(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is symmetric in $v_1,\ldots,v_n$.

Proof:

(i) is trivial.

(ii) follows from expansion of determinant.

(iii) holds since, by multiplying first row of matrix by $\sigma(qu_1v_1^{-1})\,\sigma(qu_1^{-1}v_1)$ from prefactor, & then setting $u_1v_1^{-1}=q^{\pm1}$, this row becomes $(1,0,\ldots,0)$.

(iv) holds since if v_i & v_j (for $i\neq j$) are interchanged, then determinant simply changes sign, since columns i and j of the matrix are swapped, & prefactor also changes sign. □

Evaluation of RHS of Izergin–Korepin Formula with $u_1 = \ldots = u_n = v_1 = \ldots = v_n = 1$ & $q = e^{i\pi/3}$

- Recall that $Z(\underbrace{1,\ldots,1}_{2n})|_{q=e^{i\pi/3}} = (\#\text{ of } n \times n \text{ ASMs})$.

- Cannot immediately evaluate RHS of Izergin–Korepin formula at $u_1 = \ldots = u_n = v_1 = \ldots = v_n = 1$ since setting $u_i = u_j$ or $v_i = v_j$ for $i \neq j$ gives $0\,/\,0$.

- Instead will first set $q = e^{i\pi/3}$ & then use a result of Okada.
  [This differs from the method used by Kuperberg.]

- $Z(u_1,\ldots,u_n,v_1,\ldots,v_n)|_{q=e^{i\pi/3}} =$

$$3^{-n(n-1)/2}\left(\prod_{i=1}^{n} u_i v_i\right)^{-n+1} s_{(n-1,n-1,\ldots,2,2,1,1)}(u_1^2,\ldots,u_n^2,v_1^2,\ldots,v_n^2) \qquad \textit{(Okada 2006)}$$

[where $s_\lambda(x_1,\ldots,x_k) =$ Schur function].

- A proof of this will be sketched soon.

- Now set $u_1 = \ldots = u_n = v_1 = \ldots = v_n = 1$.

- Then apply standard fact for Schur function with all variables 1:

$$s_\lambda(\underbrace{1,\ldots,1}_{k}) = \mathsf{SSYT}_\lambda(k)$$

  [where $\mathsf{SSYT}_\lambda(k) = (\#$ of semistandard Young tableaux of shape λ with entries $\leq k)$].

- Finally, apply standard product formula for # of semistandard Young tableaux:

$$\mathsf{SSYT}_\lambda(k) = \frac{\prod_{1\le i<j\le k}(\lambda_i - \lambda_j - i + j)}{\prod_{i=1}^{k-1} i!}.$$

- Gives (# of $n \times n$ ASMs) $= 3^{-n(n-1)/2}\,\mathsf{SSYT}_{(n-1,n-1,\dots,2,2,1,1)}(2n)$

$$= \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$ as required.

Sketch of Okada's proof that

$$Z(u_1,\ldots,u_n,v_1,\ldots,v_n)\big|_{q=e^{i\pi/3}} \;=\; 3^{-n(n-1)/2}\left(\prod_{i=1}^{n} u_i v_i\right)^{-n+1} s_{(n-1,n-1,\ldots,2,2,1,1)}(u_1^2,\ldots,u_n^2,v_1^2,\ldots,v_n^2)$$

- Set $q = e^{i\pi/3}$ in Izergin–Korepin formula.

- Apply identity which relates certain $n\times n$ determinant to certain $2n\times 2n$ determinant:

$$\det_{1\le i,j\le n}\left(\frac{a_i-b_j}{x_i-y_j}\right) = \frac{(-1)^{n(n+1)/2}}{\prod_{i,j=1}^{n}(x_i-y_j)}\det\begin{pmatrix} 1 & a_1 & x_1 & a_1x_1 & \ldots & x_1^{n-1} & a_1x_1^{n-1} \\ 1 & b_1 & y_1 & b_1y_1 & \ldots & y_1^{n-1} & b_1y_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & x_n & a_nx_n & \ldots & x_n^{n-1} & a_nx_n^{n-1} \\ 1 & b_n & y_n & b_ny_n & \ldots & y_n^{n-1} & b_ny_n^{n-1} \end{pmatrix} \quad \textit{(Okada 1998)}$$

  with appropriate a_i, b_i, x_i & y_i.

- Use standard determinantal expression for Schur function:

$$s_\lambda(x_1,\ldots,x_n) = \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j}\right)}{\prod_{1\le i<j\le n}(x_i-x_j)}.$$

II′. Outline of Proof of Odd-Order DASASM Formula

$$(\text{\# of } (2n+1)\times(2n+1) \text{ DASASMs}) = \prod_{i=0}^{n} \frac{(3i)!}{(n+i)!}$$

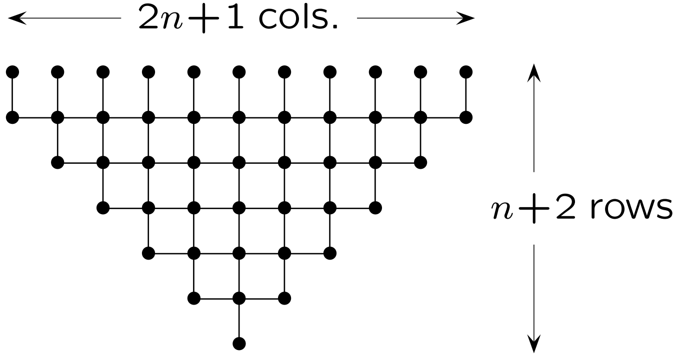
(1) Obtain bijection between $(2n+1)\times(2n+1)$ DASASMs & configurations of six-vertex model on a certain isosceles triangle.

(2) Introduce parameter-dependent bulk weights, *boundary weights* & associated partition function.

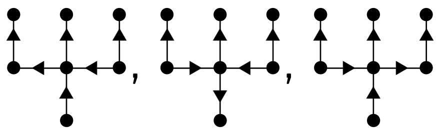
(3) Use Yang–Baxter equation, *reflection equation* & other properties to prove formula for partition function involving *sum* of *two* $(n+1)\times(n+1)$ determinants.

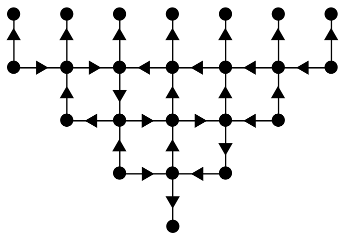
(4) Evaluate determinantal formula at values of parameters for which all weights are 1 (homogeneous limit).

Configurations of Six-Vertex Model on Isosceles Triangle

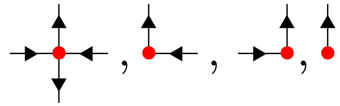
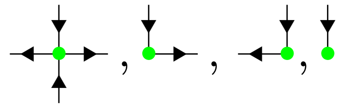
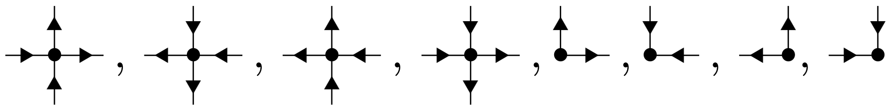
- Let $\mathcal{T}_n$ = 

$$6\text{VDA}(n) = \left\{ \begin{array}{l} \text{orientations of} \\ \text{edges of } \mathcal{T}_n \end{array} \;\middle|\; \begin{array}{l} \bullet \text{ 2 in \& 2 out arrows at each degree 4 vertex} \\ \quad (\Rightarrow \text{6 cases}) \\ \bullet \text{ no restriction at degree 2 vertices } (\Rightarrow \text{4 cases}) \\ \bullet \text{ all arrows upward on top boundary} \\ \bullet \text{ no restriction on single bottom edge } (\Rightarrow \text{2 cases}) \end{array} \right\}$$

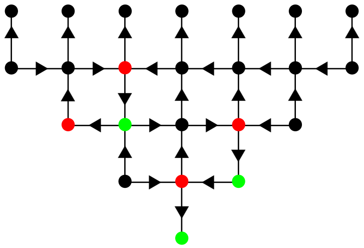
- e.g. 6VDA(1) = {  }

- e.g.  ∈ 6VDA(3)

Six-Vertex Model Configuration – DASASM Bijection

six-vertex model		DASASM
	$\longleftrightarrow$	1
	$\longleftrightarrow$	-1
	$\longleftrightarrow$	0

- Also use reflections in diagonal and antidiagonal.
- Gives bijection between 6VDA(n) & $\{(2n+1)\times(2n+1)$ DASASMs$\}$.
- e.g.

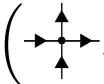
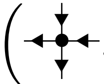


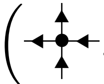
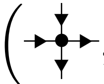
$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 0 & 1 & 0 & \\ & & 0 & 1 & -1 & & \\ & & & -1 & & & \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

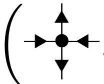
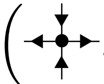
Vertex Weights

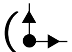
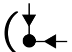
- $\sigma(x) := x - x^{-1}$ • $u =$ spectral parameter • $q =$ crossing parameter

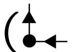
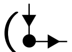
- Bulk weights (with q from previous ASM case replaced by q^2):

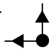
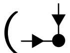
$W\Big($  $, u\Big) = W\Big($  $, u\Big) = \frac{\sigma(q^2 u)}{\sigma(q^4)}$

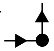
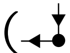
$W\Big($  $, u\Big) = W\Big($  $, u\Big) = \frac{\sigma(q^2 u^{-1})}{\sigma(q^4)}$

$W\Big($  $, u\Big) = W\Big($  $, u\Big) = 1$

- Left boundary weights: $W($  $, u) = W($  $, u) = \frac{\sigma(q\,u)}{\sigma(q)}$

$W($  $, u) = W($  $, u) = 1$

- Right boundary weights: $W($  $, u) = W($  $, u) = \frac{\sigma(q\,u^{-1})}{\sigma(q)}$

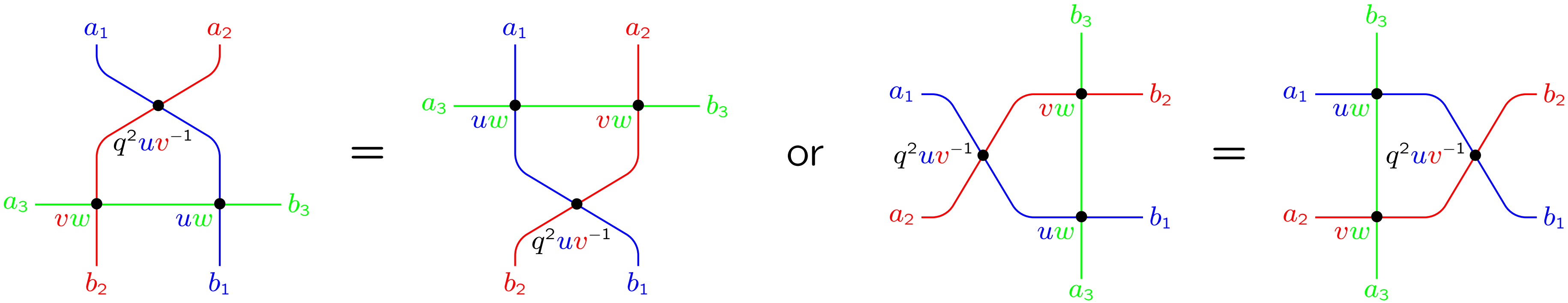
$W($  $, u) = W($  $, u) = 1$

- At $u = 1$ & $q = e^{i\pi/6}$: $W(c, 1)|_{q=e^{i\pi/6}} = 1$ for all c.

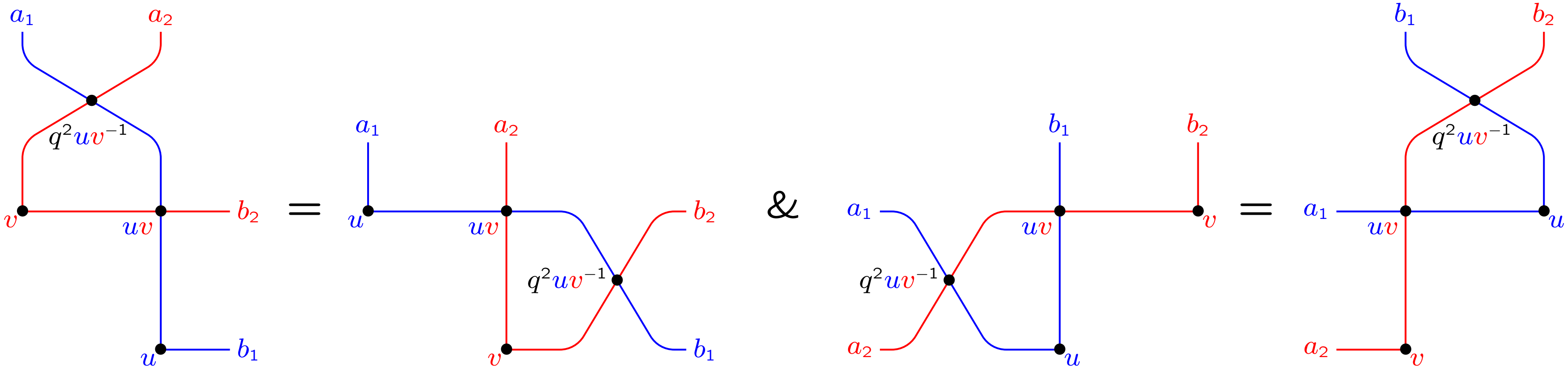
- At $u = q^{\pm 2}$: $W\left(a \overset{d}{\underset{b}{+}} c\ , q^{-2}\right) = \delta_{ab'}\,\delta_{cd'}$ & $W\left(a \overset{d}{\underset{b}{+}} c\ , q^{2}\right) = \delta_{ad'}\,\delta_{bc'}$,

  where a' denotes reversal of orientation a (i.e. in $\leftrightarrow$ out).

- Yang–Baxter equation



- Left & right reflection equations (boundary Yang–Baxter equations)



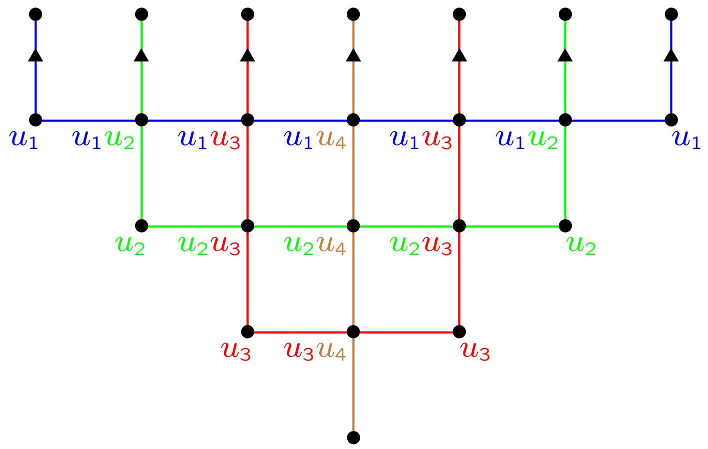
Odd-Order DASASM Partition Function

- $Z(u_1,\ldots,u_{n+1}) :=$

$$\sum_{C\in 6\text{VDA}(n)} \prod_{i=1}^{n} W(C_{ii},u_i) \left(\prod_{j=i+1}^{2n+1-i} W(C_{ij},u_i\,u_{\min(j,2n+2-j)}) \right) W(C_{i,2n+1-i},u_i)$$

[where C_{ij} = local configuration at vertex in row i & column j of $\mathcal{T}_n$]

- e.g. $Z(u_1,u_2,u_3,u_4) =$



- Therefore $Z(\underbrace{1,\ldots,1}_{n+1})|_{q=e^{i\pi/6}} = |6\text{VDA}(n)| =$ (# of $(2n+1)\times(2n+1)$ DASASMs).

Sum of Determinants Formula for Partition Function

$$Z(u_1,\ldots,u_{n+1}) =$$

$$\frac{\sigma(q^2)^n}{\sigma(q)^{2n}\,\sigma(q^4)^{n^2}}\prod_{i=1}^{n}\frac{\sigma(u_i)\sigma(qu_i)\sigma(qu_i^{-1})\sigma(q^2u_iu_{n+1})\sigma(q^2u_i^{-1}u_{n+1}^{-1})}{\sigma(u_iu_{n+1}^{-1})}\prod_{1\le i<j\le n}\left(\frac{\sigma(q^2u_iu_j)\sigma(q^2u_i^{-1}u_j^{-1})}{\sigma(u_iu_j^{-1})}\right)^2$$

$$\times\left(\det_{1\le i,j\le n+1}\left(\begin{cases}\frac{q^2+q^{-2}+u_i^2+u_j^{-2}}{\sigma(q^2u_iu_j)\,\sigma(q^2u_i^{-1}u_j^{-1})}, & i\le n\\ \frac{u_{n+1}-1}{u_j^2-1}, & i=n+1\end{cases}\right)+\det_{1\le i,j\le n+1}\left(\begin{cases}\frac{q^2+q^{-2}+u_i^{-2}+u_j^2}{\sigma(q^2u_iu_j)\,\sigma(q^2u_i^{-1}u_j^{-1})}, & i\le n\\ \frac{u_{n+1}^{-1}-1}{u_j^{-2}-1}, & i=n+1\end{cases}\right)\right)$$

Proof outline:

- Show that a function $X(u_1,\ldots,u_{n+1})$ which satisfies the following properties is uniquely determined:
  - (i) $X(u_1)=1$
  - (ii) $X(u_1,\ldots,u_{n+1})$ a Laurent polynomial in u_{n+1} of lower degree $\ge -n$, upper degree $\le n$
  - (iii) $X(u_1,\ldots,u_{n+1})|_{u_1u_{n+1}=q^2} = \frac{\sigma(qu_1)\,(\sigma(qu_1^{-1})+\sigma(q))\prod_{i=2}^{n}\sigma(q^2u_1u_i)\sigma(q^2u_iu_{n+1})}{\sigma(q)^2\,\sigma(q^4)^{2n-2}}\,X(u_2,\ldots,u_n,u_1)$
  - (iv) $X(u_1,\ldots,u_{n+1})$ symmetric in $u_1,\ldots,u_n$
  - (v) $X(u_1^{-1},\ldots,u_{n+1}^{-1})=X(u_1,\ldots,u_{n+1})$
  - (vi) $X(u_1,\ldots,u_{n+1})$ even in u_i, for $i=1,\ldots,n$
- Show that LHS & RHS of required formula both satisfy all these properties.

Sections of Proof

- If a function $X(u_1,\ldots,u_{n+1})$ satisfies following properties, then it is uniquely determined:
  (i) $X(u_1)=1$
  (ii) $X(u_1,\ldots,u_{n+1})$ a Laurent polynomial in u_{n+1} of lower degree $\geq -n$, upper degree $\leq n$
  (iii) $X(u_1,\ldots,u_{n+1})|_{u_1 u_{n+1}=q^2} = \frac{\sigma(qu_1)\,(\sigma(qu_1^{-1})+\sigma(q))\prod_{i=2}^{n}\sigma(q^2u_1u_i)\sigma(q^2u_iu_{n+1})}{\sigma(q)^2\,\sigma(q^4)^{2n-2}}\,X(u_2,\ldots,u_n,u_1)$
  (iv) $X(u_1,\ldots,u_{n+1})$ symmetric in $u_1,\ldots,u_n$
  (v) $X(u_1^{-1},\ldots,u_{n+1}^{-1}) = X(u_1,\ldots,u_{n+1})$
  (vi) $X(u_1,\ldots,u_{n+1})$ even in u_i, for $i=1,\ldots,n$

Proof:

By (ii), $X(u_1,\ldots,u_{n+1})$ is uniquely determined if known at $2n+1$ values of u_{n+1}.

Combining (iii) with (iv)–(vi) gives expressions at $4n$ ($\geq 2n+1$) values of u_{n+1}, i.e.
$X(u_1,\ldots,u_n,q^{\pm2}u_i^{-1})$ expressed in terms of $X(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_n,u_i)$ &
$X(u_1,\ldots,u_n,-q^{\pm2}u_i^{-1})$ expressed in terms of $X(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_n,-u_i)$, $\;i=1,\ldots,n$.

Result follows using recursion on n, together with (i). □

- The partition function $Z(u_1, \ldots, u_{n+1})$ satisfies

  (i) $Z(u_1) = 1$

  (ii) $Z(u_1, \ldots, u_{n+1})$ a Laurent polynomial in u_{n+1} of lower degree $\geq -n$, upper degree $\leq n$

  (v) $Z(u_1^{-1}, \ldots, u_{n+1}^{-1}) = Z(u_1, \ldots, u_{n+1})$

  (vi) $Z(u_1, \ldots, u_{n+1})$ even in u_i, for $i = 1, \ldots, n$

Proof:

(i), (ii) & (vi) follow straightforwardly from definition of weights & partition function.

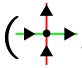
(v) follows from reflection of configurations in central vertical line of graph $\mathcal{T}_n$. $\square$

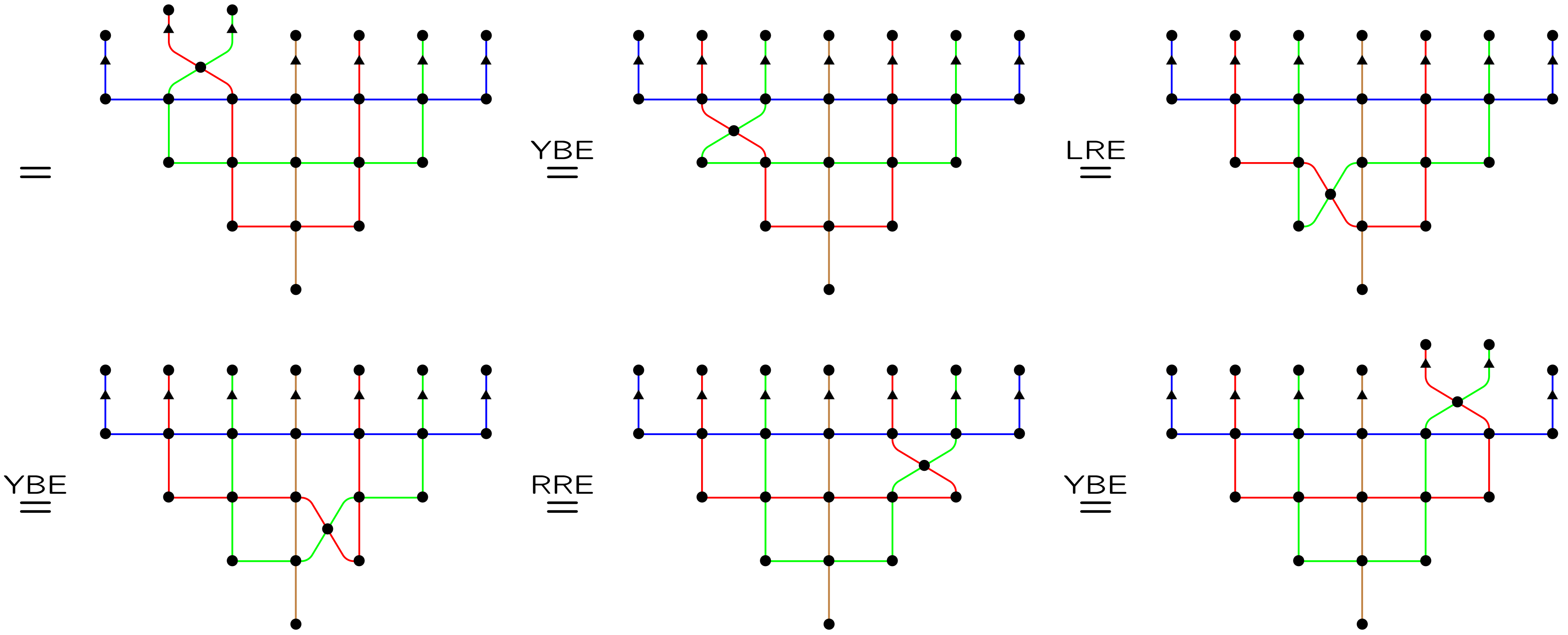
- (iv) $Z(u_1,\ldots,u_{n+1})$ is symmetric in $u_1,\ldots,u_n$

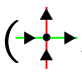
<u>Proof:</u>

Use Yang–Baxter and reflection equations (YBE, LRE, RRE) to show that $Z(u_1,\ldots,u_{n+1})$ is symmetric in u_i and u_{i+1}, $i=1,\ldots,n-1$.

e.g. $n=3$ & $i=2$:

$W($$, q^2 u_2{}^{-1} u_3)\, Z(u_1, u_2, u_3, u_4)$

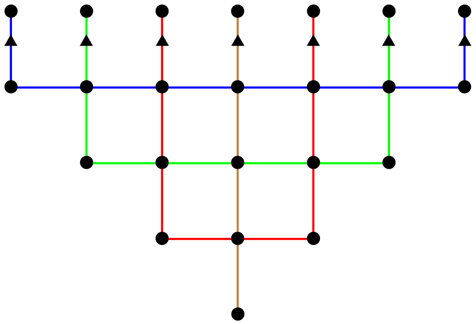
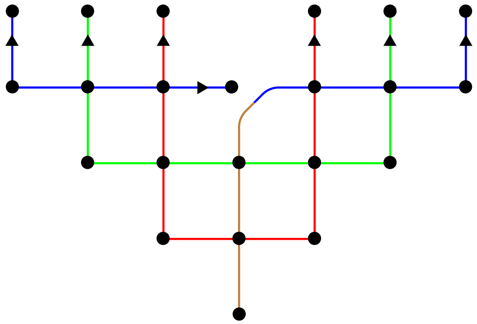


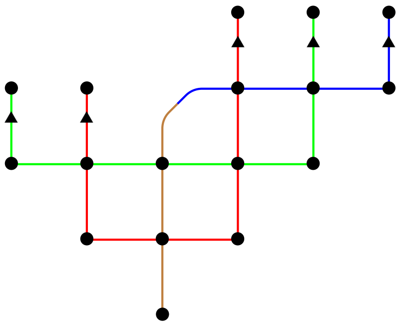
$= W($$, q^2 u_2{}^{-1} u_3)\, Z(u_1, u_3, u_2, u_4)$ □

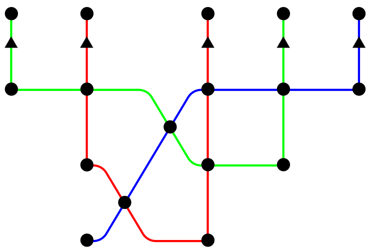
- (iii) $Z(u_1,\ldots,u_{n+1})|_{u_1u_{n+1}=q^2} = \frac{\sigma(qu_1)\,(\sigma(qu_1^{-1})+\sigma(q))\prod_{i=2}^{n}\sigma(q^2u_1u_i)\sigma(q^2u_iu_{n+1})}{\sigma(q)^2\,\sigma(q^4)^{2n-2}}\,Z(u_2,\ldots,u_n,u_1)$

Proof:

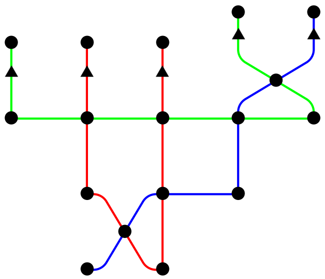
e.g. $n=3$:

$Z(u_1,u_2,u_3,u_4)|_{u_1u_4=q^2} =$  $=$ 

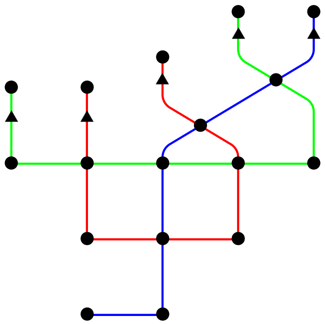
$= \frac{\sigma(qu_1)\,\sigma(q^2u_1u_2)\,\sigma(q^2u_1u_3)}{\sigma(q)\,\sigma(q^4)^2}$ 

$= \frac{\sigma(qu_1)\,\sigma(q^2u_1u_2)\,\sigma(q^2u_1u_3)}{\sigma(q)\,\sigma(q^4)^2}$ 

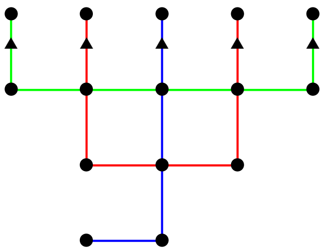
$$\overset{\text{YBE, RRE}}{=} \quad \frac{\sigma(qu_1)\,\sigma(q^2u_1u_2)\,\sigma(q^2u_1u_3)}{\sigma(q)\,\sigma(q^4)^2}$$



$$\overset{\text{RRE, YBE}}{=} \quad \frac{\sigma(qu_1)\,\sigma(q^2u_1u_2)\,\sigma(q^2u_1u_3)}{\sigma(q)\,\sigma(q^4)^2}$$



$$= \quad \frac{\sigma(qu_1)\,\sigma(q^2u_1u_2)\,\sigma(q^2u_2u_4)\,\sigma(q^2u_1u_3)\,\sigma(q^2u_3u_4)}{\sigma(q)\,\sigma(q^4)^4}$$



$$= \quad \frac{\sigma(qu_1)\,\sigma(q^2u_1u_2)\,\sigma(q^2u_2u_4)\,\sigma(q^2u_1u_3)\,\sigma(q^2u_3u_4)}{\sigma(q)\,\sigma(q^4)^4}\left(\frac{\sigma(qu_1{}^{-1})}{\sigma(q)}+1\right)\left(Z_-(u_2,u_3,u_1)+Z_+(u_2,u_3,u_1)\right)$$

$$= \quad \frac{\sigma(qu_1)\,(\sigma(qu_1{}^{-1})+\sigma(q))\,\sigma(q^2u_1u_2)\,\sigma(q^2u_2u_4)\,\sigma(q^2u_1u_3)\,\sigma(q^2u_3u_4)}{\sigma(q)^2\,\sigma(q^4)^4}\,Z(u_2,u_3,u_1) \qquad \square$$

- $Y(u_1,\dots,u_{n+1})$:=

$$\frac{\sigma(q^2)^n}{\sigma(q)^{2n}\,\sigma(q^4)^{n^2}}\prod_{i=1}^{n}\frac{\sigma(u_i)\sigma(qu_i)\sigma(qu_i^{-1})\sigma(q^2u_iu_{n+1})\sigma(q^2u_i^{-1}u_{n+1}^{-1})}{\sigma(u_iu_{n+1}^{-1})}\prod_{1\le i<j\le n}\left(\frac{\sigma(q^2u_iu_j)\sigma(q^2u_i^{-1}u_j^{-1})}{\sigma(u_iu_j^{-1})}\right)^2$$

$$\times\left(\det_{1\le i,j\le n+1}\left(\begin{cases}\frac{q^2+q^{-2}+u_i^2+u_j^{-2}}{\sigma(q^2u_iu_j)\,\sigma(q^2u_i^{-1}u_j^{-1})}, & i\le n\\ \frac{u_{n+1}-1}{u_j^2-1}, & i=n+1\end{cases}\right)+\det_{1\le i,j\le n+1}\left(\begin{cases}\frac{q^2+q^{-2}+u_i^{-2}+u_j^2}{\sigma(q^2u_iu_j)\,\sigma(q^2u_i^{-1}u_j^{-1})}, & i\le n\\ \frac{u_{n+1}^{-1}-1}{u_j^{-2}-1}, & i=n+1\end{cases}\right)\right)$$

satisfies

(i) $Y(u_1)=1$

(ii) $Y(u_1,\dots,u_{n+1})$ a Laurent polynomial in u_{n+1} of lower degree $\ge -n$, upper degree $\le n$

(iii) $Y(u_1,\dots,u_{n+1})|_{u_1u_{n+1}=q^2} = \frac{\sigma(qu_1)\,(\sigma(qu_1^{-1})+\sigma(q))\prod_{i=2}^{n}\sigma(q^2u_1u_i)\sigma(q^2u_iu_{n+1})}{\sigma(q)^2\,\sigma(q^4)^{2n-2}}\,Y(u_2,\dots,u_n,u_1)$

(iv) $Y(u_1,\dots,u_{n+1})$ symmetric in $u_1,\dots,u_n$

(v) $Y(u_1^{-1},\dots,u_{n+1}^{-1})=Y(u_1,\dots,u_{n+1})$

(vi) $Y(u_1,\dots,u_{n+1})$ even in u_i, for $i=1,\dots,n$

Proof:

Reasonably straightforward, using standard properties of determinants. □

Partition Function at $q = e^{i\pi/6}$

$$Z(u_1,\ldots,u_{n+1})|_{q=e^{i\pi/6}} = 3^{-n(n-1)/2}\Big(\tfrac{u_{n+1}^n}{u_{n+1}+1}\, s_{(n,n-1,n-1,\ldots,2,2,1,1)}(u_1^2,u_1^{-2},\ldots,u_n^2,u_n^{-2},u_{n+1}^{-2}) + \tfrac{u_{n+1}^{-n}}{u_{n+1}^{-1}+1}\, s_{(n,n-1,n-1,\ldots,2,2,1,1)}(u_1^2,u_1^{-2},\ldots,u_n^2,u_n^{-2},u_{n+1}^{2})\Big)$$

[where $s_\lambda(x_1,\ldots,x_k)$ = Schur function]

Proof outline:

- Substitute $q = e^{i\pi/6}$ into determinantal expression for partition function.
- Apply general determinant identity

$$\det_{1\le i,j\le k}\left(\frac{a_i-b_j}{x_i-y_j}\right) = \frac{(-1)^{k(k+1)/2}}{\prod_{i,j=1}^{k}(x_i-y_j)}\det\begin{pmatrix}1 & a_1 & x_1 & a_1x_1 & \ldots & x_1^{k-1} & a_1x_1^{k-1}\\ 1 & b_1 & y_1 & b_1y_1 & \ldots & y_1^{k-1} & b_1y_1^{k-1}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & a_k & x_k & a_kx_k & \ldots & x_k^{k-1} & a_kx_k^{k-1}\\ 1 & b_k & y_k & b_ky_k & \ldots & y_k^{k-1} & b_ky_k^{k-1}\end{pmatrix}$$ *(Okada 1998)*

with $a_i = \begin{cases} u_i^{\pm 2}+u_i^{\pm 4}, & i \le n\\ -1, & i = n+1,\end{cases}$ $\quad x_i = \begin{cases} u_i^{\pm 6}, & i \le n\\ 1, & i = n+1,\end{cases}$ $\quad b_j = u_j^{\mp 2}+u_j^{\mp 4}, \quad y_j = u_j^{\mp 6}.$

- Use standard determinantal expression for Schur function:

$$s_\lambda(x_1,\ldots,x_k) = \det_{1\le i,j\le k}(x_i^{\lambda_j+k-j})/\prod_{1\le i<j\le k}(x_i-x_j).$$

of Odd-Order DASASMs

$$\text{(\# of } (2n{+}1)\times(2n{+}1) \text{ DASASMs)} = 3^{-n(n-1)/2}\,\mathrm{SSYT}_{(n,n-1,n-1,\ldots,2,2,1,1)}(2n+1)$$
$$= \prod_{i=0}^{n} \frac{(3i)!}{(n+i)!}$$

[where $\mathrm{SSYT}_\lambda(k) = $ (# of semistandard Young tableaux of shape λ with entries $\le k$)]

<u>Proof:</u>

- Recall (# of $(2n{+}1)\times(2n{+}1)$ DASASMs) $= Z(\underbrace{1,\ldots,1}_{n+1})|_{q=e^{i\pi/6}}$.

- Set $u_1 = \ldots = u_{n+1} = 1$ in Schur function expression for $Z(u_1,\ldots,u_{n+1})|_{q=e^{i\pi/6}}$.

- Apply standard fact for Schur function with all variables 1:
  $$s_\lambda(\underbrace{1,\ldots,1}_{k}) = \mathrm{SSYT}_\lambda(k).$$

- Apply standard product formula for # of semistandard Young tableaux:
  $$\mathrm{SSYT}_\lambda(k) = \frac{\prod_{1\le i<j\le k}(\lambda_i - \lambda_j - i + j)}{\prod_{i=1}^{k-1} i!}.$$

of Odd-Order DASASMs with Fixed Central Entry

$$\frac{\text{\# of } (2n+1)\times(2n+1) \text{ DASASMs with central entry } -1}{\text{\# of } (2n+1)\times(2n+1) \text{ DASASMs with central entry } 1} = \frac{n}{n+1}$$

Proof outline:

- Introduce partition functions $Z_{\pm}(u_1,\ldots,u_{n+1})$ for $(2n+1)\times(2n+1)$ DASASMs with central entry ± 1.

- Show that $Z_{\pm}(u_1,\ldots,u_{n+1}) = \frac{1}{2}\big(Z(u_1,\ldots,u_n,u_{n+1}) \pm (-1)^n Z(u_1,\ldots,u_n,-u_{n+1})\big)$.

- Use previous results for $Z(u_1,\ldots,u_{n+1})$.

Special Features of Odd-Order DASASM Proof

- Six-vertex model considered on *triangle* instead of square.
- Only a *single* set $u_1, \ldots, u_{n+1}$ of spectral parameters used.
- Last spectral parameter u_{n+1} plays *special* role.
- Yang–Baxter *and* reflection equation needed (with certain boundary weights).
- Partition function formula involves *sum* of *two* determinantal terms.

III$'$. Outline of Proof of DSASM Formula

(# of $n\times n$ DSASMs)

$$= \operatorname*{Pfaffian}_{0 \text{ or } 1\le i,j\le n-1}\left(\sum_{k=0}^{\min(i,j)}(3-\delta_{k,0})\left(\binom{i+j-2k-1}{i-k}-\binom{i+j-2k-1}{j-k}\right)\right)$$

[range for i,j starts at 0 for n even, 1 for n odd]

(1) Obtain bijection between $n\times n$ DSASMs & configurations of six-vertex model on a certain triangle.

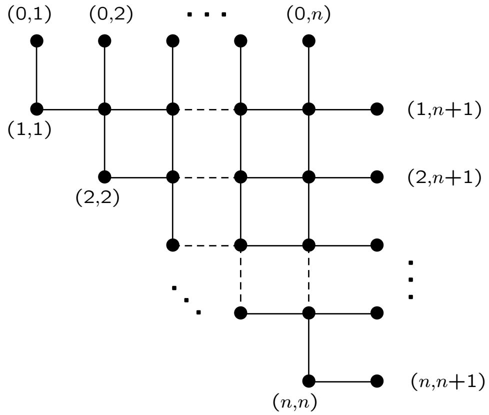
(2) Introduce parameter-dependent bulk weights, *left boundary weights* & associated partition function.

(3) Use Yang–Baxter equation, *left reflection equation* & other properties to prove formula for partition function involving Pfaffian.

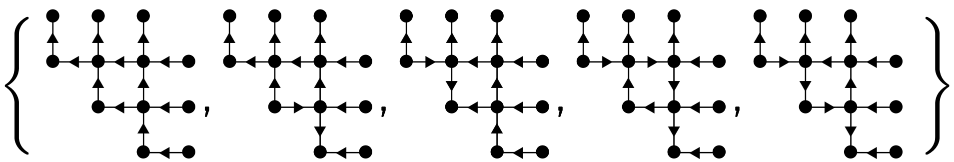
(4) Evaluate Pfaffian formula at values of parameters for which all weights are 1 (homogeneous limit).

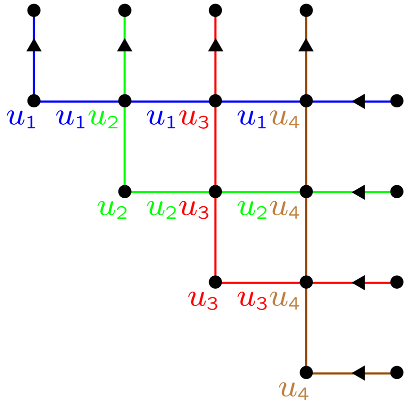
- Steps (1)–(3) are, to some extent, similar to steps (1)–(3) for odd-order DASASMs.
- In steps (2) & (3), the partition function was independently defined & the Pfaffian formula was independently obtained by *Garbali, de Gier, Mead, Wheeler 2023*.

A Few Details of DSASM Proof

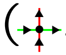
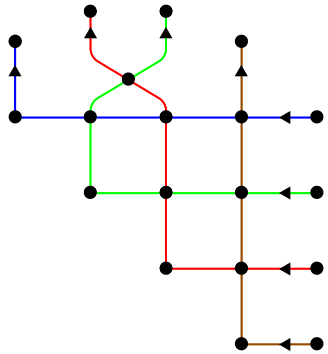
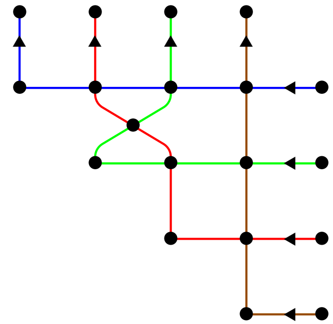
- Let $\mathcal{E}_n =$ 

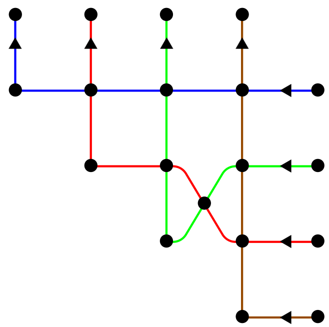
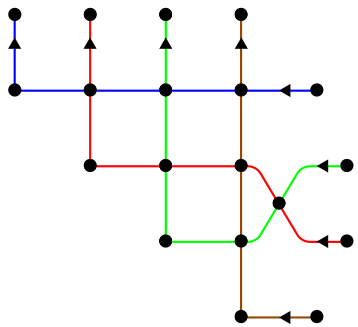
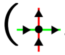
- $6\text{VD}(n) = \left\{ \begin{array}{l|l} \text{orientations of} & \bullet \text{ 2 in \& 2 out arrows at each degree 4 vertex} \\ \text{edges of } \mathcal{E}_n & \quad (\Rightarrow \text{6 cases}) \\ & \bullet \text{ no restriction at degree 2 vertices } (\Rightarrow \text{4 cases}) \\ & \bullet \text{ all arrows upward on top boundary} \\ & \bullet \text{ all arrows leftward on right boundary} \end{array} \right\}$

- e.g. 6VD(3) = 

- Partition function: e.g. $Z(u_1, u_2, u_3, u_4) \; =$ 

- Symmetry of partition function: e.g.

$W($$, q^2 u_2{}^{-1} u_3)\, Z(u_1, u_2, u_3, u_4) \; =$  $\overset{\text{YBE}}{=}$ 

$\overset{\text{LRE}}{=}$  $\overset{\text{YBE}}{=}$  $= W($$, q^2 u_2{}^{-1} u_3)\, Z(u_1, u_3, u_2, u_4),$

- Pfaffian formula for partition function:

$$Z(u_1,\ldots,u_n)=\prod_{1\le i<j\le n}\frac{\sigma(q^2u_iu_j)\,\sigma(q^2u_i^{-1}u_j^{-1})}{\sigma(q^4)\,\sigma(u_iu_j^{-1})}\ \underset{0\text{ or }1\le i<j\le n}{\text{Pfaffian}}\begin{pmatrix}Z(u_j), & i=0\\ \dfrac{\sigma(q^4)\,\sigma(u_iu_j^{-1})\,Z(u_i,u_j)}{\sigma(q^2u_iu_j)\,\sigma(q^2u_i^{-1}u_j^{-1})}, & i\ge 1\end{pmatrix}$$

(RB, Fischer, Koutschan 2023; Garbali, de Gier, Mead, Wheeler 2023)

  - Range for i,j starts at 0 for n odd, 1 for n even.
  - Only strictly upper triangular part of skew-symmetric matrix is shown for Pfaffian.
  - $Z(u_j)$ & $Z(u_i,u_j)$ in Pfaffian can easily be written explicitly using sums over 6VD(1) & 6VD(2).
  - Formula remains valid for general six-vertex model boundary weights.

- In step (4) of proof, evaluation of RHS of Pfaffian formula at $u_1=\ldots=u_n=1$ & $q=e^{i\pi/6}$ involves application of general result for homogeneous limit of certain multivariate Pfaffian expressions.

- In this case, $Z(u_1,\ldots,u_n)|_{q=e^{i\pi/6}}$ is not a Schur function.

Final Messages

- ASMs are interesting combinatorial objects.
- Many results involving ASMs have been proved using certain connections with integrability.
- The proofs are indirect & technical.
- Many aspects of ASMs are still not properly understood.

"These conjectures are of such compelling simplicity that it is hard to know how any mathematician can bear the pain of living without understanding why they are true . . . I expect that these problems will remain with us for some time."

(Robbins 1991, "The story of 1, 2, 7, 42, 429, 7436, . . .")