QQ-systems to solve Bethe Ansatz

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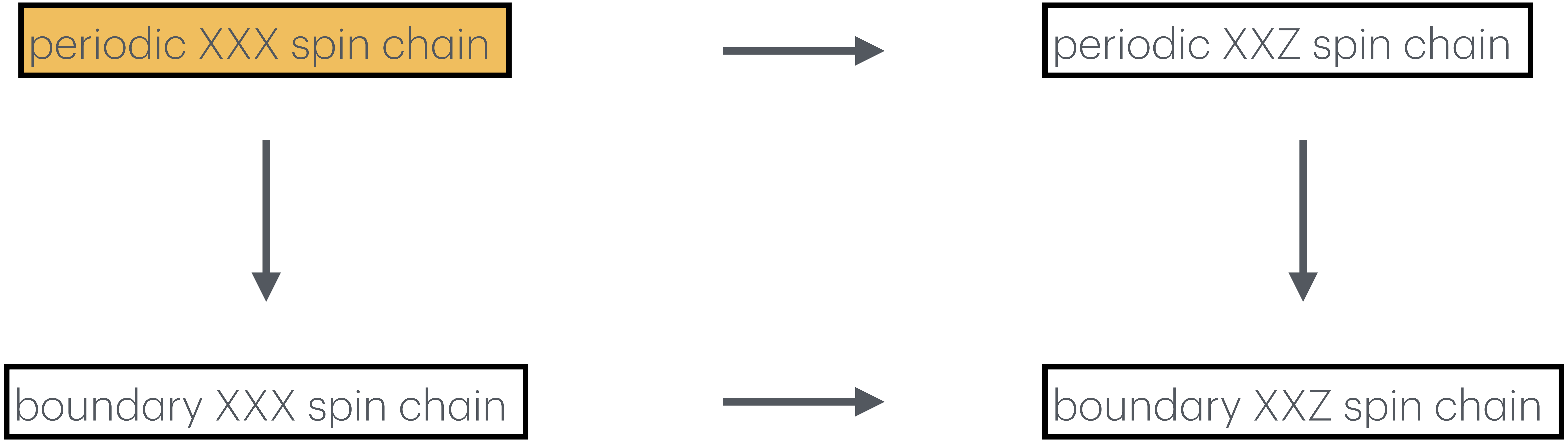
related work: Granet, Jacobsen 1910.07797

QQ-system


periodic XXX spin chain
periodic XXZ spin chain
boundary XXX spin chain
boundary XXZ spin chain

Periodic XXX spin chain: definition

spectral problem $$H = \sum_{k=1}^{N} \vec{\sigma}_k \cdot \vec{\sigma}_{k+1}\,, \qquad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1\,.$$

algebraic BA $$\mathbb{R}(u) = (u - \frac{i}{2})\mathbb{I} + i\mathbb{P} \qquad \mathbb{M}_0(u) = \mathbb{R}_{01}(u)\,\mathbb{R}_{02}(u)\ldots\mathbb{R}_{0N}(u) \qquad \mathbb{M}_0 = \begin{pmatrix} \mathbb{A}(u) & \mathbb{B}(u) \\ \mathbb{C}(u) & \mathbb{D}(u) \end{pmatrix}$$

transfer matrix $$\mathbb{T}(u) = \mathrm{tr}_0(\mathbb{M}_0(u)) \qquad \left[\mathbb{T}(u), \mathbb{T}(v)\right] = 0$$

eigenvectors $$\mathbb{B}(u_1)\ldots\mathbb{B}(u_M)\,|\,0\rangle \equiv |\,u_1, \ldots, u_M\rangle \qquad \mathbb{T}(u)\,|\,u_1, \ldots, u_M\rangle = T(u)\,|\,u_1, \ldots, u_M\rangle \qquad |\,0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes N}$$

TQ equation $$T(u)\,Q(u) = (u + i/2)^N Q(u - i) + (u - i/2)^N Q(u + i) \qquad Q(u) = \prod_{j=1}^{M} (u - u_j)$$

Bethe ansatz $$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^N = -\prod_{k=1}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,, \qquad j = 1\ldots M$$

periodic XXX spin chain: Bethe ansatz solutions

Bethe ansatz $\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^N = -\prod_{k=1}^{M}\frac{u_j-u_k+i}{u_j-u_k-i}\,, \quad j=1\ldots M \longrightarrow \{u_1,\ldots,u_M\}$

Physical solution: $\mathbb{B}(u_1)\ldots\mathbb{B}(u_M)\,|\,0\rangle \equiv |\,u_1,\ldots,u_M\rangle$ is an eigenvector of $\mathbb{T}(u)$

admissible solution: all roots are finite, pairwise distinct
and if the solution is singular $\{\frac{i}{2},-\frac{i}{2},u_1,\ldots,u_{M-2}\}$

$$\prod_{j=1}^{M-2}\frac{(u_j+\frac{i}{2})}{(u_j-\frac{i}{2})}\frac{(u_j+\frac{3i}{2})}{(u_j-\frac{3i}{2})}=(-1)^N$$

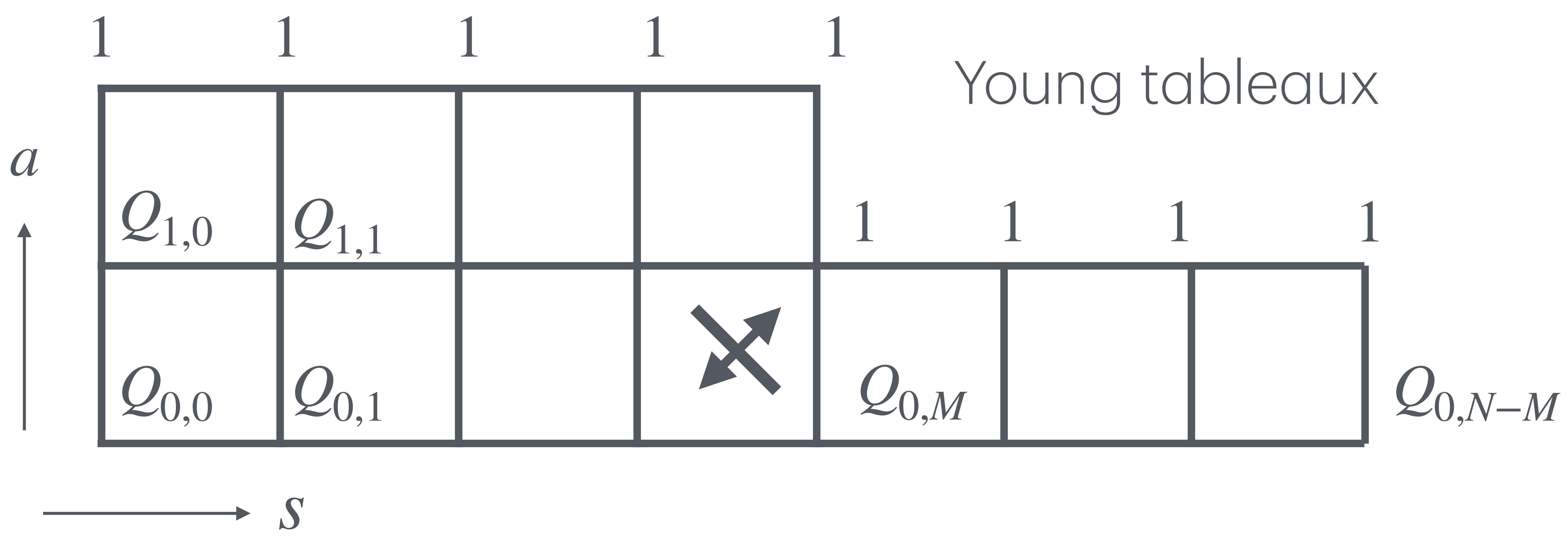
is satisfied

Admissible = physical $\mathcal{N}(N,M)=\binom{N}{M}-\binom{N}{M-1}$

periodic XXX spin chain: QQ-system

Marboe, Volin [17]

Focus on an M magnon state



QQ-equations

$$Q_{a+1,s}(u)\,Q_{a,s+1}(u) \propto Q^+_{a+1,s+1}(u)\,Q^-_{a,s}(u) - Q^-_{a+1,s+1}(u)\,Q^+_{a,s}(u)$$

$$f^{\pm}(u) = f(u \pm \frac{i}{2})$$

$$Q_{0,0}(u) = u^N, \qquad Q_{1,0}(u) = Q(u) = \prod_{j=1}^{M}(u-u_j)$$

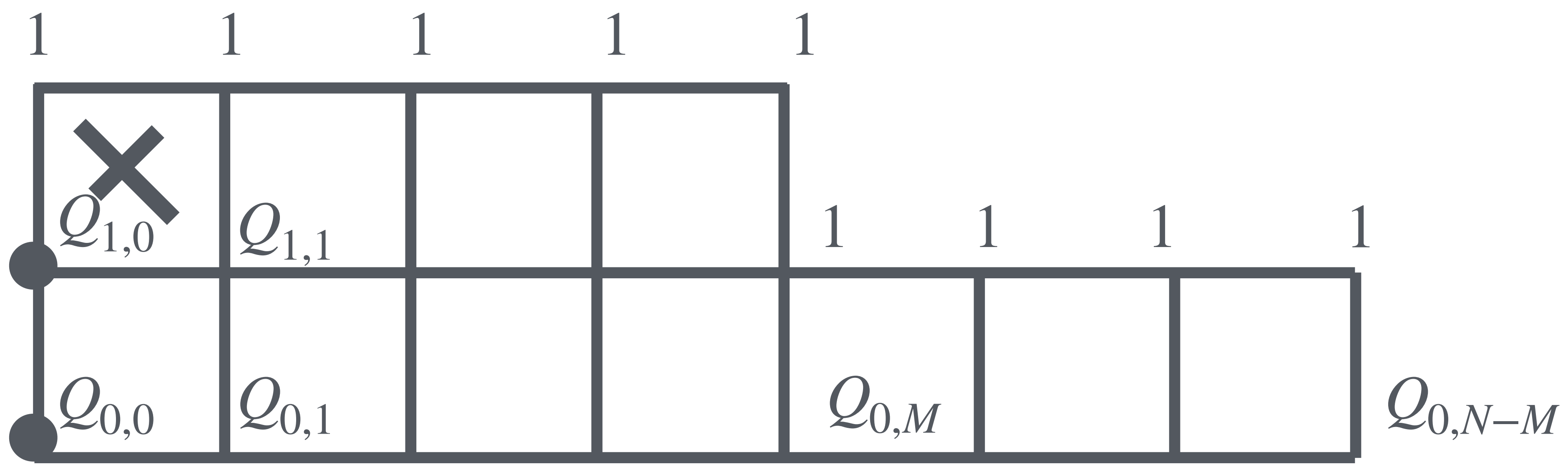
unique solution with $\mathbf{deg}(Q_{a,s})$

number of boxes to the right and top

polynomial solutions: admissible Bethe roots

periodic XXX spin chain: solving the QQ-system

$$Q_{a+1,s}(u)\,Q_{a,s+1}(u) \propto Q^{+}_{a+1,s+1}(u)\,Q^{-}_{a,s}(u) - Q^{-}_{a+1,s+1}(u)\,Q^{+}_{a,s}(u)$$



$(a,s)=(1,0)$ $\quad Q_{1,1}(u) = Q^{+}_{1,0}(u) - Q^{-}_{1,0}(u) \equiv Q'_{1,0}(u) = Q'(u)$

$$f'(u) = f^{+}(u) - f^{-}(u)$$

$(a,s)=(0,0)$ $\quad Q_{0,1}Q_{1,0} = Q^{-}_{0,0}Q^{+}_{1,1} - Q^{+}_{0,0}Q^{-}_{1,1}$ $\quad\curvearrowright\quad$ $Q_{0,1}Q = Q^{-}_{0,0}Q^{++} + Q^{+}_{0,0}Q^{--} - Q(Q^{-}_{0,0} + Q^{+}_{0,0})$

$$(Q_{0,1} + Q^{-}_{0,0} + Q^{+}_{0,0})Q = Q^{-}_{0,0}Q^{++} + Q^{+}_{0,0}Q^{--}$$

$$T = Q_{0,1} + Q^{-}_{0,0} + Q^{+}_{0,0}$$

Polynomiality of $Q_{0,1}$ is equivalent to the polynomiality of T: Bethe ansatz equations

Polynomiality of all $Q_{a,s}$ is equivalent to the admissibility of the Bethe roots

periodic XXX spin chain: the P-function

Pronko, Stroganov [99]

Define P such that $Q_{0,0} = P^+Q^- - P^-Q^+$ then

$$Q_{1,1} = Q' \qquad Q_{0,1} \propto P'^+Q'^- - P'^-Q'^+$$

$$Q_{1,2} = Q'', \qquad Q_{0,2} \propto P''^+Q''^- - P''^-Q''^+$$

$$Q_{1,n} = Q^{(n)}, \qquad Q_{0,n} \propto P^{(n)+}Q^{(n)-} - P^{(n)-}Q^{(n)+}$$

we need to integrate $R = \frac{P^+}{Q^+} - \frac{P^-}{Q^-} = \left(\frac{P}{Q}\right)'$

$$R(u) = \frac{u^N}{Q^+Q^-} = \frac{u^{N-2}}{u^{++}u^{--}\bar{Q}^+\bar{Q}^-} \qquad Q(u) = u^+u^-\bar{Q}(u)\,, \qquad \bar{Q}(u) = \prod_{u_j \neq \pm\frac{i}{2}} (u - u_j)$$

$$R = r + \frac{q_+}{\bar{Q}^+} + \frac{q_-}{\bar{Q}^-} + \frac{a_+}{u^{++}} + \frac{a_-}{u^{--}} \qquad a_\pm = \mp \frac{T(\mp\frac{i}{2})}{2i\bar{Q}(\pm\frac{i}{2})\,\bar{Q}(\mp\frac{3i}{2})} \qquad q_+ = q^+\,, \qquad q_- = -q^-$$

integrate $r = \rho' = \rho^+ - \rho^-$ $\qquad p'(u) = \frac{1}{u}\,, \qquad p(u) = -i\psi(-iu + \frac{1}{2})$

$$P = \rho Q + u^+u^-q + (a_+ - a_-)u\bar{Q} + \frac{1}{2}(a_+ + a_-)(p^{++} + p^{--})Q$$

polynomiality $a_+ = -a_-$

$$(-1)^M \frac{\bar{Q}(+\frac{i}{2})\bar{Q}(+\frac{3i}{2})}{\bar{Q}(-\frac{i}{2})\bar{Q}(-\frac{3i}{2})} = 1$$

P function satisfies $T = P^{++}Q^{--} - P^{--}Q^{++}$

periodic XXX spin chain: numerical solution of QQ

mathematica code to solve QQ-system

$Q[0,0,u\_] := u^L$

$Q[1,0,u\_] := \mathrm{Sum}[c[k]u^k, \{k,0,M-1\}] + u^M$

$Q[1,n\_,u\_] := Q[1,n-1,u+I/2] - Q[1,n-1,u-I/2];$

$Q[0,n\_,u\_] := (Q[1,n,u+I/2]Q[0,n-1,u-I/2] - Q[1,n,u-I/2]Q[0,n-1,u+I/2])/Q[1,n-1,u]$

$y[n\_,u\_] := \mathrm{PolynomialRemainder}[\mathrm{Numerator}[\mathrm{Together}[Q[0,n,u]]], \mathrm{Denominator}[\mathrm{Together}[Q[0,n,u]]], u]$

example $\quad N = L = 6 \quad M = 2$

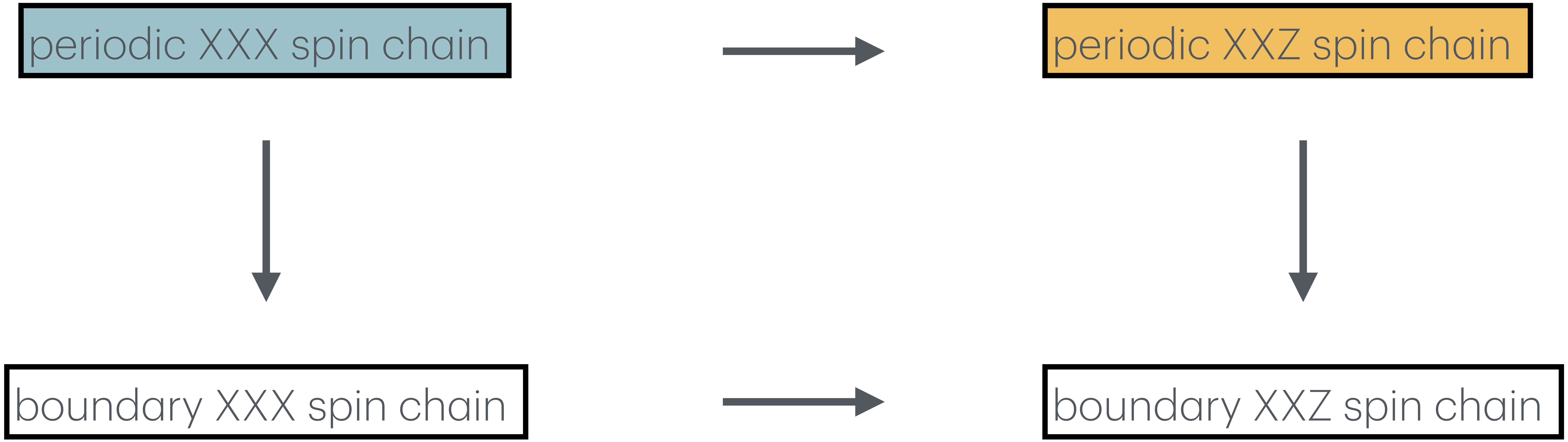
$sol = \mathrm{Solve}[\mathrm{Table}[\mathrm{CoefficientList}[y[n,u],u] == 0,\{n,1,M\}], \mathrm{Table}[c[k],\{k,0,M-1\}]]$ 9 solutions

$$\left\{c(0)\to\frac{1}{4},c(1)\to 0\right\},\left\{c(0)\to\frac{1}{20}\left(-5-2\sqrt{5}\right),c(1)\to 0\right\},\left\{c(0)\to\frac{1}{20}\left(2\sqrt{5}-5\right),c(1)\to 0\right\},\left\{c(0)\to-\frac{1}{8},c(1)\to-\frac{\sqrt{3}}{4}\right\},\left\{c(0)\to-\frac{1}{8},c(1)\to\frac{\sqrt{3}}{4}\right\},\left\{c(0)\to\frac{1}{16}\left(5-\sqrt{17}\right),c(1)\to-\frac{1}{4}\sqrt{\frac{3}{2}\left(9-\sqrt{17}\right)}\right\}$$

$$,\left\{c(0)\to\frac{1}{16}\left(5-\sqrt{17}\right),c(1)\to\frac{1}{4}\sqrt{\frac{3}{2}\left(9-\sqrt{17}\right)}\right\},\left\{c(0)\to\frac{1}{16}\left(5+\sqrt{17}\right),c(1)\to-\frac{1}{4}\sqrt{\frac{3}{2}\left(9+\sqrt{17}\right)}\right\},\left\{c(0)\to\frac{1}{16}\left(5+\sqrt{17}\right),c(1)\to\frac{1}{4}\sqrt{\frac{3}{2}\left(9+\sqrt{17}\right)}\right\}$$

$\mathrm{Solve}[(Q[1,0,u]/.\,sol[[1]]) == 0,u]//\mathrm{Flatten}$ $\quad \left\{u\to-\frac{i}{2},u\to\frac{i}{2}\right\}$ $\quad$ Bethe roots from $Q_{1,0}$

QQ-system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boundary XXX spin chain
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Periodic XXZ spin chain: definition

spectral problem

$$H = \sum_{k=1}^{N} \left[\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \frac{1}{2}(q+q^{-1}) \sigma_k^z \sigma_{k+1}^z \right], \qquad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1 \qquad q = e^{\eta}$$

transfer matrix

$$\mathbb{R}(u) = \begin{pmatrix} \sinh(u+\frac{\eta}{2}) & 0 & 0 & 0 \\ 0 & \sinh(u-\frac{\eta}{2}) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(u-\frac{\eta}{2}) & 0 \\ 0 & 0 & 0 & \sinh(u+\frac{\eta}{2}) \end{pmatrix}$$

$$\mathbb{M}_0(u) = \mathbb{R}_{01}(u)\,\mathbb{R}_{02}(u)\ldots\mathbb{R}_{0N}(u)$$

$$\mathbb{M}_0 = \begin{pmatrix} \mathbb{A}(u) & \mathbb{B}(u) \\ \mathbb{C}(u) & \mathbb{D}(u) \end{pmatrix}$$

eigenvectors

$$\mathbb{B}(u_1)\ldots\mathbb{B}(u_M)\,|0\rangle \equiv |u_1,\ldots,u_M\rangle \qquad \mathbb{T}(u)\,|u_1,\ldots,u_M\rangle = T(u)\,|u_1,\ldots,u_M\rangle \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes N}$$

TQ equation

$$T(u)\,Q(u) = \sinh^N(u+\tfrac{\eta}{2})\,Q(u-\eta) + \sinh^N(u-\tfrac{\eta}{2})\,Q(u+\eta) \qquad Q(u) = \prod_{j=1}^{M} \sinh(u-u_j)$$

Bethe ansatz

$$\left(\frac{\sinh(u_k+\frac{\eta}{2})}{\sinh(u_k-\frac{\eta}{2})} \right)^N = -\prod_{j=1}^{M} \frac{\sinh(u_k-u_j+\eta)}{\sinh(u_k-u_j-\eta)}, \qquad k = 1,\ldots,M$$

periodic XXX spin chain: Bethe ansatz solutions

Bethe ansatz $$\left(\frac{\sinh(u_k+\frac{\eta}{2})}{\sinh(u_k-\frac{\eta}{2})}\right)^N = -\prod_{j=1}^{M}\frac{\sinh(u_k-u_j+\eta)}{\sinh(u_k-u_j-\eta)}, \qquad k=1,\ldots,M \longrightarrow \{u_1,\ldots,u_M\}$$

Physical solution: $\mathbb{B}(u_1)\ldots\mathbb{B}(u_M)|0\rangle \equiv |u_1,\ldots,u_M\rangle$ is an eigenvector of $\mathbb{T}(u)$

admissible solution: all roots are finite, pairwise distinct
and if the solution is singular $\{-\frac{\eta}{2},\frac{\eta}{2},u_1,\ldots,u_{M-2}\}$

$$\frac{\bar{Q}(+\frac{\eta}{2})\bar{Q}(+\frac{3\eta}{2})}{\bar{Q}(-\frac{\eta}{2})\bar{Q}(-\frac{3\eta}{2})} = (-1)^N \qquad \bar{Q}(u) = \prod_{j=1}^{M-2}\sinh(u-u_j)$$ is satisfied

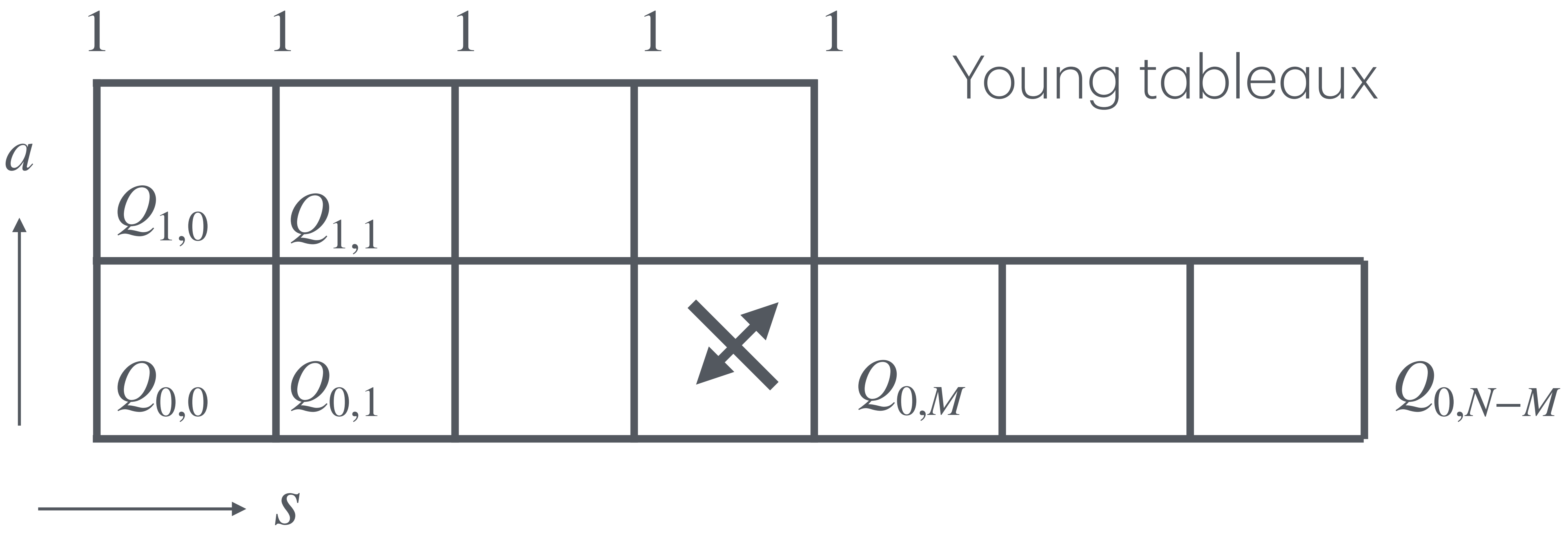
Admissible = physical $$\mathcal{N}(N,M) = \binom{N}{M}$$

periodic XXZ spin chain: QQ-system

Focus on an M magnon state

$t = e^u \qquad t^{-1} = e^{-u}$

$q = e^\eta \qquad Q_{a,s}(t, t^{-1}) \equiv Q_{a,s}(t)$



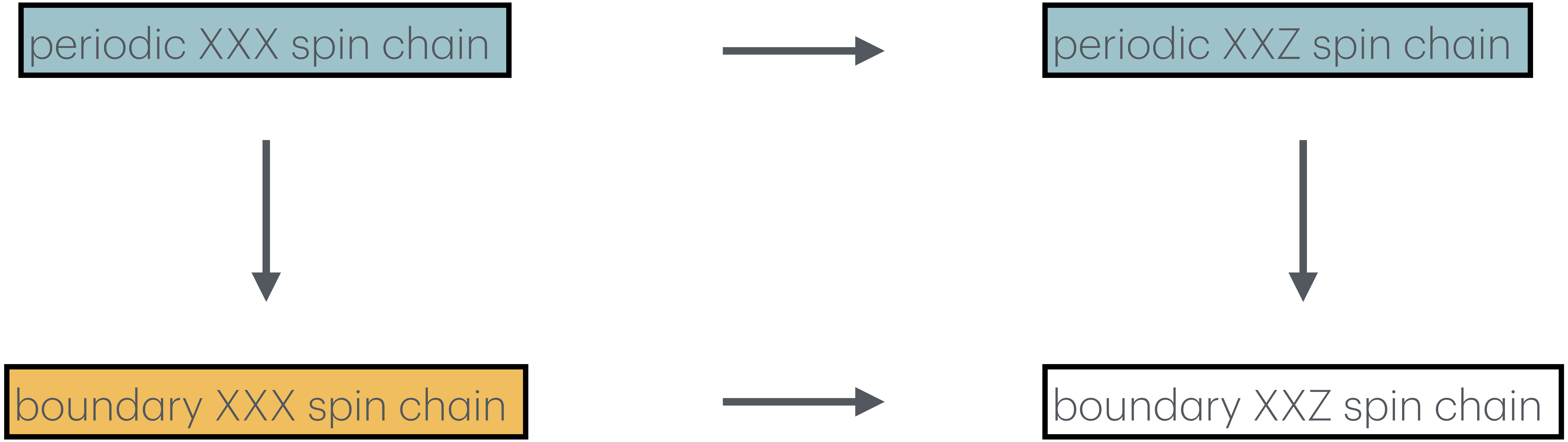
QQ-equations

$$Q_{a+1,s}(t)\,Q_{a,s+1}(t) \propto Q^+_{a+1,s+1}(t)\,Q^-_{a,s}(t) - Q^-_{a+1,s+1}(t)\,Q^+_{a,s}(t) \qquad f^\pm(t) = f(tq^{\pm\frac{1}{2}})$$

$$Q_{0,0}(t) = (t - t^{-1})^N, \qquad Q_{1,0}(t) = Q(t) = \prod_{j=1}^{M} (tt_j^{-1} - t^{-1}t_j)$$

polynomial solutions in t and t^{-1}: admissible Bethe roots. proved using $P(t)$

QQ-system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Open XXX spin chain: definition

spectral problem

$$H = \sum_{k=1}^{N-1} \vec{\sigma}_k \cdot \vec{\sigma}_{k+1}$$

$$\mathbb{K}(u) = \mathbb{I}$$

transfer matrix

$$\mathbb{T}(u) = \mathrm{tr}_0 \mathbb{U}_0(u)\,, \qquad \mathbb{U}_0(u) = \mathbb{M}_0(u)\,\widehat{\mathbb{M}}_0(u) \qquad \widehat{\mathbb{M}}_0(u) = \mathbb{R}_{0N}(u)\cdots\mathbb{R}_{02}(u)\,\mathbb{R}_{01}(u)$$

$$\left[\mathbb{T}(u), \mathbb{T}(v)\right] = 0 \qquad \mathbb{T}(u) = \mathbb{T}(-u) \qquad \mathbb{U}_0(u) = \begin{pmatrix} \mathbb{A}(u) & \mathbb{B}(u) \\ \mathbb{C}(u) & \frac{u^-}{u}\mathbb{D}(u) + \frac{i}{2u}\mathbb{A}(u) \end{pmatrix}$$

eigenvectors

$$\mathbb{B}(u_1)\ldots\mathbb{B}(u_M)\,|0\rangle \equiv |\,u_1, \ldots, u_M\rangle \qquad \mathbb{T}(u)\,|\,u_1, \ldots, u_M\rangle = T(u)\,|\,u_1, \ldots, u_M\rangle \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes N}$$

TQ equation

$$u\,T(u)\,Q(u) = (u^+)^{2N+1}\,Q^{--}(u) + (u^-)^{2N+1}\,Q^{++}(u)\,, \quad Q(u) = \prod_{k=1}^{M}\left(u - u_k\right)\left(u + u_k\right)$$

Bethe ansatz

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^{2N} = \prod_{k:k\neq j}^{M} \frac{(u_j - u_k + i)(u_j + u_k + i)}{(u_j - u_k - i)(u_j + u_k - i)} \qquad j = 1,\ldots,M$$

open XXX spin chain: Bethe ansatz solutions

Bethe ansatz $$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^{2N}=\prod_{k:k\neq j}^{M}\frac{(u_j-u_k+i)(u_j+u_k+i)}{(u_j-u_k-i)(u_j+u_k-i)} \quad j=1,\ldots,M \longrightarrow \{u_1,\ldots,u_M\}$$

Physical solution: $\mathbb{B}(u_1)\ldots\mathbb{B}(u_M)\,|0\rangle \equiv |u_1,\ldots,u_M\rangle$ is an eigenvector of $\mathbb{T}(u)$

admissible solution: all roots are finite, pairwise distinct, not equal to $\{0, i/2, -i/2\}$
and satisfies

$$\mathfrak{Re}(u_j)>0$$

or

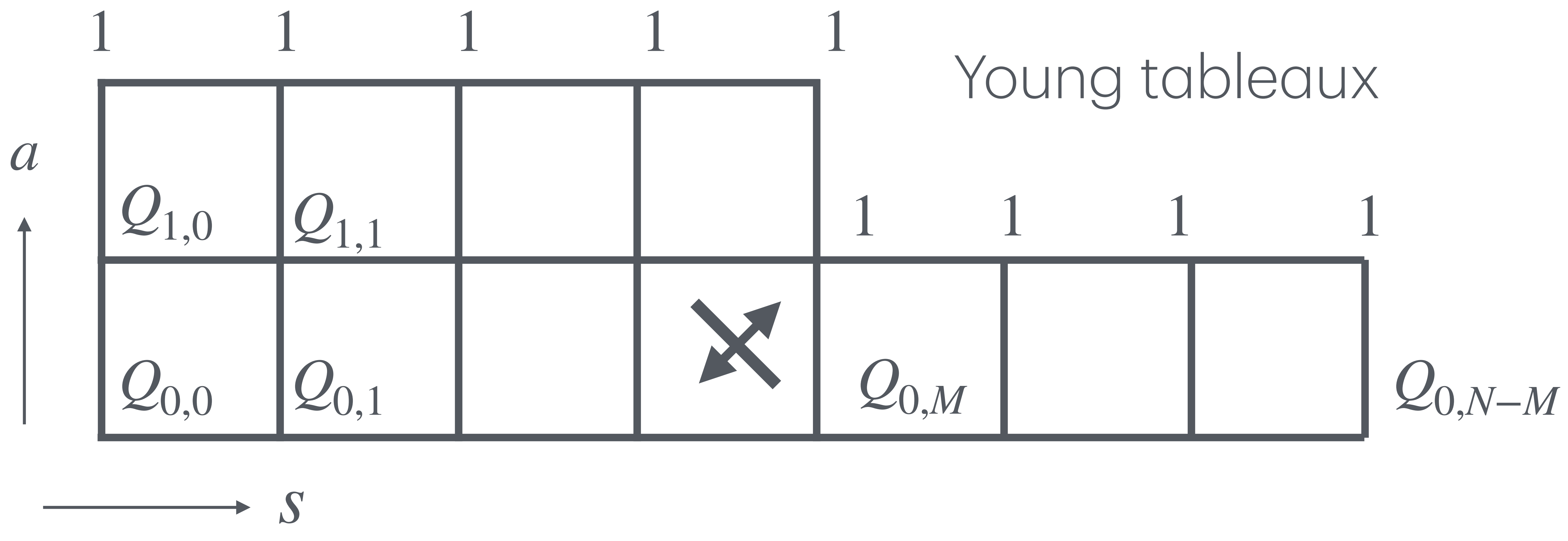
$$\mathfrak{Re}(u_j)=0 \quad \text{and} \quad \mathfrak{Im}(u_j)>0$$

Admissible = physical

$$\mathcal{N}(N,M)=\binom{N}{M}-\binom{N}{M-1}$$

open XXX spin chain: QQ-system

Focus on an M magnon state



QQ-equations

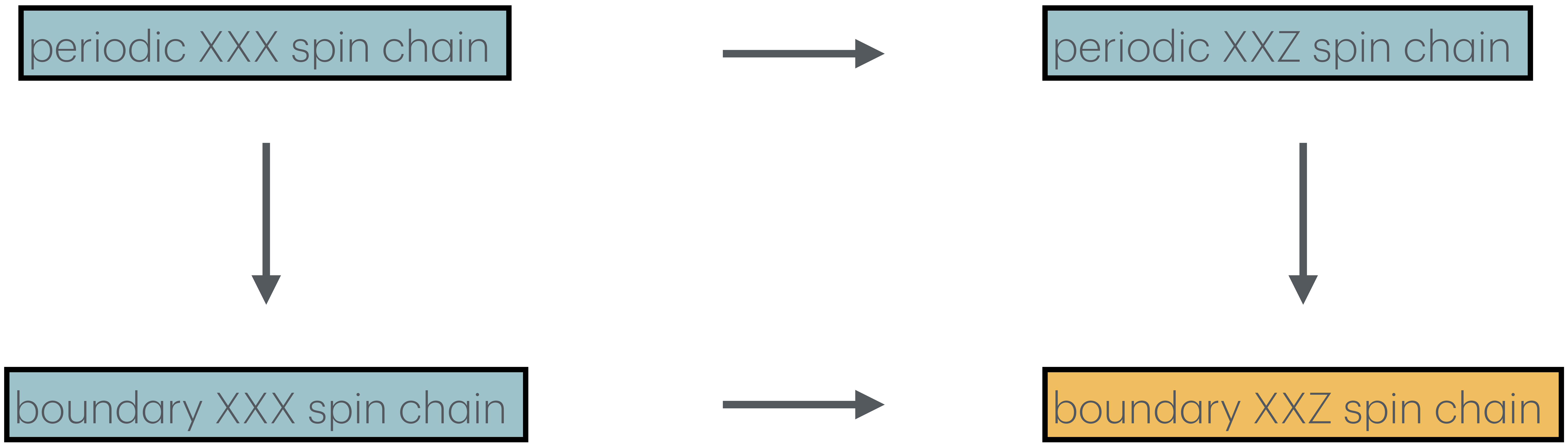
$$u\,Q_{a+1,s}(u)\,Q_{a,s+1}(u) \propto Q^{+}_{a+1,s+1}(u)\,Q^{-}_{a,s}(u) - Q^{-}_{a+1,s+1}(u)\,Q^{+}_{a,s}(u) \qquad f^{\pm}(u) = f(u \pm \frac{i}{2})$$

$$Q_{0,0}(u) = u^{2N} \qquad \text{with} \qquad Q_{1,0}(u) = Q(u) = \prod_{k=1}^{M} \left(u - u_k\right)\left(u + u_k\right)$$

unique solution with $\mathbf{deg}(Q_{a,s})$ twice the number of boxes to the right and top

polynomial solutions: admissible Bethe roots

QQ-system



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Open XXZ spin chain: definition

spectral problem
$$H = \sum_{k=1}^{N-1}\left[\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \frac{1}{2}(q+q^{-1})\sigma_k^z\sigma_{k+1}^z\right] - \frac{1}{2}(q-q^{-1})\left(\sigma_1^z - \sigma_N^z\right)$$

transfer matrix
$$\mathbb{K}^L(u) = \mathrm{d}iag(e^{-u-\frac{\eta}{2}}, e^{u+\frac{\eta}{2}})\,, \qquad \mathbb{K}^R(u) = \mathrm{d}iag(e^{u-\frac{\eta}{2}}, e^{-u+\frac{\eta}{2}})\,.$$

$$\mathbb{T}(u) = \mathrm{tr}_0\mathbb{K}_0^L(u)\,\mathbb{U}_0(u)$$

$$\mathbb{U}_0(u) = \mathbb{M}_0(u)\,\mathbb{K}_0^R(u)\,\widehat{\mathbb{M}}_0(u)$$

$$\mathbb{U}_0(u) = \begin{pmatrix} e^{u-\frac{\eta}{2}}\mathbb{A}(u) & \mathbb{B}(u) \\ \mathbb{C}(u) & \frac{e^{-u-\frac{\eta}{2}}\sinh(2u-\eta)}{\sinh(2u)}\mathbb{D}(u) + \frac{e^{u-\frac{\eta}{2}}\sinh(\eta)}{\sinh(2u)}\mathbb{A}(u) \end{pmatrix}$$

eigenvectors
$$\mathbb{B}(u_1)\dots\mathbb{B}(u_M)\,|\,0\rangle \equiv |\,u_1,\dots,u_M\rangle \qquad \mathbb{T}(u)\,|\,u_1,\dots,u_M\rangle = T(u)\,|\,u_1,\dots,u_M\rangle \qquad |0\rangle = \begin{pmatrix}1\\0\end{pmatrix}^{\otimes N}$$

TQ equation
$$\sinh(2u)\,T(u)\,Q(u) = \sinh(2u+\eta)\,\sinh^{2N}(u+\tfrac{\eta}{2})\,Q(u-\eta) + \sinh(2u-\eta)\,\sinh^{2N}(u-\tfrac{\eta}{2})\,Q(u+\eta)$$

Bethe ansatz
$$\left(\frac{\sinh(u_j+\frac{\eta}{2})}{\sinh(u_j-\frac{\eta}{2})}\right)^{2N} = \prod_{\substack{k\neq j\\k=1}}^{M}\frac{\sinh(u_j-u_k+\eta)\sinh(u_j+u_k+\eta)}{\sinh(u_j-u_k-\eta)\sinh(u_j+u_k-\eta)}\,, \quad j=1,\dots,M$$

open XXZ spin chain: QQ-system

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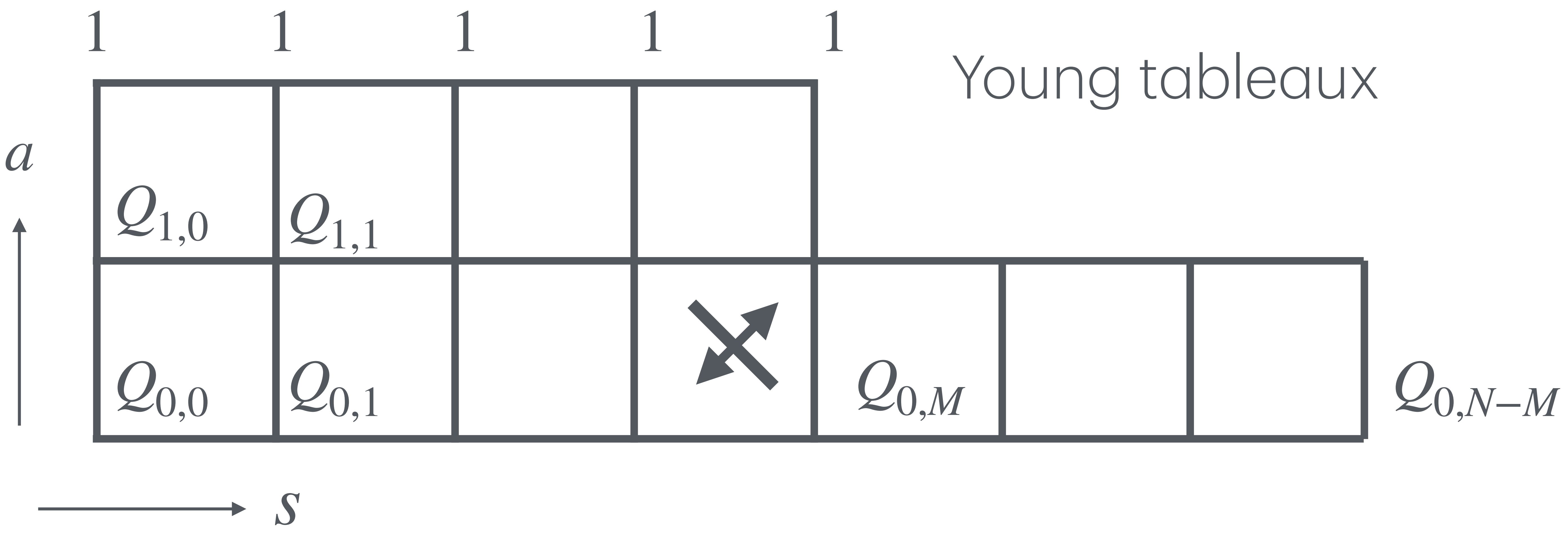

QQ-equations

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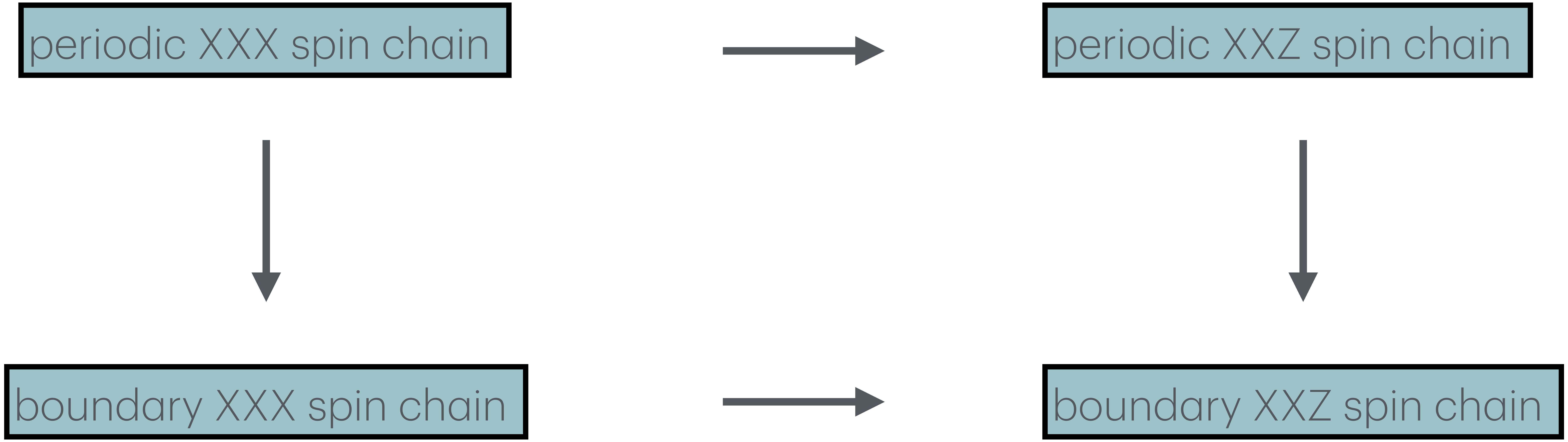
$$Q_{0,0}(t) = (t - t^{-1})^{2N} \qquad \text{with} \qquad Q_{1,0}(t) = Q(t) = \prod_{k=1}^{M} \left(tt_k^{-1} - t^{-1}t_k\right)\left(tt_k - t^{-1}t_k^{-1}\right)$$

polynomial solutions in t and t^{-1}: admissible Bethe roots. proved using $P(t)$

QQ-system



Cylinder partition function of the 6-vertex model from algebraic geometry,
Bajnok, Jacobsen,Jiang, Nepomechie, Zhang 2002.09019

Nepomechie: higher rank 2003.06823 boundary with parameters 1912.12702