A generalised multidimensional Jarník-Besicovitch theorem

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Abstract In this short note we prove a general multidimensional Jarník-Besicovitch theorem which gives the Hausdorff dimension of simultaneously approximable set of points with error of approximations dependent on continuous functions in all dimensions. Consequently, the Hausdorff dimension of the set varies along continuous functions. This resolves a problem posed by Barral-Seuret (2011).

1 Localised Jarník-Besicovitch theorem

The Jarník-Besicovitch set is of foundational nature in the theory of metric Diophantine approximation;

$$W(\tau) := \left\{ x \in [0,1) : \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \quad \text{for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

It has been generalised in various directions such as replacing the error of approximation by an arbitrary function tending to zero and hence studying the associated metrical theory has received much attention over the years. We refer the reader to [2] for a survey of metric theory of Diophantine approximation. Staying within the scope of Jarník-Besicovtich set, for any $x \in [0,1)$, let us define the approximation order of x to be

$$\delta(x) = \sup\{\tau : x \in W(\tau)\}.$$

From the asymptotic form of Dirichlet's theorem (1842), it follows that $\delta(x) \geq 2$ for all irrational numbers x. For any $\tau \geq 2$, the classical Jarník-Besicovitch theorem (1928, 1934) states that

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$$\dim_{\mathscr{H}}\{x\in[0,1):\delta(x)\geq\tau\}=\dim_{\mathscr{H}}\{x\in[0,1):\delta(x)=\tau\}=\frac{2}{\tau}.$$

Here and throughout, $\dim_{\mathscr{H}}(X)$, denotes the Hausdorff dimension of a set X. For any $s\in\mathbb{R}^+$, $\mathscr{H}^s$ denotes the s-dimensional Hausdorff measure of X. In the case $s=d$, the s-dimensional Hausdorff measure is comparable with the d-dimensional Lebesgue measure. Finally, $B(x,r)$ denotes a ball centred at x and of radius r.

In [1], Barral-Seuret investigated the structure of the set of points with their approximation order varying along a continuous function $\tau(x)\geq 2$. They called the corresponding set as the *localised Jarník-Besicovitch set*

$$W_{\mathrm{loc}}(\tau(x)):=\{x\in[0,1):\delta(x)=\tau(x)\}.$$

They proved that the Hausdorff dimension of the set $W_{\mathrm{loc}}(\tau(x))$ to be

$$\frac{2}{\min\{\tau(x):x\in\mathbb{R}\}}.$$

Roughly speaking this result gives the size of the set of real numbers with a prescribed order of approximation. For example, for real numbers $0<a<b<1$, it gives

$$\dim_{\mathscr{H}}\{x\in[a,b]:\delta(x)=2(1+x)\}=\frac{1}{1+a}.$$

The result of Barral-Seuret was further generalised to the settings of continued fractions by Wang-Wu-Xu in [8].

In the higher dimensions the analogue of Jarník-Besicovitch set has been well studied specifically by Rynne in a sequence of papers in the 90's. To state the most relevant result, we introduce a little notation first. Let $\underline{\tau}=(\tau_1,\ldots,\tau_d)$ be a vector of strictly positive numbers and let $W_d(\underline{\tau})$ denote the set of all $\mathbf{x}\in[0,1)^d$ for which the system of inequalities

$$|qx_i-p_i|<q^{-\tau_i},\quad 1\leq i\leq d,$$

are satisfied for infinitely many $(p_1,\ldots,p_d,q)\in\mathbb{Z}^d\times\mathbb{N}$. The Hausdorff dimension of this set was determined by Rynne [5].

Theorem 1 (Rynne, 1998). *Let* $\frac{1}{d}\leq\tau_1\leq\tau_2\ldots\leq\tau_d$. *Then*

$$\dim_{\mathscr{H}}W_d(\underline{\tau})=\min_{1\leq j\leq d}\left\{\frac{1+d+j\tau_j-\sum_{i=1}^{j}\tau_i)}{1+\tau_j}\right\}.$$

In this paper, we replace the constant vector $\underline{\tau}$ in the set $W_d(\underline{\tau})$ with the function

$$\underline{\tau}(\mathbf{x}):=\{(\tau_1(x_1),\ldots,\tau_d(x_d)):x_1,\ldots,x_d\in[0,1]\},$$

where every function $\tau_i(x_i)$ is a continuous function on $[0,1]$. To be precise, let $W_d(\underline{\tau}(\mathbf{x}))$ denote the set of all $\mathbf{x} \in [0,1)^d$ for which the system of inequalities

$$|qx_i - p_i| < q^{-\tau_i(x_i)}, \quad 1 \le i \le d,$$

are satisfied for infinitely many $(p_1, \ldots, p_d, q) \in \mathbb{Z}^d \times \mathbb{N}$. We calculate the Hausdorff dimension of this set and, thus, answer a question [1, §6] raised by Barral-Seuret of extending their one dimensional result to higher dimensions.

Theorem 2. *Let*

$$\frac{1}{d} \le \min_{x_1 \in [0,1]} \tau_1(x_1) \le \min_{x_2 \in [0,1]} \tau_2(x_2) \le \cdots \le \min_{x_d \in [0,1]} \tau_d(x_d).$$

Then

$$\dim_{\mathscr{H}} W_d(\underline{\tau}(\mathbf{x})) = \min_{1 \le j \le d} \left\{ \frac{d+1+j \min\limits_{x_j \in [0,1]} \tau_j(x_j) - \sum_{i=1}^{j} j \min\limits_{x_i \in [0,1]} \tau_i(x_i)}{1 + \min\limits_{x_j \in [0,1]} \tau_j(x_j)} \right\}.$$

2 Proof

2.1 The upper bound

The upper bound relies on the natural covering of the set $W_d(\underline{\tau}(\mathbf{x}))$. Here we prove for $d=2$ for clarity by showing that the s-dimensional Hausdorff measure of this set is zero whenever $s > \dim_{\mathscr{H}} W_2(\underline{\tau}(\mathbf{x}))$. The general case $d>2$ follows similarly.

$$\begin{aligned} W_2(\underline{\tau}(\mathbf{x})) &= \left\{ (x_1, x_2) \in [0,1]^2 : \begin{array}{c} |qx_1 - p_1| < q^{-\tau_1(x_1)}, \ |qx_2 - p_2| < q^{-\tau_2(x_2)} \\ \text{for infinitely many} \quad (p_1, p_2, q) \in \mathbb{Z}^2 \times \mathbb{N} \end{array} \right\} \\ &\subseteq \bigcup_{q=N}^{\infty} \bigcup_{p_1, p_2 \le q} \left\{ (x_1, x_2) \in [0,1]^2 : |qx_1 - p_1| < q^{-\tau_1(x_1)}, \ |qx_2 - p_2| < q^{-\tau_2(x_2)} \right\} \\ &= \bigcup_{q=N}^{\infty} \bigcup_{p_1, p_2 \le q} B\left(\frac{p_1}{q}, q^{-1-\tau_1(x_1)}\right) \times B\left(\frac{p_2}{q}, q^{-1-\tau_2(x_2)}\right) \\ &\subseteq \bigcup_{q=N}^{\infty} \bigcup_{p_1, p_2 \le q}^{\infty} B\left(\frac{p_1}{q}, q^{-1-\min\limits_{x_1 \in [0,1]} \tau_1(x_1)}\right) \times B\left(\frac{p_2}{q}, q^{-1-\min\limits_{x_2 \in [0,1]} \tau_2(x_2)}\right) \end{aligned}$$

So, $W_2(\underline{\tau}(\mathbf{x}))$ is a subset of a collection of rectangles and each one of them

$$R = B\left(\frac{p_1}{q}, \frac{1}{q^{1+\min\limits_{x_1\in[0,1]}\tau_1(x_1)}}\right) \times B\left(\frac{p_2}{q}, \frac{1}{q^{1+\min\limits_{x_2\in[0,1]}\tau_2(x_2)}}\right)$$

can be covered in two ways: either by collection of squares formed by shorter side lengths or by a bigger square of side length as the longer side of the rectangle.

Case I. Since $\min\limits_{x_1\in[0,1]}\tau_1(x_1) \leq \min\limits_{x_2\in[0,1]}\tau_2(x_2)$, the rectangle R can be covered by at most

$$2q^{\left(\min\limits_{x_2\in[0,1]}\tau_2(x_2) - \min\limits_{x_1\in[0,1]}\tau_1(x_1)\right)}$$

squares of side length $q^{-1-\min\limits_{x_2\in[0,1]}\tau_2(x_2)}$. Hence the s-dimensional Hausdorff measure of $W_2(\underline{\tau}(\mathbf{x}))$ can be estimated as

$$\begin{aligned}
\mathscr{H}^s\Big(W_2(\underline{\tau}(\mathbf{x}))\Big) &\leq 2\liminf_{N\to\infty}\sum_{q=N}^{\infty} q^2 q^{\left(\min\limits_{x_2\in[0,1]}\tau_2(x_2) - \min\limits_{x_1\in[0,1]}\tau_1(x_1)\right)} q^{-s\left(1+\min\limits_{x_2\in[0,1]}\tau_2(x_2)\right)} \\
&\leq 2\liminf_{N\to\infty}\sum_{q=N}^{\infty} q^{2+\min\limits_{x_2\in[0,1]}\tau_2(x_2) - \min\limits_{x_1\in[0,1]}\tau_1(x_1) - s\left(1+\min\limits_{x_2\in[0,1]}\tau_2(x_2)\right)}.
\end{aligned}$$

Therefore, for any

$$s > \frac{3+\min\limits_{x_2\in[0,1]}\tau_2(x_2) - \min\limits_{x_1\in[0,1]}\tau_1(x_1)}{1+\min\limits_{x_2\in[0,1]}\tau_2(x_2)},$$

$\mathscr{H}^s\Big(W_2(\underline{\tau}(\mathbf{x})\Big) = 0$. This shows that

$$\dim_{\mathscr{H}} W_2(\underline{\tau}(\mathbf{x}) \leq \frac{3+\min\limits_{x_2\in[0,1]}\tau_2(x_2) - \min\limits_{x_1\in[0,1]}\tau_1(x_1)}{1+\min\limits_{x_2\in[0,1]}\tau_2(x_2)}.$$

Case II. The second case concerns covering the rectangle R by the square formed by the longer side length $q^{-1-\min\limits_{x_1\in[0,1]}\tau_1(x_1)}$. Hence the s-dimensional Hausdorff measure of $W_2(\underline{\tau}(\mathbf{x}))$ can be estimated as

$$\mathscr{H}^s\Big(W_2(\underline{\tau}(\mathbf{x}))\Big) \leq \liminf_{N\to\infty}\sum_{q=N}^{\infty} q^{2-s\left(1+\min\limits_{x_1\in[0,1]}\tau_1(x_1)\right)}.$$

Therefore, for any $s > \frac{3}{1+\min\limits_{x_1\in[0,1]}\tau_1(x_1)}$, $\mathscr{H}^s\Big(W_2(\underline{\tau}(\mathbf{x})\Big) = 0$. This shows that

$$\dim_{\mathscr{H}} W_2(\underline{\tau}(\mathbf{x})) \leq \frac{3}{1+\min\limits_{x_1\in[0,1]} \tau_1(x_1)}.$$

Hence combining both the above cases, we have

$$\dim_{\mathscr{H}} W_2(\underline{\tau}(\mathbf{x})) \leq \min\left(\frac{3}{1+\min\limits_{x_1\in[0,1]} \tau_1(x_1)}, \frac{3+\min\limits_{x_2\in[0,1]} \tau_2(x_2)-\min\limits_{x_1\in[0,1]} \tau_1(x_1)}{1+\min\limits_{x_2\in[0,1]} \tau_2(x_2)}\right).$$

Following similar line of covering, as above, for arbitrary d, we have

$$\dim_{\mathscr{H}} W_d(\underline{\tau}(\mathbf{x})) \leq \min_{1\leq j\leq d}\left\{\frac{d+1+j\min\limits_{x_j\in[0,1]} \tau_j(x_j)-\sum_{i=1}^{j}\min\limits_{x_i\in[0,1]} \tau_i(x_i)}{1+\min\limits_{x_j\in[0,1]} \tau_j(x_j)}\right\}.$$

2.2 The lower bound

The main ingredient in proving the lower bound of Theorem 2 is the following mass transference principle, from balls to rectangles, proved by Wang-Wu-Xu in [7]. We refer the reader to [3] for more intricate result regarding the generalised Hausdorff measure criterion which also, of course, implies the Hausdorff dimension results. To state the Wang-Wu-Xu result we need a bit more notation. Let $\{x_n\}_{n\in\mathbb{N}}\subset[0,1]^d$ with $d\geq 1$ be a sequence of rationals and let $\{r_n\}_{n\geq 1}$ be a sequence of positive numbers tending to zero. Define the limsup set generated by balls

$$W := \left\{x\in[0,1]^d : x\in B(x_n,r_n) \text{ for i.m. } n\in\mathbb{N}\right\} = \limsup_{n\to\infty} B(x_n,r_n).$$

For any $\mathbf{a}=(a_1,\ldots,a_d)$, with $1\leq a_1\leq\ldots\leq a_d$, define the limsup set generated by rectangles

$$W^{\mathbf{a}} := \left\{x\in[0,1]^d : x\in B^{\mathbf{a}}(x_n,r_n) \text{ for i.m. } n\in\mathbb{N}\right\} = \limsup_{n\to\infty} B^{\mathbf{a}}(x_n,r_n)$$

where $B^{\mathbf{a}}(x,r)$ denotes a rectangle with center x and side length $(r^{a_1}\ldots,r^{a_d})$.

The main result of [7] is the following mass transference principle (see also [6, Theorem 2.4].

Theorem 3 (Wang-Wu-Xu, 2015). *Let $\{B_i : i\geq 1\}$ be a sequence of balls such that for any ball $B\subset[0,1]^d$, $\mathscr{H}^d(B\cap\limsup_{i\to\infty} B_i)=\mathscr{H}^d(B)$. Let $\mathbf{a}=(a_1,\ldots,a_d)$, with $1\leq a_1\leq\ldots\leq a_d$. Then we have*

$$\dim_{\mathscr{H}} W^{\mathbf{a}} \geq \min_{1 \leq j \leq d} \left\{ \frac{d + ja_j - \sum_{i=1}^{j} a_i}{a_j} \right\} := s(\mathbf{a})$$

and for any ball $B \subset [0,1]^d$,

$$\mathscr{H}^{s(\mathbf{a})}(B \cap W^{\mathbf{a}}) = \mathscr{H}^{s(\mathbf{a})}(B).$$

It is worth stressing that the Lebesgue measure of the set $W^{\mathbf{a}}$ being full i.e. $\mathscr{H}^d(W^{\mathbf{a}}) = 1$, in general settings, follows from the Khintchine-Groshev type theorem proved in [4, 2]. Having Theorem 3 at our disposal, we are in a position to prove the lower bound of Theorem 2. First note that the sequence of rectangles in $[0,1]^d$ can be written as

$$B^{\mathbf{a}}(x_n, r_n) := \prod_{i=1}^{d} B\left(x_{n,i}, r_n^{a_i}\right)$$

It is also clear from the definition of Hausdorff measure that, since $s(\mathbf{a}) \leq d$, we have $\mathscr{H}^{s(\mathbf{a})}(B \cap W^{\mathbf{a}}) = \mathscr{H}^{s(\mathbf{a})}(B) > 0$.

Now consider a localised limsup set by replacing the constant exponents a_i with continuous functions $a_i(x_i)$ for all $1 \leq i \leq d$. Given a sequence of balls $\{B^{\mathbf{a}(\mathbf{x})}(x_n, r_n)\}_{n \geq 1}$ in a compact bounded cube $\mathscr{C} = C_1 \times \cdots \times C_d$ of $[0,1]^d$, where

$$\mathbf{a}(\mathbf{x}) = (a_1(x_1), \ldots, a_d(x_d))$$

is a d-dimensional continuous function with

$$1 \leq a_1(x_1) \leq \ldots \leq a_d(x_d).$$

Consider the limsup set

$$W^{\mathbf{a},L} = \left\{ \mathbf{x} \in \mathscr{C} : \mathbf{x} \in B^{\mathbf{a}(\mathbf{x})}(x_n, r_n) \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

Let

$$a_0 = \min_{\mathbf{x} \in \mathscr{C}} \{a_1(x_1), \ldots, a_d(x_d)\}.$$

Then there exists a ball $B := B^{\mathbf{a}(\mathbf{x})}(x_n, r_n) \subset \mathscr{C}$ such that

$$\min(a_1(x_1), \ldots, a_d(x_d)) \leq a_0 + \varepsilon \quad \forall \mathbf{x} \in B. \tag{1}$$

Then

$$W^{a_0+\varepsilon} = \limsup_{n \to \infty} B^{a_0+\varepsilon}(x_n, r_n) = \limsup_{n \to \infty} \prod_{i=1}^{d} B\left(x_{n,i}, r_n^{a_0+\varepsilon}\right).$$

Now for $\mathbf{a}(\mathbf{x})$ satisfying (1), define

$$s(\mathbf{a}(\mathbf{x})) := \min_{1\le j\le d} \left\{ \frac{d + j \min\limits_{x_j\in C_j} a_j(x_j) - \sum_{i=1}^{j} \min\limits_{x_i\in C_i} a_i(x_i)}{\min\limits_{x_j\in C_j} a_j(x_j)} \right\}.$$

Then

$$\begin{aligned}\mathscr{H}^{s(\mathbf{a}(\mathbf{x}))}(W^{\mathbf{a},L}) &\ge \mathscr{H}^{s(\mathbf{a}(\mathbf{x}))}(B\cap W^{\mathbf{a},L}) \\ &\ge \mathscr{H}^{s(\mathbf{a}(\mathbf{x}))}(B\cap W^{a_0+\varepsilon}) \\ &\ge \mathscr{H}^{s(\mathbf{a}(\mathbf{x}))}(B) \quad \text{by letting } \varepsilon\to 0 \\ &> 0.\end{aligned}$$

Hence from the definition of Hausdorff dimension, it follows that

$$\dim_{\mathscr{H}} W^{\mathbf{a},L} \ge s(\mathbf{a}(\mathbf{x})).$$

The lower bound of the proof of Theorem 2 follows by identifying

$$\mathscr{C} = [0,1]^d, \mathbf{a}(\mathbf{x}) = \underline{\tau}(\mathbf{x}), B(x_i, r_i) = B\left(\frac{p_i}{q}, \frac{1}{q^{1+\tau_i(x_i)}}\right)$$

to yield that

$$\dim_{\mathscr{H}} W_d\left(\underline{\tau}(\mathbf{x})\right) \ge \min_{1\le j\le d} \left\{ \frac{d + 1 + j \min\limits_{x_j\in[0,1]} \tau_j(x_j) - \sum_{i=1}^{j} \min\limits_{x_i\in[0,1]} \tau_i(x_i)}{1 + \min\limits_{x_j\in[0,1]} \tau_j(x_j)} \right\}.$$

Remark 1. It is worth stressing that if we look at the set $W^{\mathbf{a},L}$ locally, then the power functions $(a_i(x_i))$ in the above proof are almost constants. In this sense the Hausdorff measure result of Theorem 3 is applicable. In comparison, the proof of Barral-Seuret [1] is much more involved as they tackled the problem of exact approximation order and Hausdorff dimension of level sets.

Finally we would like to point out that, very recently, Wang-Wu has introduced a mass transference principle [6] from rectangles to rectangles for the linear form settings. Given this new avatar, we envisage that the main result of this paper may be extended to the dual linear forms but it would require careful synthesis of the framework introduced in their paper.

Acknowledgements The author is supported by the ARC DP200100994. Part of this work was carried out during the workshop "Ergodic Theory, Diophantine approximation and related topics" sponsored by the MATRIX Research Institute. The author thank Baowei Wang and Weiliang Wang for useful discussions on this topic.

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