A new perspective on SU(N) long-range spin models

Thomas Quella (University of Melbourne)

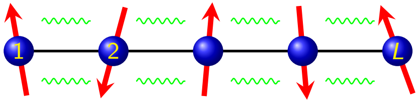
MATRIX Workshop
"Integrability in low-dimensional quantum systems"
27/6/2017

Based on work with Roberto Bondesan (arXiv:1405.2971)
and work in preparation with Roberto Bondesan and Jochen Peschutter

See also related work by Tu, Nielsen and Sierra (arXiv:1405.2950)

THE UNIVERSITY OF MELBOURNE

Spin models: Mathematical setup



Spin operators: $\vec{S}_i \in \mathfrak{g}$
(rep matrices of $\mathfrak{g}$ on $\mathcal{H}_i$)

Ingredients

- Symmetry (here: a simple Lie algebra $\mathfrak{g}$)
- Hilbert space $\mathcal{H} = \bigotimes_i \mathcal{H}_i$ (a unitary rep of $\mathfrak{g}$)
- Hamiltonian $H \in \mathrm{End}(\mathcal{H})$
  (hermitean, commuting with action of $\mathfrak{g}$)

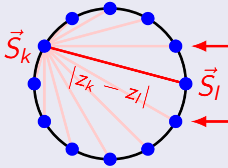
Quantities of interest

- The spectrum of H
- The properties as $L \to \infty$ (thermodynamic limit)

The Haldane-Shastry Model as a Paradigm

The Haldane-Shastry Model

SU(N) quantum spins on a circle



SU(N) spins in the fundamental representation

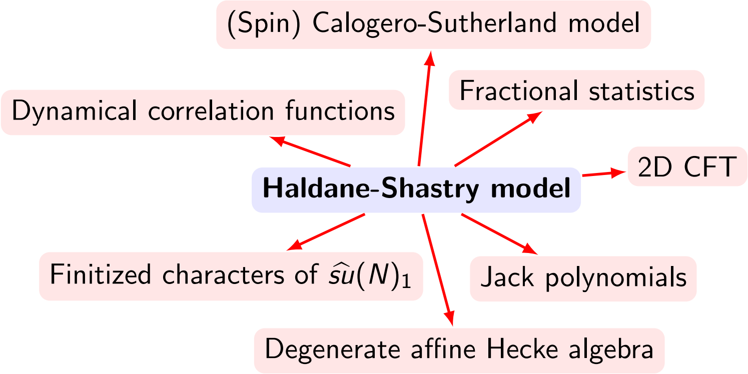
Equidistant positions $z_k = e^{\frac{2\pi i}{L} k}$

Hamiltonian

[Haldane] [Shastry]

$$H = \left(\frac{2\pi}{L}\right)^2 \sum_{k<l} \frac{\vec{S}_k \cdot \vec{S}_l}{|z_k - z_l|^2} = \left(\frac{2\pi}{L}\right)^2 \sum_{k<l} \frac{\vec{S}_k \cdot \vec{S}_l}{\sin^2 \frac{\pi}{L}(k-l)}$$

What's your interest in the Haldane-Shastry model?



Properties of the Haldane-Shastry model

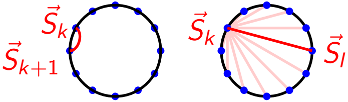
Features

- Yangian symmetry [Haldane,Ha,Talstra,Bernard,Pasquier]
- Quantum integrability [Bernard,Gaudin,Haldane,Pasquier]
- Physical insights:
  - Elementary excitations are spinons
  - Generalized exclusion principle: Haldane statistics [Haldane]

Specifically

- The full spectrum and all eigenstates are explicitly known
- They are related to correlators of the $SU(N)_1$ WZW model

Comparison to the SU(N) Heisenberg model



	Heisenberg	**Haldane-Shastry**
Continuum limit	$SU(N)_1$	$SU(N)_1$
Quantum integrability	✓	✓
Yangian symmetry for $L \to \infty$	✓	✓
Yangian symmetry for $L < \infty$	✗	✓
Logarithmic corrections as $L \to \infty$	✓	✗

$\Rightarrow$ The Haldane-Shastry model realizes the CFT almost perfectly

The goals of this talk

Limitations of the Haldane-Shastry model

The nice properties of the Haldane-Shastry model are bound to

- SU(N) symmetry
- spins in the fundamental representation
- 1D arrangements
- equidistant positions

Goals

- Suggest a systematic generalization of the Haldane-Shastry model
- Investigate which of the features survive

Tool: Entanglement

Matrix Product States

Designing quantum states

Question

How to construct quantum systems (or quantum states) with predefined properties?

Answer

Use their entanglement features to decode/encode this information

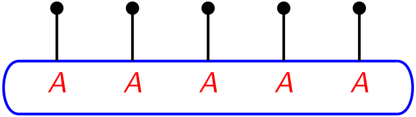
Matrix product states

Ingredients

- Two local Hilbert spaces $\mathcal{V}$ (physical) and $\mathcal{B}$ (auxiliary)
- $\mathcal{V}$ and $\mathcal{B}$ should be representations of the symmetry group
- An intertwiner $A : \mathrm{End}(\mathcal{B}) \to \mathcal{V}$

Definition of matrix product states (MPS)

$$\text{Periodic BC:} \quad |\psi\rangle = \mathrm{tr}(A \otimes \cdots \otimes A) \in \mathcal{H} = \mathcal{V}^{\otimes L}$$

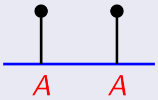


$$= \sum_{k_1,\ldots,k_L} \mathrm{tr}\Big[\underbrace{A^{k_1} \cdots A^{k_L}}_{\text{matrix product}}\Big]\,|k_1 \cdots k_L\rangle$$

Properties of matrix product states

Essential properties

- The dimension of V encodes the entanglement of $|\psi\rangle$
- There exists a so-called "parent Hamiltonian" H with
  - H is a sum of local projectors (onto the complement of $A \otimes A$)
  - H is gapped
  - $H \geq 0$
  - $H|\psi\rangle = 0$

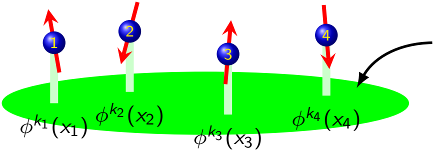


- If A satisfies certain natural properties, one also has
  - $|\psi\rangle$ is the unique ground state

Problem

Critical systems are gapless, so they require infinite size of $\mathcal{B}$

Matrix product states for critical systems



Auxiliary quantum field theory (QFT)

ϕ: Set of fields $\phi^k(x_k)$

Strategy

Replace $\mathcal{B}$ by the Hilbert space of a QFT:

$$|\psi\rangle = \sum_{\{k_i\}} \underbrace{\langle \phi^{k_1}(x_1) \cdots \phi^{k_L}(x_L) \rangle}_{\text{QFT correlator}} |k_1, \ldots, k_L\rangle$$

The goal of the talk

Strategy

1. Start from a 2D CFT/vertex operator algebra
2. Construct a quantum spin model (in either 1D or 2D)
3. Try to solve it in the thermodynamic limit
4. Investigate its relation to the original CFT

Concrete choice of CFT here: WZW models

- Based on affine Lie algebra
- Natural realization of $\mathfrak{g}$-symmetry

From CFT to long-range spin models

Outline of the idea

Philosophy (based on a given CFT)

[Cirac,Sierra] [Nielsen,Cirac,Sierra]

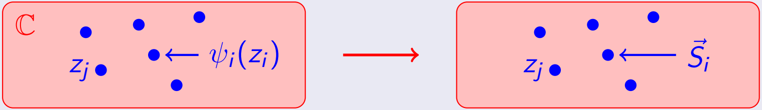
CFT correlation function $\psi(z_1, \dots, z_L)$

State $|\psi\rangle$ (wave function) of a quantum mechanical lattice model

Hamiltonian with $|\psi\rangle$ as a (unique) groundstate

Sketch

Conformal field theory — **Quantum spin model**



Field insertions — Spin locations

Outline of the idea

Philosophy (based on a given CFT)

[Cirac,Sierra] [Nielsen,Cirac,Sierra]

Correlator $\psi_{m_1\cdots m_L}(z_1,\ldots,z_L) = \langle \psi_{m_1}(z_1)\cdots\psi_{m_L}(z_L)\rangle$

State $|\psi\rangle = \sum_{\{m_i\}} \psi_{m_1\cdots m_L}(z_1,\ldots,z_L)\,|m_1\cdots m_L\rangle$

Hamiltonian $H \geq 0$ with $H|\psi\rangle = 0$

Sketch

Conformal field theory **Quantum spin model**

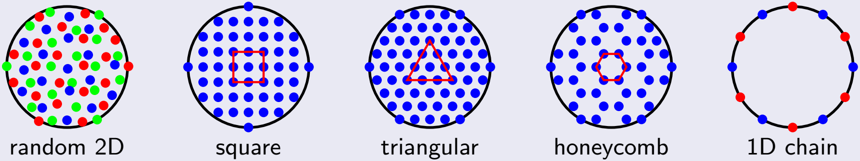
Field insertions Spin locations

Features and natural questions

Features

- Interpretation as an Infinite Matrix Product State (∞MPS)
- Freedom: Type and position of field insertions can be chosen at will

Examples



Questions

- Why a lattice model? $\rightarrow$ Potential cold atom implementation
- Thermodynamic limit: What is the relation to the original CFT?

Application to SU(N) spin models

The SU(N) WZW model

Starting point: The SU(N)$_1$ WZW model

- Based on affine Lie algebra $\widehat{su}(N)_1$ extending su(N)
- It has $N-1$ basic fields (integrable reps) with abelian fusion
- We will only work with the fundamental field $\psi(z)$

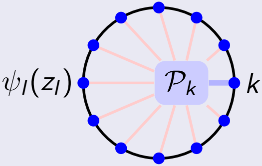
Affine Lie algebra: Centrally extended loop algebra

$$[J_m^a, J_n^b] \;=\; if^{ab}{}_c\, J_{m+n}^c + km\kappa^{ab}\delta_{m+n}$$

Construction of the Hamiltonian

Strategy

- Find operators $\mathcal{P}_k$ that annihilate $\langle\psi_1(z_1)\cdots\psi_L(z_L)\rangle$
- Define $H = \sum_k \mathcal{P}_k^* \mathcal{P}_k$
- By construction:
  - H is hermitean
  - $H \geq 0$
  - $|\psi\rangle$ is a groundstate

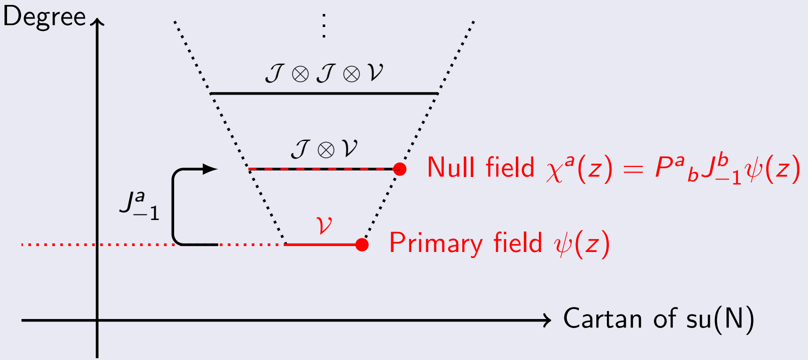


Relevant result

The operators $\mathcal{P}_k$ arise from null vectors in Verma modules over the affine Lie algebra $\widehat{su}(N)$ after making use of WZW Ward identities

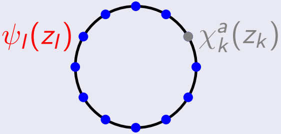
Null vectors in affine Verma modules

Structure of the relevant Verma module over $\widehat{su}(N)_1$



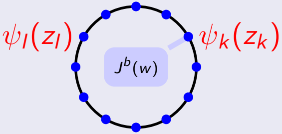
Construction of the null operators $\mathcal{P}_k$

$0 =$



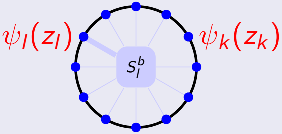
$$= \left\langle \psi_1(z_1)\cdots \underbrace{\chi_k^a(z_k)}_{P^a{}_b J^b_{-1}\psi_k(z_k)} \ldots \psi_L(z_L)\right\rangle$$

$$= \oint_{z_k} \frac{dw/w}{2\pi i} \, (\mathcal{P}_k)^a{}_b$$



"mode $\to$ current"

$$= \sum_{l(\neq k)} \frac{(\mathcal{P}_k)^a{}_b}{z_k - z_l}$$



"WZW Ward identity"

Evaluation of the Hamiltonian

Null operator

$$\mathcal{P}_k^a\big(\{z_l\}\big)\,\big\langle\psi(z_1)\cdots\psi(z_L)\big\rangle \stackrel{\text{def}}{=} \sum_{l(\neq k)} \frac{(\mathcal{P}_k)^a{}_b\,S_l^b}{z_k - z_l}\,\big\langle\psi(z_1)\cdots\psi(z_L)\big\rangle = 0$$

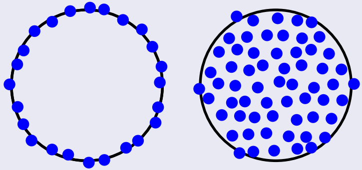
Corresponding Hamiltonian

$$\begin{aligned} H &= \sum_k \mathcal{P}_k^a\big(\{z\}\big)^\dagger\,\kappa_{ab}\,\mathcal{P}_k^b\big(\{z\}\big) \\ &= \sum_k \sum_{i,j(\neq k)} \bar{w}_{ki}w_{kj}\Big\{-\tfrac{i}{4}\tfrac{N+2}{N+1}\,f_{abc}\,S_i^aS_j^bS_k^c - \tfrac{N(-1)^{d_k}}{4(N+1)}\,d_{abc}\,S_i^aS_j^bS_k^c \\ &\qquad + \tfrac{N+2}{2(N+1)}\,\vec{S}_i\cdot\vec{S}_j\Big\} \end{aligned}$$

Discussion

Thermodynamic limit

Expectation:
– Gapless
– Same CFT
(actually, no...)



Generic 1D   Generic 2D

Expectation/hope:
– Gapped spin liquid
– 1D edge CFT
– Anyonic excitations

For the general case an analytic solution is beyond reach

One analytic result

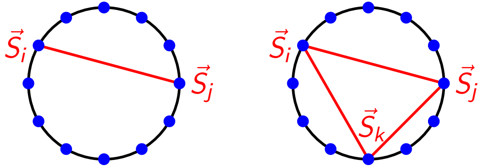
[Bondesan,TQ] [Tu,Nielsen,Sierra]

The exact groundstate is defined in terms of

$$\left\langle \psi_{1,\vec{q}_1}(z_1)\cdots\psi_{L,\vec{q}_L}(z_L)\right\rangle = \delta_{\vec{q},0}\,\underbrace{e^{if(\{\vec{q}_l\})}}_{\text{known}}\prod_{i<j}(z_i-z_j)^{\langle\vec{q}_i,\vec{q}_j\rangle}$$

where $\vec{q}_l$ are quantum numbers (weights) with respect to SU(N)

The Hamiltonian for the uniform chain



Useful quantity:

$$w_{ij} = \frac{z_i + z_j}{z_i - z_j}$$

The Hamiltonian

[Bondesan,TQ] [Tu,Nielsen,Sierra]

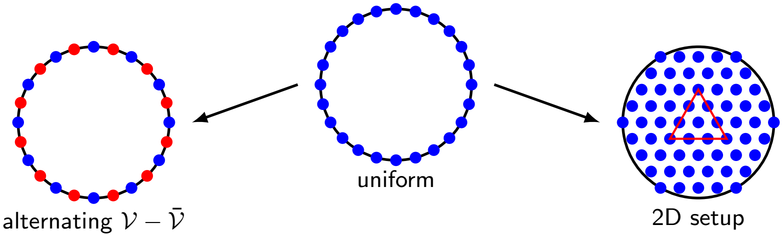
$$H = C_1 \sum_{k \neq l} \frac{\vec{S}_k \cdot \vec{S}_l}{|z_k - z_l|^2} + \underbrace{C_2\, \vec{S}^2 + C_3\, d_{abc} S^a S^b S^c}_{\text{coupling to total spin } \vec{S}} + C_4$$

Result

- Reduction to Haldane-Shastry model plus coupling to total spin
- Exact solution despite absence of Yangian symmetry

Generalizations

Overview: Generalizations



Other generalizations

- Supersymmetrization: SU(N) $\to$ SU(M+N|M)
- A long range loop model (limit $M \to \infty$)

Discussion of the alternating Hamiltonian

Discussion

- The three-spin couplings do not decouple, not even for special arrangements of spins
- An analytic solution is (currently) not available
- Numerical evidence: The thermodynamic limit of an equidistant alternating chain on a circle is described by a (yet unidentified) CFT

Numerical implementation

- The rewriting in terms of a loop model reduces the numerical complexity drastically
- The number N only arises as a parameter of the loop model but does not affect the complexity

Embedding into supersymmetric setups

Extension [Bondesan,Peschutter,TQ: work in preparation]

The construction of the Hamiltonian generalizes to supergroups of the form SU(M+N|M)

Comments

- The WZW theories for supergroups are much more intricate than for ordinary groups ($\rightarrow$ log CFT) [Schomerus,Saleur] [TQ,Schomerus] [...]
- Lattice discretizations of these theories are highly desired

A long-range loop model

Basic idea

- The Hilbert space admits a multiplicity free decomposition

$$\mathcal{H} = (\mathcal{V} \otimes \bar{\mathcal{V}})^{\otimes \ell} = \bigoplus_\lambda \mathcal{V}_\lambda \otimes \mathcal{S}_\lambda$$

  into irreps of SU(N) and the walled Brauer algebra $\mathrm{WB}_{\ell,\ell}(N)$
- The Hamiltonian is an element of $\mathrm{WB}_{\ell,\ell}(N)$
- The latter can be studied on an arbitrary representation of $\mathrm{WB}_{\ell,\ell}(\delta)$, including those which define loop models with arbitrary fugacity δ

Comments

- The loop model provides a faithful representation of the spectrum of the SU(M+N|M) spin model as $M \to \infty$

Summary and Outlook

Summary and Outlook

Summary

[Bondesan,TQ] [Tu,Nielsen,Sierra]

Long-range SU(N) spin models on arbitrary lattices in 1D or 2D can be constructed based on the null vectors in the $SU(N)_1$ WZW model

Concrete results

[Bondesan,TQ] [Tu,Nielsen,Sierra]

- The 1D uniform case can be reduced to the Haldane-Shastry model
- All eigenstates and their energies are known explicitly

Outlook

- The 1D alternating case leads to a yet to be identified CFT
- Exploration of various 2D setups [Tu,Nielsen,Cirac]
- Application to higher levels and other symmetry groups

see [Tu] [Bondesan,Peschutter,TQ] [TQ,Tu] for SO(N), GL(M|N) and SP(N)