# Aspects of defects and integrability

Ed Corrigan

Department of Mathematics, University of York

Integrability in Low-Dimensional Quantum Systems

MATRIX - University of Melbourne

July 2017

## Contents

- Boundaries and defects (eg impurities, shocks, dislocations) are ubiquitous in nature
- What are their properties within an integrable field theory in two-dimensional space-time?
  - Examples of integrable defects and the special role played by energy-momentum conservation and Bäcklund transformations
  - Solitons scattering with defects and some curious effects
  - Defects in integrable quantum field theory and transmission matrices
  - Scattering defects
  - Boundaries revisited
- Ideas developed with: P Bowcock, C Robertson (Durham-Maths)<br>C Zambon (Durham-Physics)<br>D Hills, R Parini (York)

## An integrable discontinuity - Bowcock, EC, Zambon (2003)

Start with a single selected point on the $x$-axis, say $x_0$, and denote the field to the left ($x < x_0$) by $u$, and to the right ($x > x_0$) by $v$:

$$\cdots \qquad u(x,t) \qquad \bullet\, x_0 \qquad v(x,t) \qquad \cdots$$

Field equations in separated domains:

$$\partial^2 u = -\frac{\partial U}{\partial u}, \quad x < x_0, \quad \partial^2 v = -\frac{\partial V}{\partial v}, \quad x > x_0, \quad \partial^2 = \partial_t^2 - \partial_x^2$$

- How can the fields $u, v$ be ‘sewn’ together at $x_0$?

- If the wave equations are nonlinear but ‘integrable’ are there sewing conditions that preserve the integrability?

- Not so easy: see, for example Goodman, Holmes, Weinstein (2002)

- sine-Gordon, KdV, nonlinear Schrödinger, affine Toda field theories ...

- A simple example ($\delta$-impurity) would be to put

$$u(x_0,t) = v(x_0,t), \quad u_x(x_0,t) - v_x(x_0,t) = 2\lambda u(x_0,t),$$

with linear wave equations for $u$ and $v$.

- Typically, there is reflection and transmission:

$$u = e^{-i\omega t}\left(e^{ikx} + R\,e^{-ikx}\right), \quad v = e^{-i\omega t}\,T\,e^{ikx}, \quad \omega^2 = k^2$$

with

$$R = -\frac{\lambda e^{2ikx_0}}{ik+\lambda}, \quad T = \frac{ik}{ik+\lambda}$$

- There is a distinguished point - translation symmetry is lost and momentum is not conserved while total energy is preserved including contribution from the impurity.

- Could an alternative type of defect also compensate for momentum and other conservation laws?

- Consider the field contributions to energy-momentum:

$$P^\mu = \int_{-\infty}^{x_0} dx\, T^{0\mu}(u) + \int_{x_0}^{\infty} dx\, T^{0\mu}(v), \quad \partial_\nu T^{\nu\mu} = 0$$

where the components of $T^{\nu\mu}(u)$ are (similarly with $v$)

$$T^{00} = \frac{1}{2}\left(u_t^2 + u_x^2\right) + U,\ T^{01} = T^{10} = -u_t u_x,\ T^{11} = \frac{1}{2}\left(u_t^2 + u_x^2\right) - U$$

Using the field equations, can we arrange

$$\frac{dP^\mu}{dt} = -\left[T^{1\mu}(u)\right]_{x=x_0} + \left[T^{1\mu}(v)\right]_{x=x_0} = -\frac{dD^\mu(u,v)}{dt}$$

with the right hand side depending only on the fields at $x = x_0$?

If so, $P^\mu + D^\mu$ is conserved with $D^\mu$ being the defect contribution.

- It turns out that only a few possible sewing conditions (and bulk potentials $U,\ V$) are permitted for this to work.

- Consider the field contribution to energy and calculate

$$\frac{dP^0}{dt} = [u_x u_t]_{x_0} - [v_x v_t]_{x_0}.$$

Choosing sewing conditions of the form

$$u_x = v_t + X(u,v), \quad v_x = u_t + Y(u,v), \quad \text{at} \quad x = x_0$$

we find

$$\frac{dP^0}{dt} = u_t X - v_t Y.$$

This is a total time derivative if

$$X = -\frac{\partial D^0}{\partial u}, \quad Y = \frac{\partial D^0}{\partial v},$$

for some $D^0$. Then

$$\frac{dP^0}{dt} = -\frac{dD^0}{dt}.$$

- Expected anyway since time translation remains good.

On the other hand, for momentum

$$\frac{dP^1}{dt} = -\left[\frac{u_t^2+u_x^2}{2} - U(u)\right]_{x_0} + \left[\frac{v_t^2+v_x^2}{2} - V(v)\right]_{x_0}$$

$$= \left[-v_t X + u_t Y - \frac{X^2-Y^2}{2} + U - V\right]_{x_0} = -\frac{dD^1(u,v)}{dt}$$

This is a total time derivative provided the first piece is a perfect differential and the second piece vanishes. Thus

$$X = -\frac{\partial D^0}{\partial u} = \frac{\partial D^1}{\partial v}, \quad Y = \frac{\partial D^0}{\partial v} = -\frac{\partial D^1}{\partial u},$$

In other words the fields at the defect should satisfy:

$$\frac{\partial^2 D^0}{\partial v^2} = \frac{\partial^2 D^0}{\partial u^2}, \quad \frac{1}{2}\left(\frac{\partial D^0}{\partial u}\right)^2 - \frac{1}{2}\left(\frac{\partial D^0}{\partial v}\right)^2 = U(u) - V(v).$$

Highly constraining - just a few possible combinations for $U, V, D^0$ ...

- sine-Gordon, Liouville, massless free, or, massive free.

For example, if $U(u) = m^2u^2/2,\ V(v) = m^2v^2/2$, $D^0$ turns out to be

$$D^0(u,v) = \frac{m\sigma}{4}(u+v)^2 + \frac{m}{4\sigma}(u-v)^2,$$

and $\sigma$ is a free parameter.

- Note: the Tzitzéica (aka BD, MZS, $a_2^{(2)}$ affine Toda) potential

$$U(u) = e^u + 2e^{-u/2}$$

is **not** possible.

- There is a Lagrangian description of this type of defect (type I):

$$\mathcal{L} = \theta(-x+x_0)\mathcal{L}(u) + \delta(x-x_0)\left(\frac{uv_t - u_t v}{2} - D^0(u,v)\right) + \theta(x-x_0)\mathcal{L}(v)$$

In the free case ($m \neq 0$), with a wave incident from the left half-line

$$u = \left(e^{ikx} + Re^{-ikx}\right) e^{-i\omega t}, \quad v = T\, e^{ikx} e^{-i\omega t}, \quad \omega^2 = k^2 + m^2,$$

we find:

$$R = 0, \quad T = -\frac{(i\omega - m \sinh\eta)}{(ik + m\cosh\eta)} = -i\,\frac{\sinh\left(\frac{\theta-\eta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta-\eta}{2} + \frac{i\pi}{4}\right)}, \quad \sigma = e^{-\eta}$$

- By design, conserves energy/momentum (no dependence on $x_0$).
- No bound state (provided $\eta$ is real).
- for comparison recall for $\delta$-impurity:

$$u(x_0, t) = v(x_0, t), \quad u_x(x_0, t) - v_x(x_0, t) = 2\lambda u(x_0, t)),$$

$$R = -\frac{\lambda e^{2ix_0}}{\lambda + ik}, \quad T = \frac{ik}{\lambda + ik}$$

- bound state at $k = i\lambda$ if $m > \lambda > 0$.

- the $\delta$-impurity preserves energy but not momentum.

## sine-Gordon - Bowcock, EC, Zambon (2003, 2004, 2005)

Choosing $u, v$ to be sine-Gordon fields (and scaling the coupling and mass parameters to unity), the allowed possibilities are:

$$D^0(u,v) = -2\left(\sigma\cos\frac{u+v}{2} + \sigma^{-1}\cos\frac{u-v}{2}\right),$$

where $\sigma$ is a constant, to find

$$\begin{aligned}
x < x_0: \quad \partial^2 u &= -\sin u, \\
x > x_0: \quad \partial^2 v &= -\sin v, \\
x = x_0: \quad u_x &= v_t - \sigma\sin\frac{u+v}{2} - \sigma^{-1}\sin\frac{u-v}{2}, \\
x = x_0: \quad v_x &= u_t + \sigma\sin\frac{u+v}{2} - \sigma^{-1}\sin\frac{u-v}{2}.
\end{aligned}$$

- The final two are a Bäcklund transformation ‘frozen’ at $x_0$.
- The defect could be anywhere - essentially topological
- Higher spin charges, via an adapted Lax pair, are also conserved.

## Solitons and defects - Bowcock, EC, Zambon (2005)

The sine-Gordon model has solitons and antisolitons.

Consider a soliton incident from $x < 0$ (putting $x_0 = 0$).

It will not be possible to satisfy the sewing conditions (in general, for all times) unless a similar soliton emerges into the region $x > 0$:

$$x < 0: \quad e^{iu/2} = \frac{1 + iE}{1 - iE},$$

$$x > 0: \quad e^{iv/2} = \frac{1 + izE}{1 - izE},$$

$$E = e^{ax+bt+c}, \qquad a = \cosh\theta, \ \ b = -\sinh\theta, \ \ \theta > 0$$

where $z$ is to be determined. It is also useful to set $\sigma = e^{-\eta}$.

- To find....

$$z = \coth\left(\frac{\eta - \theta}{2}\right)$$

$$z = \coth\left(\frac{\eta - \theta}{2}\right) \qquad \theta > 0$$

Remarks:

- $\eta < \theta$ implies $z < 0$; ie the soliton emerges as a (shifted) anti-soliton.

  - the final state will contain a discontinuity of magnitude $4\pi$ at $x = 0$.

- $\eta = \theta$ implies $z = \infty$ and there is **no** emerging soliton.

  - the energy-momentum of the soliton is captured by the 'defect'.

  - the topological charge is also captured by a discontinuity $2\pi$.

- $\eta > \theta$ implies $z > 0$; ie the soliton is shifted but retains its character.

## Comments

- Defects at $x = x_1 < x_2 < x_3 < \cdots < x_n$ behave independently

  - each contributes a factor $z_i$ for a total $z = z_1 z_2 \ldots z_n$.

- Each component of a multisoliton is affected separately

  - thus at most one can be 'filtered out'.

- Since a soliton can be absorbed, could a starting configuration with $u = 0,\ v = 2\pi$ decay into a soliton?

  - needs quantum mechanics to provide the probability for decay.

- Contrast previous uses Estabrook - Wahlquist (1973)

  - a Bäcklund transformation 'creates' a soliton.

- Defects can also move (with constant speed), and scatter.

- What about 'finite gap' solutions of sine-Gordon? EC, Parini (2017)

  - generally quite complicated....

General solutions of sine-Gordon in terms of generalised theta functions - see for example Dubrovin, 1981; Mumford, 1984 - are defined over Riemann sufaces of genus $g$:

$$\theta(z,B)=\sum_{n\in\mathbb{Z}^g} e^{\frac{1}{2}n\cdot Bn+n\cdot z}, \quad z\in\mathbb{C}^g, \quad \mathrm{Re}(B)<0$$

An example - for $g=1$ these are the Jacobi theta functions:

$$\vartheta_1(z)=-\vartheta_2(z+i\pi), \quad \vartheta_2(z)=\sum_{n=-\infty}^{\infty} e^{\frac{B}{2}\left(n+\frac{1}{2}\right)^2+z\left(n+\frac{1}{2}\right)}$$

$$\vartheta_3(z)=\theta(z,B), \quad \vartheta_4(z)=\theta(z+i\pi,B)$$

In terms of these the two solutions to left and right of the defect are:

$$e^{iu/2}=\frac{\vartheta_3(z)}{\vartheta_4(z)}, \quad e^{iv/2}=\frac{\vartheta_3(z+\Delta)}{\vartheta_4(z+\Delta)}, \quad z=\frac{\cosh\theta x-\sinh\theta t}{\vartheta_3(0)\vartheta_4(0)}+z_0$$

Then, $\Delta$ is determined via the sewing conditions and given by

$$e^{\theta-\eta}=i\frac{\vartheta_1(\Delta)}{\vartheta_2(\Delta)}\to\tanh\left(\frac{\Delta}{2}\right), \quad B\to-\infty.$$

The previous result is obtained in the single soliton limit.

## Generalisations

- What about Tzitzéica ($a_2^{(2)}$ affine Toda)?
- Multi-component fields - what about other affine Toda field theories?

  - only the $a_n^{(1)}$ affine Toda theories can work - EC, Zambon (2009)

  - Bäcklund transformations are similar - Fordy, Gibbons (1980)
- What about nonlinear Schrödinger, KdV, mKdV, etc, etc? Caudrelier, Mintchev, Ragoucy (2004,) EC, Zambon (2005), Caudrelier (2008), ...
- Is the setup genuinely integrable? For an alternative (algebraic) approach see Avan, Doikou (2012, 2013); Doikou (2014, 2016)
- What about SUSY? See, for example, Gomes, Ymai, Zimerman (2008); Aguirre, Gomes, Spano, Zimerman (2015)
- What about models in $2+1$ dimensions, for example Kadomtsev-Petviashvili, Davey-Stewartson, etc?

## Classical type II defect - EC, Zambon (2009)

Consider two relativistic field theories with fields $u$ and $v$, and add a new degree of freedom $\lambda(t)$ at the defect location ($x_0 = 0$):

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left((u-v)\lambda_t - D^0(\lambda, u, v)\right)$$

Then the usual Euler-Lagrange equations lead to

- equations of motion:

$$\partial^2 u = -\frac{\partial U}{\partial u} \quad x < 0, \qquad \partial^2 v = -\frac{\partial V}{\partial v} \quad x > 0$$

- defect conditions at $x = 0$

$$u_x = \lambda_t - D^0_u \qquad v_x = \lambda_t + D^0_v \qquad (u-v)_t = -D^0_\lambda.$$

- Note: the quantity $\lambda$ is conjugate to the discontinuity $u - v$ at the defect location.

As before, consider momentum

$$P^1 = -\int_{-\infty}^{0} dx\, u_t u_x - \int_0^{\infty} dx\, v_t v_x ,$$

and seek a functional $D^1(u, v, \lambda)$ such that $P^1_t \equiv -D^1_t$.

As before, implies constraints on $U,\ V,\ D^1$.

Putting $q = (u-v)/2,\ p = (u+v)/2$ these are:

$$D^0_p = -D^1_\lambda \qquad D^0_\lambda = -D^1_p$$

implying

$$D^0 = f(p+\lambda, q) + g(p-\lambda, q) \qquad D^1 = f(p+\lambda, q) - g(p-\lambda, q)$$

and

$$\frac{1}{2}(D^0_\lambda D^1_q - D^0_q D^1_\lambda) = U(u) - V(v)$$

- Powerful constraint on $f, g$ since $\lambda$ does not enter the right side
  - what is the general solution?

Note:

- Now possible to choose $f, g$ for potentials $U, V$ any one of sine-Gordon, Liouville, Tzitzéica, or free massive or massless.
- Tzitzéica:

$$U(u) = (e^u + 2\,e^{-u/2} - 3), \quad V(v) = (e^v + 2\,e^{-v/2} - 3)$$

and the defect potential $D^0(\lambda, p, q)$ is given by

$$\begin{aligned} D^0 = \quad & 2\sigma\left(e^{(p+\lambda)/2} + e^{-(p+\lambda)/4}\left(e^{q/2} + e^{-q/2}\right)\right) \\ & + \frac{1}{\sigma}\left(8\,e^{-(p-\lambda)/4} + e^{(p-\lambda)/2}\left(e^{q/2} + e^{-q/2}\right)^2\right) \end{aligned}$$

- In sine-Gordon the type-II defect has two free parameters
  - in a sense it is two 'fused' type-I defects - EC, Zambon (2009, 2010)
- Other affine Toda field theories?
  - $a_r^{(1)}, \ (c_n^{(1)}, \ d_{n+1}^{(2)}), \ a_{2n}^{(2)}, d_n^{(1)}$ - Robertson (2014); Bowcock, Bristow (2017)
  - needs unifying idea?

For example, $d_4^{(1)}$ is not a straightforward generalisation - the defect part of the Lagrangian is given by Bowcock and Bristow

$$\mathcal{L}_D = \sum_1^4 u_k v_{kt} + 2\lambda_2(u_2 - v_2)_t + 2\lambda_3(u_3 - v_3)_t - (D + \bar{D})$$

and

$$2(U(u) - V(v)) = D_{p_1}\bar{D}_{q_1} + D_{q_2}\bar{D}_{\lambda_2} - D_{\lambda_2}\bar{D}_{q_2} + D_{q_3}\bar{D}_{\lambda_3} - D_{\lambda_3}\bar{D}_{q_3} + D_{p_4}\bar{D}_{q_4}$$

$$q_k = (u_k - v_k)/2, \;\; p_k = (u_k + v_k)/2,$$

with the set of relevant roots given in terms of the orthonormal vectors $e_k, \; k = 1, 2, 3, 4$ by

$$\alpha_0 = -e_1 - e_2, \; \alpha_1 = e_1 - e_2, \; \alpha_2 = e_2 - e_3, \; \alpha_3 = e_3 - e_4, \; \alpha_4 = e_3 + e_4,$$

so that $\alpha_2$ is the central dot in the $d_4^{(1)}$ root diagram.

## Defects in quantum field theory

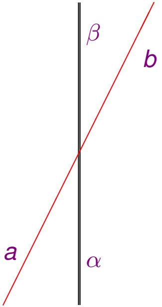
- Expect Soliton-defect scattering - pure transmission compatible with the bulk S-matrix
- Expect Topological charge will be preserved but may be exchanged with the defect
- Expect For each type of defect there may be several types of transmission matrix (eg in sine-Gordon expect two different transmission matrices since the topological charge on a defect can only change by $\pm 2$ as a soliton/anti-soliton passes).
- - More generally, expect transmission matrices to be labelled by weight lattices.
- Expect Not all transmission matrices need be unitary (eg in sine-Gordon one is a ‘resonance’ of the other)
- Questions Relationship between different types of defect; assemblies of defects, defect-defect scattering; fusing defects; ...

A transmission matrix is intrinsically infinite-dimensional:

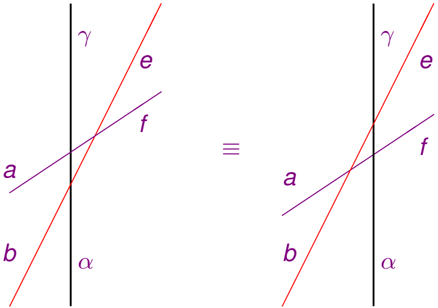
$$T_{a\alpha}^{b\beta}(\theta,\eta), \quad a+\alpha=b+\beta$$

where $\alpha, \beta$ and $a, b$ are defect and particle (eg soliton) labels respectively (typically they will be sets of weights); and $\eta$ is a collection of defect parameters.

Schematically:



Schematic compatibility relation - Delfino, Mussardo, Simonetti (1994)




$$S_{ab}^{cd}(\Theta)\, T_{d\alpha}^{f\beta}(\theta_a) T_{c\beta}^{e\gamma}(\theta_b) = T_{b\alpha}^{d\beta}(\theta_b) T_{a\beta}^{c\gamma}(\theta_a) S_{cd}^{ef}(\Theta)$$

With $\Theta = \theta_a - \theta_b$ and sums over the ‘internal’ indices $\beta, c, d$.

- For sine-Gordon a solution was known - Konik, LeClair (1999)

## Zamolodchikov's sine-Gordon soliton-soliton S-matrix - reminder

$$S_{ab}^{cd}(\Theta) = \rho(\Theta) \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & C & B & 0 \\ 0 & B & C & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

where

$$A(\Theta) = \frac{qx_2}{x_1} - \frac{x_1}{qx_2}, \; B(\Theta) = \frac{x_1}{x_2} - \frac{x_2}{x_1}, \; C(\Theta) = q - \frac{1}{q}$$

$$x_a = e^{\gamma\theta_a}, \; a = 1, 2, \; \Theta = \theta_1 - \theta_2, \; q = e^{i\pi\gamma}, \; \gamma = \frac{8\pi}{\beta^2} - 1,$$

and

$$\rho(\Theta) = \frac{\Gamma(1+z)\Gamma(1-\gamma-z)}{2\pi i} \prod_1^\infty R_k(\Theta) R_k(i\pi - \Theta)$$

$$R_k(\Theta) = \frac{\Gamma(2k\gamma + z)\Gamma(1 + 2k\gamma + z)}{\Gamma((2k+1)\gamma + z)\Gamma(1 + (2k+1)\gamma + z)}, \; z = i\gamma/\pi.$$

Useful to define the variable $Q = e^{4\pi^2 i/\beta^2} = \sqrt{-q}$.

- K-L solutions have the form

$$T^{b\beta}_{a\alpha}(\theta) = f(q,x)\begin{pmatrix} Q^{\alpha}\,\delta^{\beta}_{\alpha} & q^{-1/2}e^{\gamma(\theta-\eta)}\,\delta^{\beta-2}_{\alpha} \\ q^{-1/2}\,e^{\gamma(\theta-\eta)}\,\delta^{\beta+2}_{\alpha} & Q^{-\alpha}\,\delta^{\beta}_{\alpha} \end{pmatrix}$$

where $f(q,x)$ is not uniquely determined but, for a unitary transmission matrix, should satisfy

$$\begin{aligned} \bar{f}(q,x) &= f(q,qx) \\ f(q,x)f(q,qx) &= \left(1+e^{2\gamma(\theta-\eta)}\right)^{-1} \end{aligned}$$

- A 'minimal' solution has the following form

$$f(q,x) = \frac{e^{i\pi(1+\gamma)/4}}{1+ie^{-2\pi iy}}\,\frac{r(x)}{\bar{r}(x)},$$

where it is convenient to put $y = i\gamma(\theta-\eta)/2\pi$ and

$$r(x) = \prod_{k=0}^{\infty}\frac{\Gamma(k\gamma+1/4-y)\Gamma((k+1)\gamma+3/4-y)}{\Gamma((k+1/2)\gamma+1/4-y)\Gamma((k+1/2)\gamma+3/4-y)}$$

$$T_{a\alpha}^{b\beta}(\theta) = f(q,x)\begin{pmatrix} Q^{\alpha}\,\delta_{\alpha}^{\beta} & q^{-1/2}e^{\gamma(\theta-\eta)}\,\delta_{\alpha}^{\beta-2} \\ q^{-1/2}\,e^{\gamma(\theta-\eta)}\,\delta_{\alpha}^{\beta+2} & Q^{-\alpha}\,\delta_{\alpha}^{\beta} \end{pmatrix}$$

Remarks (supposing $\theta > 0$) - Bowcock, EC, Zambon (2005):

Tempting to suppose $\eta$ (possibly renormalized) is the same parameter as in the type I classical model.

- $\eta < 0$ - the off-diagonal entries dominate;
- $\theta > \eta > 0$ - the off-diagonal entries dominate;
- $\eta > \theta > 0$ - the diagonal entries dominate.

• Similar features to the classical soliton-defect scattering.

• The different behaviour of solitons versus anti-solitons (diagonal terms) is a direct consequence of the defect term in the Lagrangian proportional to

$$\delta(x - x_0)(uv_t - vu_t)/2$$

• $\theta = \eta$ is not special (neither is $y = -1/4$) but there is a simple pole nearby at $y = 1/4$:

$$\theta = \eta - \frac{i\pi}{2\gamma} \to \eta, \text{ as } \beta \to 0$$

This pole is like a resonance, with complex energy,

$$E = m_s \cosh\theta = m_s(\cosh\eta\cos(\pi/2\gamma) - i\sinh\eta\sin(\pi/2\gamma))$$

and a 'width' proportional to $\sin(\pi/2\gamma)$.

• The Zamolodchikov S-matrix has 'breather' poles corresponding to soliton-anti-soliton bound states at

$$\Theta = i\pi(1 - n/\gamma), \; n = 1, 2, ..., n_{\max};$$

use the bootstrap to define the transmission factors for breathers and find for the lightest breather:

$$T(\theta) = -i\frac{\sinh\left(\frac{\theta-\eta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta-\eta}{2} + \frac{i\pi}{4}\right)}$$

## Type II transmission matrix for sine-Gordon - EC, Zambon (2010)

There is another, more general, set of solutions to the quadratic relations for the transmission matrix:

$$\rho(\theta)\begin{pmatrix} (a_+ Q^\alpha + a_- Q^{-\alpha}\, x^2)\,\delta_\alpha^\beta & x\,(b_+ Q^\alpha + b_- Q^{-\alpha})\,\delta_\alpha^{\beta-2} \\ x\,(c_+ Q^\alpha + c_- Q^{-\alpha})\,\delta_\alpha^{\beta+2} & (d_+ Q^\alpha\, x^2 + d_- Q^{-\alpha})\,\delta_\alpha^\beta \end{pmatrix}$$

where $x = e^{\gamma\theta}$.

The free constants satisfy the two constraints

$$a_\pm\, d_\pm - b_\pm\, c_\pm = 0$$

These and $\rho(\theta)$are constrained further by crossing and unitarity.

- For a range of parameters this describes a type II defect.

- With $a_- = d_+ = 0$ and $b_+ = c_- = 0$ or $b_- = c_+ = 0$ (after a similarity transformation), reduces to the type I solution.

- For another choice of parameters reduces to a direct sum of the Zamolodchikov S-matrix and two infinite dimensional pieces.

## Alternative formulation - Weston (2010)

Summary: for Type II

$$T = \rho(x)\begin{pmatrix} xa_+Q^{-N} + x^{-1}a_-Q^N & A \\ A^* & xd_+Q^N + x^{-1}d_-Q^{-N} \end{pmatrix},$$

where $A^*$ and $A$ are 'generalised' raising and lowering operators, respectively,

$$A^*|k\rangle = |k+2\rangle \quad A|k\rangle = F(k)|k-2\rangle \quad N|k\rangle = k|k\rangle,\ k \in \mathbb{Z}$$

$$F(N) = f_0 + f_+Q^{2N} + f_-Q^{-2N}, \quad f_+ = Q^{-2}a_-d_+, \quad f_- = Q^2a_+d_-$$

- $T$ intertwines the coproducts of finite (soliton) and infinite (defect) representations of the Borel subalgebra of $U_q(a_1^{(1)})$.

- Idea extends to all other quantum algebras allowing (in principle) calculations of associated defect matrices. For some examples see EC, Zambon (2010), Boos et al. (2011).

- How to construct $A, A^*$ in terms of fields?

## Defect-defect scattering - type I

$$T_{1\,a\alpha}^{\;\;b\gamma}\,T_{2\,b\beta}^{\;\;c\delta}\,U_{\gamma\delta}^{\rho\sigma}=U_{\alpha\beta}^{\delta\gamma}\,T_{2\,a\delta}^{\;\;b\rho}\,T_{1\,b\gamma}^{\;\;c\sigma}.$$

$$T_i\approx\begin{pmatrix} Q^{N_i} & \beta_i\,x\,A_i \\ \beta_i\,x\,A_i^* & Q^{-N_i}\end{pmatrix},\quad i=1,2$$

where

$$x=e^{\gamma\theta},\ q=e^{i\pi\gamma},\ Q^2=-q;\quad \beta_i^*=\beta_i.$$

Data carried by $\beta_i,\ A_i,\ A_i^*$, $i=1,2$, $F(N)=f_0$, with two sets of mutually commuting generalised annihilation and creation operators.

$U$ is independent of $x$: equating terms in powers of $x$ leads to the following four equations:

$$\left(\beta_2\,Q^{N_1}\,A_2+\beta_1\,Q^{-N_2}\,A_1\right)\,U=U\,\left(\beta_1\,Q^{N_2}\,A_1+\beta_2\,Q^{-N_1}\,A_2\right)$$

$$\left(\beta_1\,Q^{N_2}\,A_1^*+\beta_2\,Q^{-N_1}\,A_2^*\right)\,U=U\,\left(\beta_2\,Q^{N_1}\,A_2^*+\beta_1\,Q^{-N_2}\,A_1^*\right)$$

$$Q^{N_1+N_2}\,U=U\,Q^{N_1+N_2},\quad A_1\,U\,A_1=A_2\,U\,A_2$$

$$U = \sum_{k=-\infty}^{\infty} A_1^k A_2^{-k} U_k(N_1, N_2, \lambda), \quad \lambda = \beta_1/\beta_2$$

Then

$$U_{k+2}(N_1, N_2, \lambda) = U_k(N_1 - 2, N_2 + 2, \lambda)$$

$$U_{2l}(N_1, N_2, \lambda) = U_0(N_1 - 2l, N_2 + 2l, \lambda)$$

$$U_{2l+1}(N_1, N_2, \lambda) = U_1(N_1 - 2l, N_2 + 2l, \lambda).$$

and

$$\left(\lambda Q^{-N_2} A_1 + Q^{N_1} A_2\right) U = U \left(Q^{-N_1} A_2 + \lambda Q^{N_2} A_1\right),$$

$$Q^{N_1-2} U_1^{(N_1,N_2)} + \lambda Q^{-N_2} U_0^{(N_1,N_2)} = \lambda Q^{N_2} U_0^{(N_1-2,N_2)} + Q^{-N_1} U_1^{(N_1,N_2-2)}$$

$$Q^{N_1} U_0^{(N_1,N_2+2)} + \lambda Q^{-N_2} U_1^{(N_1+2,N_2)} = \lambda Q^{N_2+2} U_1^{(N_1,N_2)} + Q^{-N_1} U_0^{(N_1,N_2)}$$

Formal generating functionals

$$U(x,y) = \sum_{n,m} x^n y^m U_0(n,m), \quad V(x,y) = \sum_{n,m} x^n y^m U_1(n,m)$$

Then

$$\begin{aligned} \lambda U(x, y/Q) + Q^{-2} V(Qx, y) &= \lambda x^2 U(x, Qy) + y^2 V(x/Q, y) \\ (\lambda/x^2) V(x, y/Q) + (1/y^2) U(Qx, y) &= \lambda Q^2 V(x, Qy) + U(x/Q, y). \end{aligned}$$

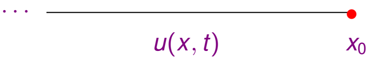
These can be written slightly more symmetrically by rearranging and putting $r = Q,\ s = \lambda Q$:

$$\begin{aligned} x\, U(x, ry) - x^{-1}\, U(x, r^{-1}y) &= \frac{y}{xs} \left( (ry)^{-1}\, V(rx, y) - ry\, V(r^{-1}x, y) \right) \\ y^{-1}\, U(rx, y) - y\, U(r^{-1}x, y) &= \frac{ys}{x} \left( rx\, V(x, ry) - (rx)^{-1}\, V(x, r^{-1}y) \right). \end{aligned}$$

- What is the general solution?

## Boundaries revisited: the sine-Gordon model EC, Zambon (2012)

Start with a single selected point on the $x$-axis, say $x_0 = 0$, and denote the field to the left ($x < 0$) by $u$:




- The sine-Gordon model with a general (two-parameter) integrable boundary condition was analyzed by Ghoshal, Zamolodchikov (1994), ...

- ...and sine-Gordon model with dynamical boundary was considered by Baseilhac, Delius (2001), Baseilhac, Koizumi (2003)

- A defect (or several defects) can be placed in front of the boundary and generate a new boundary (as seen from $x \ll 0$); for the sinh-Gordon example, see Bajnok, Simon (2008).

But...

- The defect will introduce dependence on topological charge in the modified reflection matrix.
- Generally, the boundary should be considered as carrying topological charge, which may change as a soliton reflects.
- Ansatz

$$R_{a\,\alpha}^{b\,\beta}(\theta) = \begin{pmatrix} r_+(\alpha, x)\,\delta_\alpha^\beta & s_+(\alpha, x)\,\delta_\alpha^{\beta-2} \\ s_-(\alpha, x)\,\delta_\alpha^{\beta+2} & r_-(\alpha, x)\,\delta_\alpha^\beta \end{pmatrix}$$

- Boundary Yang-Baxter equation Cherednik (1984)

$$R_{a\,\alpha}^{q\,\beta}(\theta_a)\,S_{b\,q}^{p\,s}(\Theta_+)R_{p\,\beta}^{r\,\gamma}(\theta_b)\,S_{s\,r}^{d\,c}(\Theta_-) = S_{b\,a}^{p\,q}(\Theta_-)R_{p,\alpha}^{r\,\beta}(\theta_b)S_{q\,r}^{s\,c}(\Theta_+)R_{s\,\beta}^{d\,\gamma}(\theta_a),$$

with $\Theta_+ = (\theta_b + \theta_a)$ and $\Theta_- = (\theta_b - \theta_a)$.

- Ghoshal-Zamolodchikov solution reformulated

$$R_{a\,\alpha}^{b\,\beta}(\theta) = \sigma(\theta) \begin{pmatrix} (r_1 x + r_2/x)\,\delta_\alpha^\beta & k_0\left(x^2 - 1/x^2\right)\delta_\alpha^{\beta-2} \\ l_0\left(x^2 - 1/x^2\right)\delta_\alpha^{\beta+2} & (r_2 x + r_1/x)\,\delta_\alpha^\beta \end{pmatrix}$$

and $l_0 = k_0$, $r_1 r_2 = 1$.

- General solution

$$r_+(\alpha, x) = \left(x^2 - 1/x^2\right)\left(r_3 q^{\alpha+1} x - r_4 q^{-\alpha-1}/x\right) + r_1 x + r_2/x,$$

$$r_-(\alpha, x) = \left(x^2 - 1/x^2\right)\left(r_4 q^{-\alpha+1} x - r_3 q^{\alpha-1}/x\right) + r_2 x + r_1/x,$$

$$s_+(\alpha, x) = \left(x^2 - 1/x^2\right)(k_0 + k_1 q^\alpha + k_2 q^{-\alpha}),$$

$$s_-(\alpha, x) = \left(x^2 - 1/x^2\right)(l_0 + l_1 q^\alpha + l_2 q^{-\alpha}),$$

$$k_1 l_1 = -r_3^2, \quad k_2 l_2 = -r_4^2, \quad k_1 l_0 + q^2 k_0 l_1 = q r_2 r_3, \quad k_0 l_2 + q^2 k_2 l_0 = q r_1 r_4.$$

- A defect placed in front of a boundary generalises $R$ according to

$$R_{a\,\alpha\,\tilde{\alpha}}^{b\,\beta\,\tilde{\beta}}(\theta) = T_{a\,\tilde{\alpha}}^{c\,\tilde{\gamma}}(\theta) R_{c\,\alpha}^{d\,\beta}(\theta) \hat{T}_{d\,\tilde{\gamma}}^{b\,\tilde{\beta}}(\theta)$$

where $\hat{T}(\theta) = T^{-1}(-\theta)$.

Begin with an $R$ matrix corresponding to a Dirichlet boundary condition,

$$R^{(0)}{}_{c\,\alpha}^{d\,\beta}(\theta) = \sigma(\theta)\begin{pmatrix} (rx + x^{-1}r^{-1})\,\delta_\alpha^\beta & 0 \\ 0 & (rx^{-1} + xr^{-1})\,\delta_\alpha^\beta \end{pmatrix}$$

- $T_{II}R^{(0)}\,\hat{T}_{II}$ is equivalent to the general solution given above when $T_{II}$ is the general type II transmission matrix;
- $T_I R^{(0)}\,\hat{T}_I$ equivalent to the G-Z solution when $T_I$ is restricted to the type I (Konik-LeClair) transmission matrix.

• Is there a Lagrangian description of the generalised boundary condition corresponding to the general solution? For example

$$\mathcal{L}_B(u,\lambda) = \theta(-x)\,\mathcal{L}_{sG} + \delta(x)(u\lambda_t - B(u,\lambda)),$$

with

$$B(u,\lambda) = e^{\lambda/2}f(u) + e^{-\lambda/2}g(u),$$

and

$$f(u)g(u) = h_+e^{u/2} + h_-e^{-u/2} + 2\,(e^u + e^{-u}) + h_0$$

Thank you!